# Dynamics of Structures

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If we restrict ourselves to analysis of *linear systems*, however, it is so convenient to use the principle of superposition to study the combined effects of static and dynamic loadings that different methods, of different character, are applied to these different loadings.

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**Dynamic Loading** a Loading that varies over time

**Dynamic Response** the Response of a structural system to a dynamic loading, expressed in terms of stresses and/or deflections

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Our focus will be on *deterministic analysis*

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**Non Periodic Load** here we see two sub-cases,

- the load is described in terms of analytic functions, $p(t) = p_o f(t)$,
- the load is experimentally measured, and is known only in a discrete set of instants; in this case, we say that we have a *time-history*.

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More fundamentally, a dynamical problem is characterized by the relevance of *inertial forces*, arising from the motion of structural or serviced masses.

A dynamic analysis is *required* only when the inertial forces represent a significant portion of the total load, otherwise a static analysis will suffice, even if the loads are (slowly) varying over time.

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If we can assume that the mass is concentrated in a discrete set of *lumped masses*, the analytical problem is greatly simplified, because the inertial forces are applied only at the lumped masses, and the deflections can be computed at these points only, consenting the formulation of the problem in terms of a set of ordinary differential equations, one for each component of the inertial forces.

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Of course, a continuous system has an infinite number of degrees of freedom.

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When the mass is distributed, we can simplify our problem using *generalized coordinates*. The deflections are expressed in terms of a linear combination of assigned functions of position, with the coefficients of the linear combination being the generalized coordinates. E.g., the deflectlions of a rectilinear beam can be expressed with a trigonometric series.

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To fully describe a displacement field, we need to combine an infinity of linearly indipendent *base functions*, but in practice a good approximation can be achieved using only a small number of functions and degrees of freedom.
Even if the method of generalized coordinates has its beauty, we must recognise that for each different problem we should derive an *ad hoc* formulation, without generality.

# Finite Element Method



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in FEM, we use a *piecewise approximation* to displacements which depend, in a finite portion of the structure, from the displacements component of the *nodal points* that surround that particular portion or *element*, using *interpolation functions*.

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The desired level of approximation can be achieved simply, by further subdiving the structure. Another nice feature is that the resulting equations are only loosely coupled, leading to an easier computer solution

# Writing the eq. of motion



In a deterministic d. analysis, given a prescribed load, we want to evaluate the displacements in each instant of time.

In most cases, a limited number of DDOFs gives a sufficient accuracy, and in general the d. problem can be reduced to the determination of the time-histories of some selected component of displacements,

The mathematical expression that define the dynamic displacements are known as the *Equations of Motion* (EOM), the solution of the EOM gives the requested displacements.

The formulation of the EOM is the most important, often the most difficult part of our task of dynamic analysts.

# Writing the EOM, cont.



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# 1 DOF System

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Structural dynamics is all about a motion in the neighbourhood of a point of equilibrium.

We'll start by studying a generic single degree of freedom system, with *constant* mass $m$, subjected to a non-linear generic force $F = F(y, \dot{y})$, where $y$ is the displacement and $\dot{y}$ the velocity of the particle. The equation of motion is

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It is difficult to integrate the above equation in the general case, but it's easy when the motion occurs in a small neighbourhood of the equilibrium position.

# 1 DOF System, cont.

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In a position of equilibrium, $y_{\text{eq}}$, the velocity and the acceleration are zero, and hence $f(y_{\text{eq}}, 0) = 0$.
The force can be linearized in a neighbourhood of $y_{\text{eq}}, 0$:

$$f(y, \dot{y}) = f(y_{\text{eq}}, 0) + \frac{\partial f}{\partial y}(y - y_{\text{eq}}) + \frac{\partial f}{\partial \dot{y}}(\dot{y} - 0) + O(y, \dot{y}).$$

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Assuming that $O(y, \dot{y})$ is small in a neighbourhood of $y_{\text{eq}}$,

$$\ddot{x} + a\dot{x} + bx = 0$$

where $x = y - y_{\text{eq}}$, $a = -\frac{\partial f}{\partial \dot{y}}$ and $b = -\frac{\partial f}{\partial y}$.

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where $x = y - y_{\text{eq}}$, $a = -\frac{\partial f}{\partial \dot{y}}$ and $b = -\frac{\partial f}{\partial y}$.
In an infinitesimal nb of $y_{\text{eq}}$, the equation of motion can be studied in terms of a linear differential equation of second order!

A linear constant coefficient differential equation has the integral $x = A\exp(st)$, that substituted in the equation of motion gives

$$s^2 + as + b = 0$$

whose solutions are

$$s_{1,2} = -\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b}.$$

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The nature of the solution depends on the sign of the real part of $s_1,\ s_2$.

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If we write $s_i = r_i + \imath q_i$, then we have

$$\exp(s_i t) = \exp(\imath q_i t) \exp(r_i t).$$

If one of the $r_i > 0$, the response grows infinitely over time, even for an infinitesimal perturbation of the equilibrium, so that in this case we have an *unstable equilibrium*.

If both $r_i < 0$, the response decrease over time, so we have a *stable equilibrium*.

Finally, if both $r_i = 0$ the $s$'s are imaginary, the response is harmonic with constant amplitude.

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