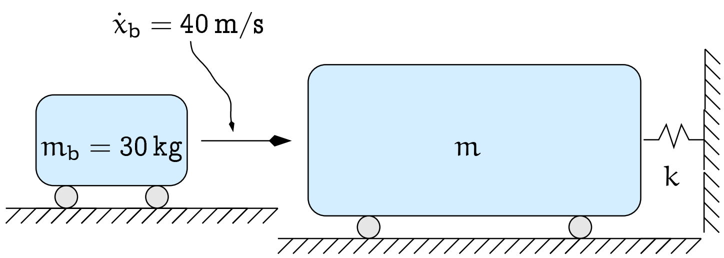
Dynamics of Structures 2009-2010

1st home assignment

1 Impact



A body of mass $m_b = 30\,kg$ hits an undamped *SDOF* system, of unknown characteristics k and m, with velocity $\dot{x}_b = 40\,m\,s^{-1}$.

After collision the two masses are «glued» together and a measurement of the ensuing free oscillations give the following results:

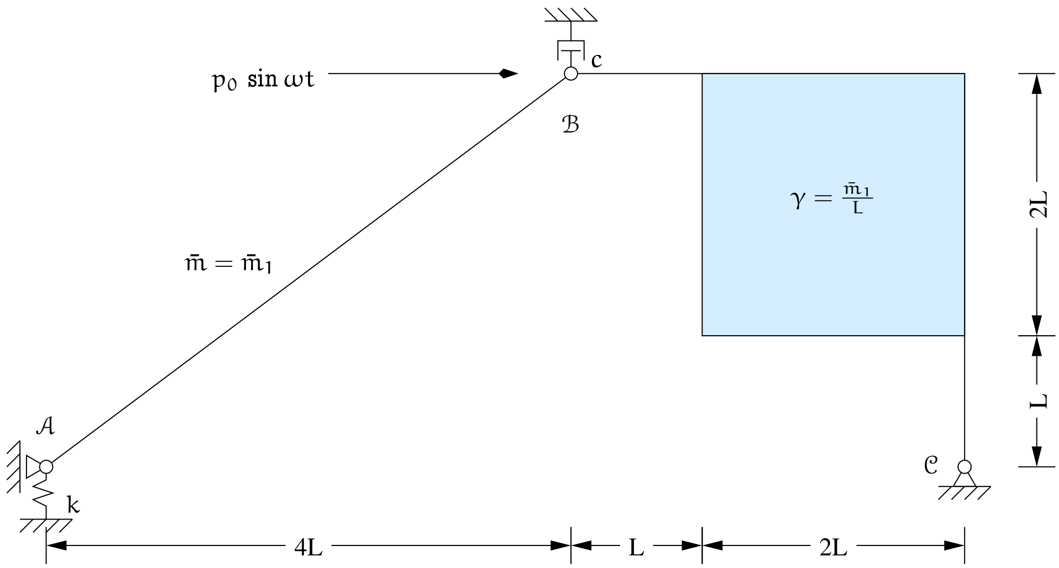
$$x_{max} = 48\,mm, \qquad \dot{x}_{max} = 240\,mm\,s^{-1}.$$

What is the natural frequency of vibration of the *original* single degree of freedom system?

Solution

The mass of the incoming body	30.000	kg
Its velocity	40.000	m/s
Its momentum	1200.000	kg m /s
The total momentum	1200.000	kg m /s
Maximum compound velocity	0.240	m/s
Compound mass	5000.000	kg
Original mass	4970.000	kg
max compound displ.	0.048	m
omega=vel_mx/displ_mx	5.000	rad/s
Omega^2	25.000	
k/m_tot	25.000	
k	125000.000	N/m
Omega_orig	5.015	rad/s
Freq	0.798	Hz
T_n	1.253	s

2 Generalised Coordinates (rigid bodies)



The articulated rigid system in figure is composed by two rigid bars,

- $\overline{\mathcal{AB}}$, with unit mass $\bar{m}_1$,
- the massless $\overline{\mathcal{BC}}$;

and by a rigid square body that is solidal to $\overline{\mathcal{BC}}$, with unit mass $\gamma = \bar{m}_1/L$.

The fixed constraints are a vertical roller in $\mathcal{A}$, a hinge in $\mathcal{C}$ and an internal hinge in $\mathcal{B}$, the deformable constraints are a vertical spring in $\mathcal{A}$, its stiffness $= k$ and a vertical dashpot in $\mathcal{B}$, its damping coefficient $= c$.

The system is excited by an horizontal force, $p(t) = p_o \sin \omega t$.

Using preferably the rotation of $\overline{\mathcal{AB}}$ about $\mathcal{A}$ as the generalised coordinate, write the equation of equilibrium of the system using the Principle of Virtual Displacements.

oOo

Solution

Using the anti-clockwise rotation θ of $\overline{\mathcal{AB}}$ about $\mathcal{C}$ (sorry for the misunderstanding) as the generalised coordinate, the displacements (and velocities and accelerations, writing $\dot{\theta}$ and $\ddot{\theta}$ in place of θ) of the points where there are applied forces are computed as in the following table

	x	y	u	v
$\mathcal{A}$	$-7L$	0	0	$-7\theta L$
$\mathcal{B}$	$-3L$	$3L$	$-3\theta L$	$-3\theta L$
G_{sq}	$-L$	$2L$	$-2\theta L$	$-\theta L$
G_{AB}	$-5L$	$\frac{3}{2}L$	$-\frac{3}{2}\theta L$	$-5\theta L$

All the rotations, particularly the rotations of the centres of mass, are of course equal to θ.

A similar table can be written for the virtual displacements, and the equation of motion is finally

$$\begin{aligned}
&- M_{sq}\left[(-2\ddot{\theta}L)(-2\delta\theta L) + (-\ddot{\theta}L)(-\delta\theta L)\right] - J_{sq}(\ddot{\theta})(\delta\theta) \\
&\quad - M_{AB}\left[(-\frac{3}{2}\ddot{\theta})(-\frac{3}{2}\delta\theta) + (-5\ddot{\theta}L)(-5\delta\theta L)\right] - J_{AB}(\ddot{\theta})(\delta\theta) \\
&\quad\quad - c(-3\dot{\theta}L)(-3\delta\theta L) - k(-7\theta L)(-7\delta\theta L) + p_0(-3\theta L) = 0
\end{aligned}$$

Simplifying $\delta\theta$, collecting $\ddot{\theta}$, $\dot{\theta}$ and θ and rearranging, the equation of motion can now be written as

$$\left(5M_{sq}L^2 + \frac{109}{4}M_{AB}L^2 + J_{sq} + J_{AB}\right)\ddot{\theta} + 9c\dot{\theta}L^2 + 49k\theta L^2 = -3p_0L.$$

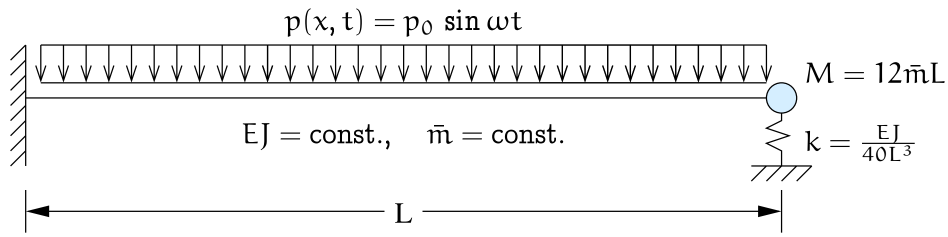
The values of masses and inertias can be expressed in terms of the unit mass and unit length, that is

$$\begin{aligned}
M_{AB} &= 5mL, & J_{AB} &= M_{AB}\frac{(5L)^2}{12} = \frac{125}{12}mL^3 \\
M_{sq} &= 4mL, & J_{sq} &= M_{sq}\frac{(2L)^2 + (2L)^2}{12} = \frac{8}{3}mL^3,
\end{aligned}$$

so that substituting and simplifying in the equation of motion we finally find

$$(\frac{508}{3}mL^3)\ddot{\theta} + (9L^2c)\dot{\theta} + (49L^2k)\theta = -3p_0L$$

3 Generalised Coordinates (flexible systems)



The beam in figure, clamped at the left and supported by a spring at the right, supports a dimensionless body at the right end.

The bending stifness and unit mass of the beam, EJ and $\bar{m}$, are constants, the supported body has mass $M = 12\bar{m}L$ and the spring has stiffness $k = \frac{EJ}{40L^3}$.

The beam-mass-spring system is excited by a spatially uniform, distributed load $p(x,t) = p_0 \sin \omega t$.

Using an appropriate shape function write the equation of motion of the equivalent *SDOF* system.

Solution

As a first shot, lets use the dhape function appropriate for a cantilever beam,

$$\phi(x) = 1 - \cos \frac{\pi x}{2L}$$

so that, for harmonic free vibration, we have

$$v(x) = Z \sin \omega t \, \phi(x) = Z \sin \omega t \, (1 - \cos \frac{\pi x}{2L})$$

and the maximum of the kinetic energy and of the elastic energy can be written

$$T_{max} = \frac{1}{2} \left(\omega^2 Z^2 \int_0^L m\phi^2(x)\,dx + \omega^2 Z^2 M \right),$$

$$V_{max} = \frac{1}{2} \left(Z^2 \int_0^L EJ\phi''^2(x)\,dx + Z^2 k \right),$$

so that, equating the two values, simplifying Z and solving for ω^2 we have

$$\omega^2 = \frac{\int_0^L EJ\phi''^2(x)\,dx + k}{\int_0^L m\phi^2(x)\,dx + M}.$$

We have, either from simple computations or from problem statement, that

$$\int_0^L m\phi^2(x)\,dx = \left(\frac{3}{2} - \frac{4}{\pi}\right) mL = 0.2268mL, \qquad M = 12mL,$$
$$\int_0^L EJ\phi''^2(x)\,dx = \frac{\pi^4}{32}\frac{EJ}{L^3} = 3.044\frac{EJ}{L^3}, \qquad k = \frac{1}{40}\frac{EJ}{L^3},$$

so that, substituting into the previous equation, we find

$$\omega^2 = \frac{3.069}{12.23}\frac{EJ}{mL^4} = 0.2510\frac{EJ}{mL^4}.$$

The equation of motion can be written as

$$\begin{cases} 12.23mL\,\ddot{Z} + 3.069\frac{EJ}{L^3}\,Z = p_o\int_0^L \phi\,dx = (1 - \frac{2}{\pi})p_0L = 0.3634p_0L \\ v(x,t) = Z(t)\,(1 - \cos\frac{\pi x}{2L}). \end{cases}$$

4 Numerical Integration

A *SDOF* has the following characteristics:

$$\begin{aligned} k &= 32\,\mathrm{kN\,m^{-1}}, \\ m &= 1800\,\mathrm{kg}, \\ \zeta &= 7\%, \\ f_y &= 2.5\,\mathrm{kN} \end{aligned}$$

and is subjected to a loading $p(t)$,

$$p(t) = 30\,\mathrm{kN}\begin{cases} at + 12(at)^2 - 64(at)^3, & 0 \leqslant t \leqslant 0.25\,\mathrm{s},\ \text{with } a = 1\,\mathrm{s}^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Disregarding the non-linear behaviour, for initial rest conditions, give the exact equation of motion, $x = x(t)$ and integrate numerically[1] the equation of motion with

1. the algorithm of central differences,

[1] Do not print all the intermediate results for every time step for every procedure.

2. the algorithm of constant acceleration and
3. the algorithm of linear acceleration,

with time step $h = 0.005\,\mathrm{s}$ in all three cases.

Plot the results of the numerical procedures and the exact solution.

Solution

The circular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{32000}{1800}} = \sqrt{\frac{160}{9}} = 4.21637\frac{\text{rad}}{\text{s}},$$

the damped circular frequency is

$$\omega_D = \omega\sqrt{1-\zeta^2} = 4.206027\frac{\text{rad}}{\text{s}}$$

and the damping coefficient is

$$c = 2\zeta\omega m = 2 \cdot 0.07 \cdot 4.21637 \cdot 1800 = 1062.53\,\mathrm{N\,s\,m^{-1}}$$

The particular integral is written as

$$\xi(t) = A + Bt + Ct^2 + Dt^3$$

Deriving repeatedly ξ with respect to t, substituting in the equation of motion and finally equating the coefficients of the powers of t in the two members, one finds (with $a = 1\,\mathrm{s}^{-1}$)

$$\begin{aligned} Dk &= -64P_0a^3 & D &= -60.0000\,a^3 \\ Ck &= +12P_0a^2 - 3Dc & C &= +17.2267\,a^2 \\ Bk &= \ +1P_0a^1 - 2Cc - 6Dm & B &= +20.0435\,a^1 \\ Ak &= \ +0P_0a^0 - 1Bc - 2Cm & A &= \ -2.6035\,a^0 \end{aligned}$$

and the particular integral and its time derivative can be written as

$$\begin{aligned} \xi(t) &= -2.6035 + 20.0435\,at + 17.2267\,(at)^2 - 60.0\,(at)^3, \\ \dot{\xi}(t) &= a\left[20.0435119028 + 34.4534095554\,at - 180\,(at)^2\right]. \end{aligned}$$

The general integral and its first time derivative are

$$x(t) = \exp(-\zeta\omega t)\,(A\cos\omega_D t + B\sin\omega_D t) + \xi(t),$$
$$\dot{x}(t) = \exp(-\zeta\omega t)\,[(B\cos\omega_D t - A\sin\omega_D t)\omega_D - \zeta\omega\,(A\cos\omega_D t + B\sin\omega_D t)] + \dot{\xi}(t).$$

Evaluating the general integral and its time derivative for $t = 0$, for rest initial conditions one has

$$A - 2.6035 = 0, \qquad A = 2.6035,$$
$$\omega_D B - \zeta\omega A + 20.0435 = 0. \qquad B = -4.58273.$$

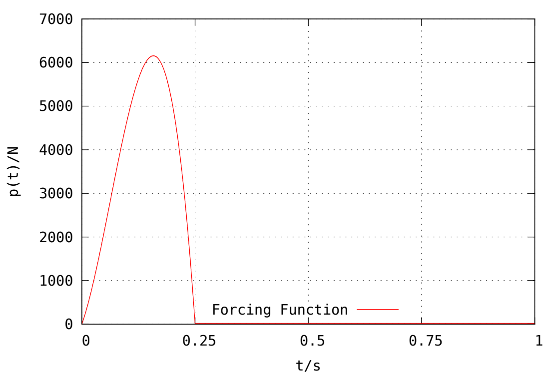
Substituting the integration constants and all the numerical values in the general integral,

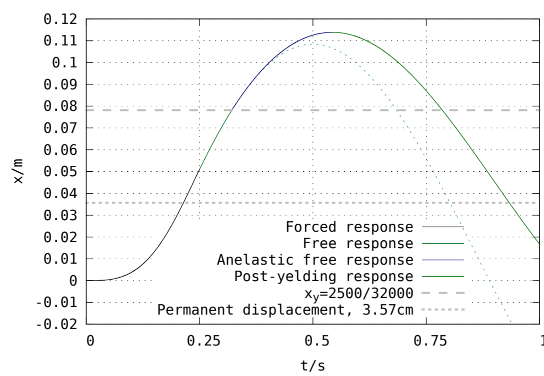
$$x(t) = \exp(-0.29t)\,(2.60\cos(4.21t) - 4.58\sin(4.21t)) + 20.04t + 17.23t^2 - 60t^3 - 2.60$$
$$\dot{x}(t) = \exp(-0.29t)(-9.60\sin(4.21t) - 20.04\cos(4.21t)) + 34.45t - 180t^2 + 20.04$$

The equations above are valid for $0.00\,\text{s} \leqslant t \leqslant 0.25\,\text{s}$ and it is $x(0.25\,\text{s}) = 0.051\,\text{m}$ and $\dot{x}(0.25\,\text{s}) = 0.428\,\text{m}\,\text{s}^{-1}$, so that imposing these values as the initial conditions for the free response we easily find

$$x_f(t) = \exp(-\zeta\omega t)(0.051\cos(\omega_D(t - 0.25)) + 0.105\sin(\omega_D(t - 0.25_))))$$

In the plot below, note that the spring experiences yielding during the free response phase.





Optional

Repeat the exercise keeping into account non-linear behaviour.

Solution

We have seen that yielding occurs in the free response phase, so the equation of motion is

$$m\,\ddot{x} + c\,\dot{x} = -k\,x_y.$$

The integral of the homogeneous associate is

$$x_h(t) = A\exp(-\frac{c}{m}t) + B\exp 0t = A\exp(-\frac{c}{m}t) + B,$$

the particular integral is

$$\xi(t) = -\frac{k\,x_y}{c}t$$

and the general integral is

$$x(t) = x_h(t) + \xi(t) = A\exp(-\frac{c}{m}t) + B - \frac{k\,x_y}{c}t.$$

Yielding occurs when $x_f(t) = 2\,500\,/32\,000\,\text{m} = 0.078\,125\,\text{m}$, hence for $t = t_y = 0.321\,075\,\text{s}$, and it is $\dot{x}_f(0.3210748) = 0.329\,239\,\text{m}\,\text{s}^{-1}$.

Imposing this initial conditions, it is

$$x_y(t) = 4.621850 - 4.543725\exp(-0.590292(t-t_y)) - 2.352885(t-t_y),$$
$$\dot{x}_y(t) = 2.682124\exp(-0.590292t) - 2.352885.$$

We leave the plastic phase when $\dot{x}_y(t) = 0$, numerically $t_e = 0.542\,943$ second, and the initial conditions for the elastic response are

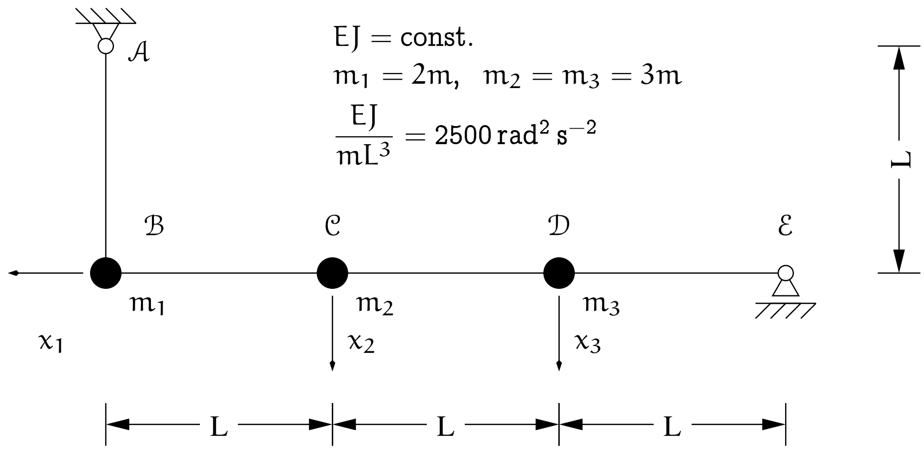
$$x(t-t_e) = x_{max} = 0.113\,852\,\text{m}, \qquad \dot{x}(t-t_e) = 0.$$

It must be noted that the elastic force is $f_S = kx - k(x_{pl})$, with $x_{pl} = x_{max} - x_{y0} = 0.113\,852\,\text{m} - 0.078\,125\,\text{m} = 0.035\,727\,\text{m}$.

With the above consideration taken into account, bimposing the initial conditions it is

$$(0.078125\cos\omega_D(t-t_e) + 0.0054822\sin\omega_D(t-t_e))\exp -\zeta\omega(t-t_e) + 0.035726745004$$

5 *MDOF* System



In the structure depicted above, the structural mass is negligible with respect to the masses of the three supported bodies, so it is correct to use the three *DOF*'s in the figure as the dynamical degrees of freedom of the system.

1. Compute the flexibilty matrix[2] $\mathbf{F}$ and the mass matrix[3] $\mathbf{M}$.

2. Compute all the eigenvalues and the eigenvectors of the 3 *DOF* system using a method of your choice.

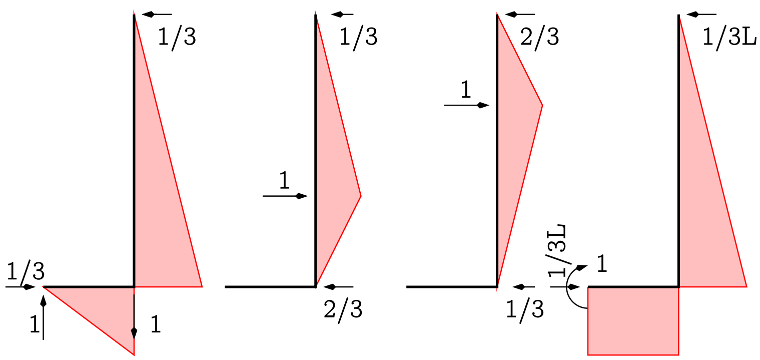
3. For non-null initial velocities

$$\dot{x}_0 = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix}^T \frac{L}{2500\,\text{s}}$$

compute the rotations in $\mathcal{A}$.

[2]Disregarding axial deformability.
[3]Be careful, $m_{11} \neq m_1$!

Solution



1. The flexibility coefficients, see figure, are given by

$$
\begin{aligned}
f_{11} &= \int_0^L x\frac{x}{EJ}\,dx + \int_0^{3L}\frac{x}{3}\frac{x}{3EJ}\,dx &&= \frac{4}{3}\frac{L^3}{EJ}\\
f_{21} &= \int_0^{3L}\frac{x}{3}\frac{x}{3EJ}\,dx - \int_{2L}^{3L}\frac{x}{3}\frac{x-2L}{EJ}\,dx &&= \frac{5}{9}\frac{L^3}{EJ}\\
f_{31} &= \int_0^{3L}\frac{x}{3}\frac{x}{3EJ}\,dx - \int_{L}^{3L}\frac{x}{3}\frac{x-L}{EJ}\,dx &&= \frac{4}{9}\frac{L^3}{EJ}\\
f_{22} &= \int_0^{L}\frac{2x}{3}\frac{2x}{3EJ}\,dx + \int_0^{2L}\frac{x}{3}\frac{x}{3EJ}\,dx &&= \frac{4}{9}\frac{L^3}{EJ}\\
f_{32} &= 2\int_0^{L}\frac{x}{3}\frac{2x}{3EJ}\,dx + \int_0^{L}\frac{L+x}{3}\frac{2L-x}{3EJ}\,dx &&= \frac{7}{18}\frac{L^3}{EJ}\\
f_{33} &= f_{22}\\
f_{41} &= \int_0^{L}1\frac{1x}{EJ}\,dx + \int_0^{3L}\frac{x}{3L}\frac{x}{3EJ}\,dx &&= \frac{3}{2}\frac{L^2}{EJ}\\
f_{42} &= \int_0^{L}\frac{3L-x}{3L}\frac{2x}{3EJ}\,dx + \int_0^{2L}\frac{x}{3L}\frac{x}{3EJ}\,dx &&= \frac{5}{9}\frac{L^2}{EJ}\\
f_{43} &= \int_0^{L}\frac{x}{3L}\frac{2x}{3EJ}\,dx + \int_0^{2L}\frac{3L-x}{3L}\frac{x}{3EJ}\,dx &&= \frac{4}{9}\frac{L^2}{EJ}
\end{aligned}
$$

The flexibility matrix is

$$
\mathbf{F} = \frac{L^3}{18\,EJ}\begin{bmatrix} 24 & 10 & 8\\ 10 & 8 & 7\\ 8 & 7 & 8 \end{bmatrix}
$$

and, by inversion, the stiffness matrix is

$$\mathbf{K} = \frac{EJ}{28\,L^3}\begin{bmatrix} 45 & -72 & 18 \\ -72 & 384 & -264 \\ 18 & -264 & 276 \end{bmatrix}.$$

Appropriately lumping masses in the first degree of freedom, the mass matrix can be written as

$$\mathbf{M} = m\begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The dynamic matrix is $\mathbf{D} = \mathbf{F}\,\mathbf{M}$,

$$\mathbf{D} = \frac{m\,L^3}{18\,EJ}\begin{bmatrix} 192 & 30 & 24 \\ 80 & 24 & 21 \\ 64 & 21 & 24 \end{bmatrix}.$$

2. The eigenvectors and the normalised eigenvalues $\bar{\omega}_j^2$ (normalized with respect to EJ/mL^3 that is, $\omega_j^2 = \bar{\omega}_j^2\omega^0 = \bar{\omega}_j^2\frac{EJ}{mL^3}$) can be calculated using this short program

```
import scipy as sp

M = sp.mat('8 0 0 ; 0 3 0 ; 0 0 3')
F = sp.mat('24 10 8 ; 10 8 7 ; 8 7 8')/18.
K = sp.mat('45 -72 18 ; -72 384 -264 ; 18 -264 276')/28.
D = sp.mat('192 30 24 ; 80 24 21 ; 64 21 24')/18.
S = sp.eye(3)

Psi = sp.mat('0. 0. 0.;0. 0. 0.;0. 0. 0.')
L   = sp.mat('0. 0. 0.;0. 0. 0.;0. 0. 0.')

for n in (1, 2, 3):
    x0 = sp.mat('1.;1.;1.')
    Dn = D*S
    for j in range(10):
        x1 = Dn*x0
        w2 = x0[0,0]/x1[0,0]
        x0 = x1*w2
        print w2, x0.T
    x0 = x0/sp.sqrt((x0.T*M*x0)[0,0])
    Psi[:,n-1] = x0
```

```
    L[n-1,n-1] = w2
    print 'Normalized'
    print w2, x0.T
    print "-"*72
    S = S - x0*x0.T*M


dotx = sp.mat('1.;0.;0.')
dotq = Psi.T*M*dotx
Fred = sp.mat('27 10 8')/18.

print Fred*M*Psi*sp.sqrt(L)*sp.diagflat(dotq)
```

that, when run, gives (with some omissis)

```
0.0731707317073 [[ 1.          0.50813008  0.44308943]]
0.0826150229486 [[ 1.          0.46585693  0.3915258 ]]
[...]
0.0836876786857 [[ 1.          0.46103734  0.38559605]]
0.0836876788132 [[ 1.          0.46103734  0.38559604]]
Normalized
0.0836876788132 [[ 0.33179371  0.15296929  0.12793834]]
------------------------------------------------------------------------
-5.04900698097 [[ 1.        -2.78850732 -3.58162557]]
0.804082367184 [[ 1.        -2.77579565 -3.59682426]]
[...]
0.803412108385 [[ 1.        -2.77419027 -3.59874374]]
0.803412108253 [[ 1.        -2.77419027 -3.59874374]]
Normalized
0.803412108253 [[ 0.11957304 -0.33171836 -0.43031272]]
------------------------------------------------------------------------
-5378.53603549 [[  1.        -18.02570814  14.6366017 ]]
7.17084583162 [[  1.        -18.02547142  14.63642813]]
[...]
7.17093592722 [[  1.        -18.02547142  14.63642813]]
Normalized
7.17093592722 [[ 0.02480368 -0.44709804  0.3630373 ]]
------------------------------------------------------------------------
```

3. The initial velocities vector can be written

$$\dot{q x}_0 = \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix}^T \frac{L}{2500\,\mathrm{s}}$$
$$= \begin{Bmatrix} 1 & 0 & 0 \end{Bmatrix}^T \alpha L \omega_0$$

where $\omega_0^2 = \frac{EJ}{mL^3}$ and $\alpha = \frac{1}{\omega_0 2500\,\mathrm{s}}$, hence the initial conditons in terms of modal coordinates are given by

$$\dot{q}_0 = \mathbf{\Psi}^T \mathbf{M}\dot{x}_0 = \begin{Bmatrix} 2.6543 \\ 0.95658 \\ 0.19842 \end{Bmatrix} \alpha L \omega_0$$

and it is

$$\mathbf{q}(t) = \mathbf{\Lambda}^{-\frac{1}{2}} \begin{bmatrix} 2.654 & 0 & 0 \\ 0 & 0.95658 & 0 \\ 0 & 0 & 0.19842 \end{bmatrix} \begin{Bmatrix} \sin \omega_1 t \\ \sin \omega_2 t \\ \sin \omega_3 t \end{Bmatrix} \alpha L \omega_0 = \mathbf{\Lambda}^{-\frac{1}{2}} \mathbf{Q} \cdot \mathbf{s}(t) \alpha L \omega_0$$

the pseudo accelerations in modal coordinates, $\mathbf{a}_m(t) = \mathbf{\Lambda}\, \mathbf{q}(t)$ are

$$\mathbf{a}_m(t) = \mathbf{\Lambda}^{+\frac{1}{2}} \mathbf{Q} \cdot \mathbf{s}(t) \alpha L \omega_0$$

the nodes' pseudo accelerations are

$$\mathbf{a}_n(t) = \mathbf{\Psi}\mathbf{\Lambda}^{\frac{1}{2}} \mathbf{Q} \cdot \mathbf{s}(t) \alpha L \omega_0$$

the equivalent static forces are

$$\mathbf{f}^{st}(t) = \mathbf{M}\mathbf{a}_n = \mathbf{M}\mathbf{\Psi}\mathbf{\Lambda}^{\frac{1}{2}} \mathbf{Q} \cdot \mathbf{s}(t) \alpha L \omega_0.$$

The rotation in A, $\theta_{\mathcal{A}}$, positive if clockwise, can be computed in terms of equivalent static forces, as the matricial product

$$\frac{L^2}{18EJ} \begin{bmatrix} 27 & 10 & 8 \end{bmatrix} \cdot \mathbf{f}^{st} = \mathbf{F}_{red} \cdot \mathbf{f}^{st},$$

substituting our expression for the equivalent static force

$$\phi_{\mathcal{A}} = \mathbf{F}_{red} \mathbf{M}\mathbf{\Psi}\mathbf{\Lambda}^{\frac{1}{2}} \mathbf{Q} \cdot \mathbf{s}(t) \alpha L \omega_0.$$

With numerical values, it is

$$\phi_{\mathcal{A}} = \frac{L^2}{EJ} \begin{bmatrix} 3/2 & 5/9 & 4/9 \end{bmatrix} m \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot$$

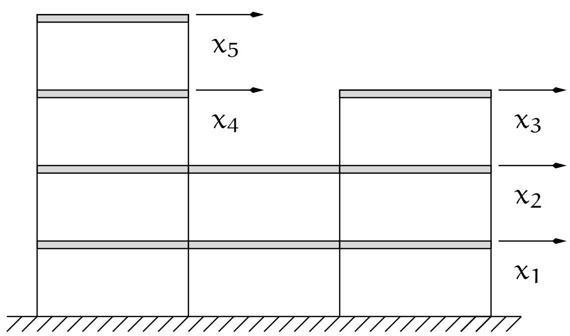
$$\begin{bmatrix} 0.3318 & 0.1196 & 0.0248 \\ 0.1530 & -0.3317 & -0.4471 \\ 0.1279 & -0.4303 & 0.3630 \end{bmatrix} \omega_0 \begin{bmatrix} 0.2893 & 0. & 0. \\ 0. & 0.8963 & 0. \\ 0. & 0. & 2.6779 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 2.654 & 0 & 0 \\ 0 & 0.95658 & 0 \\ 0 & 0 & 0.19842 \end{bmatrix} \cdot \mathbf{s}(t) \alpha L \omega_0.$$

Observing that $\omega_0^2 \frac{mL^3}{EJ} = 1$ and performing all the numerical matrix multiplications,

$$\phi_{\mathcal{A}}(t) = \alpha \left\{ 3.3840568 \quad 0.26430948 \quad 0.01941107 \right\} \begin{Bmatrix} \sin \omega_1 t \\ \sin \omega_2 t \\ \sin \omega_3 t \end{Bmatrix}.$$

6 Rayleigh-Ritz & Subspace Iteration



The structure depicted above can be analyzed as a shear type building. All columns are equal, each with a lateral stiffness indicated by k. One of the consequences of the previous statement is that the direct stiffness k_{11} is equal to $8k$, because the number of column that must be deformed to have a unit displacement in x_1 is 8.
The mass matrix is diagonal, with $m_{11} = m_{22} = 3m$ and $m_{33} = m_{44} = m_{55} = m$.

oOo

Find the first three eigenvalues and the first three eigenvectors of the structure using the Rayleigh-Ritz procedure with the Ritz base $\hat{\mathbf{\Phi}}_0$ indicated on the right, denoting the Ritz coordinates eigenvector matrix with $\mathbf{Z}$.

$$\hat{\mathbf{\Phi}}_0 = \begin{bmatrix} +1 & +1 & +1 \\ +2 & +1 & +1 \\ +3 & +0 & -1 \\ +3 & +0 & +1 \\ +4 & -1 & +1 \end{bmatrix}$$

oOo

Do one subspace iteration, deriving a new set of Ritz base vectors,

$$\hat{\mathbf{\Phi}}_1 = \mathbf{K}^{-1}\mathbf{M}\mathbf{\Phi}\mathbf{Z}_0.$$

oOo

Find the first three eigenvalues and the first three eigenvectors of the structure using the Rayleigh-Ritz procedure with the Ritz base $\hat{\mathbf{\Phi}}_1$.

oOo

Discuss the two set of results.

Solution

The structural matrices are given by

$$\mathbf{M} = m \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{K} = k \begin{bmatrix} +8 & -4 & 0 & 0 & 0 \\ -4 & +8 & -2 & -2 & 0 \\ 0 & -2 & +2 & 0 & 0 \\ 0 & -2 & 0 & +4 & -2 \\ 0 & 0 & 0 & -2 & +2 \end{bmatrix}.$$

The reduced matrices are given by

$$\mathbf{M}^z = \hat{\mathbf{\Phi}}_0^T \mathbf{M} \hat{\mathbf{\Phi}}_0 = m \begin{bmatrix} 49 & 5 & 13 \\ 5 & 7 & 5 \\ 13 & 5 & 9 \end{bmatrix}, \qquad \mathbf{K}^z = \hat{\mathbf{\Phi}}_0^T \mathbf{K} \hat{\mathbf{\Phi}}_0 = k \begin{bmatrix} 14 & -2 & 0 \\ -2 & 10 & 8 \\ 0 & 8 & 12 \end{bmatrix}$$

and solving for the Ritz eigenvectors give

$$\mathbf{\Lambda}^z = \frac{k}{m} \begin{bmatrix} 0.2568 & 0 & 0 \\ 0 & 1.0962 & 0 \\ 0 & 0 & 2.3680 \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} 0.1325 & 0.0342 & -0.1252 \\ 0.0362 & 0.4942 & 0.0287 \\ 0.0206 & -0.3548 & 0.4022 \end{bmatrix},$$

$$\mathbf{\Psi}^z = \hat{\mathbf{\Phi}}_0 \mathbf{Z} = \begin{bmatrix} 0.1893 & 0.1736 & 0.3057 \\ 0.3219 & 0.2078 & 0.1804 \\ 0.3771 & 0.4575 & -0.7779 \\ 0.4182 & -0.2521 & 0.0265 \\ 0.5145 & -0.7121 & -0.1275 \end{bmatrix}.$$

For reference, the exact solutions are

$$\mathbf{\Lambda} = \frac{k}{m}\begin{bmatrix} 0.2527 & 0 & 0 & 0 & 0 \\ 0 & 1.0731 & 0 & 0 & 0 \\ 0 & 0 & 2.2957 & 0 & 0 \\ 0 & 0 & 0 & 4.0000 & 0 \\ 0 & 0 & 0 & 0 & 5.7119 \end{bmatrix},$$

$$\mathbf{\Psi} = \begin{bmatrix} 0.1785 & -0.1805 & 0.3825 & 0.3333 & -0.1072 \\ 0.3232 & -0.2157 & 0.1064 & -0.3333 & 0.2447 \\ 0.3700 & -0.4654 & -0.7197 & 0.3333 & -0.1319 \\ 0.4434 & 0.3107 & 0.0140 & -0.3333 & -0.7717 \\ 0.5075 & 0.6705 & -0.0945 & 0.3333 & 0.4158 \end{bmatrix}.$$

To do a subspace iteration, first we compute the new base

$$\hat{\mathbf{\Phi}}_1 = \mathbf{K}^{-1}\mathbf{M}\hat{\mathbf{\Phi}}_0\mathbf{Z} = \begin{bmatrix} 0.7108 & 0.1594 & 0.1448 \\ 1.2796 & 0.1886 & 0.0604 \\ 1.4681 & 0.4173 & -0.3285 \\ 1.7459 & -0.2936 & 0.0099 \\ 2.0031 & -0.6496 & -0.0539 \end{bmatrix}$$

the new reduced matrices are,

$$\mathbf{M}^z = \hat{\mathbf{\Phi}}_1^T\mathbf{M}\hat{\mathbf{\Phi}}_1 = m\begin{bmatrix} 15.6443 & -0.1373 & -0.0320 \\ -0.1373 & +0.8652 & -0.0016 \\ -0.0320 & -0.0016 & +0.1848 \end{bmatrix},$$

$$\mathbf{K}^z = \hat{\mathbf{\Phi}}_0^T\mathbf{K}\hat{\mathbf{\Phi}}_0 = k\begin{bmatrix} +3.9535 & -0.0270 & -0.0068 \\ -0.0270 & +0.9281 & -0.0013 \\ -0.0068 & -0.0013 & +0.4282 \end{bmatrix}$$

and the solutions are

$$\mathbf{\Lambda}^z = \frac{k}{m}\begin{bmatrix} 0.2527 & 0.0000 & 0.0000 \\ 0.0000 & 1.0740 & 0.0000 \\ 0.0000 & 0.0000 & 2.3178 \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} 0.2528 & 0.0101 & 0.0049 \\ -0.0028 & 1.0758 & 0.0063 \\ -0.0009 & -0.0030 & 2.3266 \end{bmatrix},$$

$$\mathbf{\Psi}^z = \hat{\mathbf{\Phi}}_0\mathbf{Z} = \begin{bmatrix} 0.1791 & 0.1782 & 0.3415 \\ 0.3229 & 0.2156 & 0.1481 \\ 0.3703 & 0.4647 & -0.7544 \\ 0.4422 & -0.2982 & 0.0298 \\ 0.5082 & -0.6785 & -0.1196 \end{bmatrix}$$