SDOF linear oscillator

Response to Harmonic Loading

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Undamped Oscillator

The Equation of Motion

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Substituting x with ξ and simplifying, we get

$$C\,(k - \omega^2 m) = p_0.$$

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Starting from our last equation,

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We can now write the particular solution, with the dependencies on β singled out in the second term:

$$\xi(t) = \frac{p_0}{k} \frac{1}{1 - \beta^2} \sin \omega t.$$

Response Ratio

Denoting with Δ_{st} the static deformation, $\Delta_{st} = p_0/k$, we may write the particular solution in terms of Δ_{st} and the *Response Ratio*, $R(t;\,\beta)$, whose amplitude depends *only* on the *frequency ratio* $\beta = \frac{\omega}{\omega_n}$,

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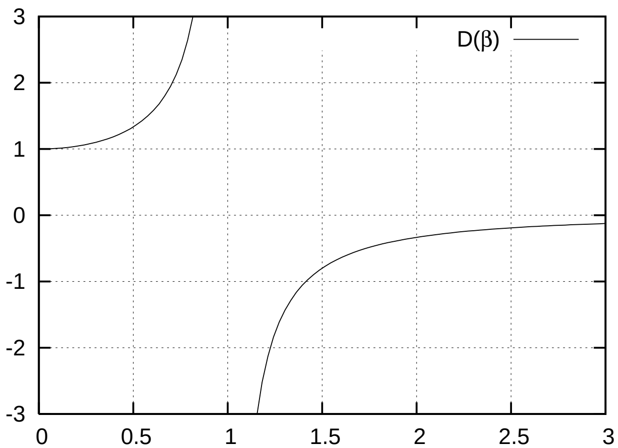

$D(\beta)$ is stationary and almost equal to 1 when $\omega << \omega_n$ (this is a *quasi*-static behaviour), it grows out of bound when $\beta \Rightarrow 1$ (resonance), it is negative for $\beta > 1$ and goes to 0 when $\omega >> \omega_n$ (high-frequency loading).

Response from Rest

We write $x(t)$, $x(0)$ and finally equate to 0:

$$x(t) = A\,\sin\omega_n t + B\,\cos\omega_n t + \Delta_{st}\,\frac{1}{1-\omega^2/\omega_n^2}\,\sin\omega t,$$
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We do as above for the velocity:

$$\dot{x}(t) = \omega_n A \cos \omega_n t - \omega_n B \sin \omega_n t + \Delta_{st} \frac{\omega}{1 - \omega^2/\omega_n^2} \cos \omega t,$$

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Solving for A and substituting, we have the response from rest:

$$x(t) = \Delta_{st}\;D(\beta)\left(\sin\omega t - \beta\,\sin\omega_n t\right).$$

Resonant Response

We have seen that the response to harmonic loading with zero initial conditions is

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As you can see, there is a term in quadrature with the loading, whose amplitude grows linearly and without bounds.

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A particular solution to this equation is a harmonic function $G\sin(\omega t - \theta)$ not in phase with input; it is however equivalent and convenient to write :

$$\begin{aligned}
\xi(t) &= G_1\,\sin\omega t + G_2\,\cos\omega t,\\
\dot{\xi}(t) &= \omega\,(G_1\,cos\omega t - G_2\,\sin\omega t),\\
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Substituting x with ξ and simplifying, we get

$$(G_2(\omega_n^2 - \omega^2) + 2G_1\omega\omega_\zeta)\cos\omega t + (G_1(\omega_n^2 - \omega^2) - 2G_2\omega\omega_n\zeta - p_0/m)\sin\omega t = 0.$$

The Particular Equation

Evaluting the previous equation for $t = 0$ and $t = \frac{\pi}{2\omega}$, we get the following linear system in $G_1,\ G_2$:

$$\begin{aligned} G_2(\omega_n^2 - \omega^2) + G_1 2\zeta\omega\omega_n &= 0, \\ G_1(\omega_n^2 - \omega^2) - G_2 2\zeta\omega\omega_n &= p_0/m. \end{aligned}$$

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Solving and doing the usual substitutions for $\Delta_{st}$and β, we can write:

$$\xi(t) = \Delta_{st}\left(\frac{1-\beta^2}{(2\beta\zeta)^2 + (1-\beta^2)^2}\sin\omega t + \frac{-2\beta\zeta}{(2\beta\zeta)^2 + (1-\beta^2)^2}\cos\omega t\right).$$

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To write the *stationary response* in terms of a *dynamic amplification factor*, it is convenient to reintroduce the phase difference θ:

$$\xi(t) = \Delta_{st}\,R(t;\ \beta,\zeta), \quad R = D(\beta,\zeta)\,sin\,(\omega t - \theta)\,.$$

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Let's start analyzing the phase difference $\theta(\beta, \zeta)$. Its expression is:

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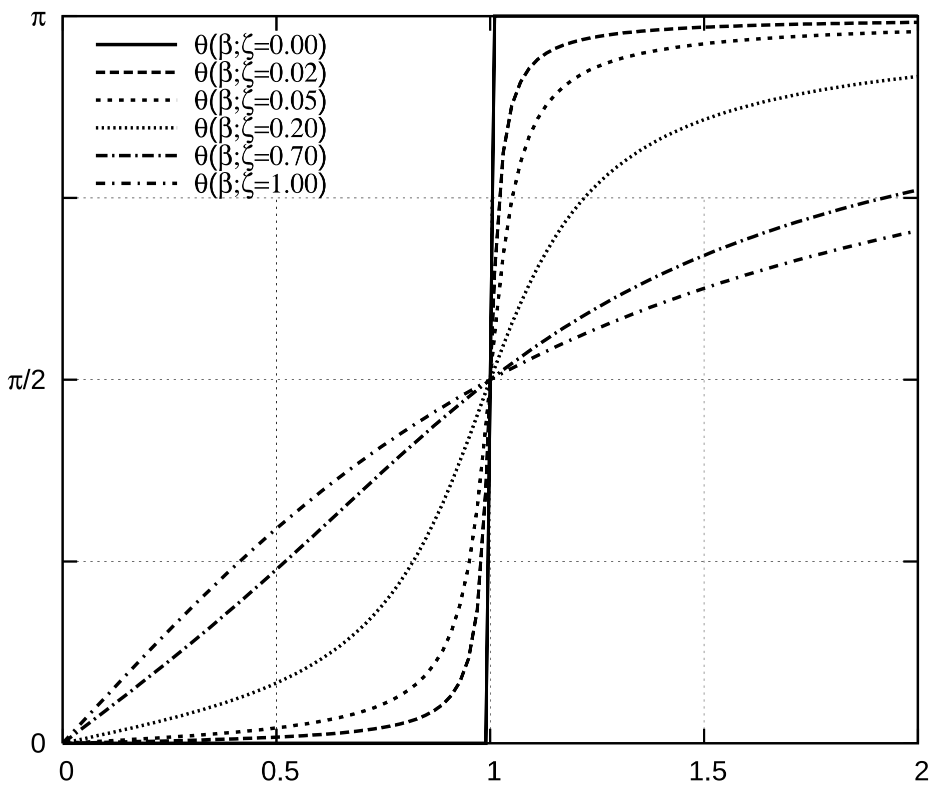

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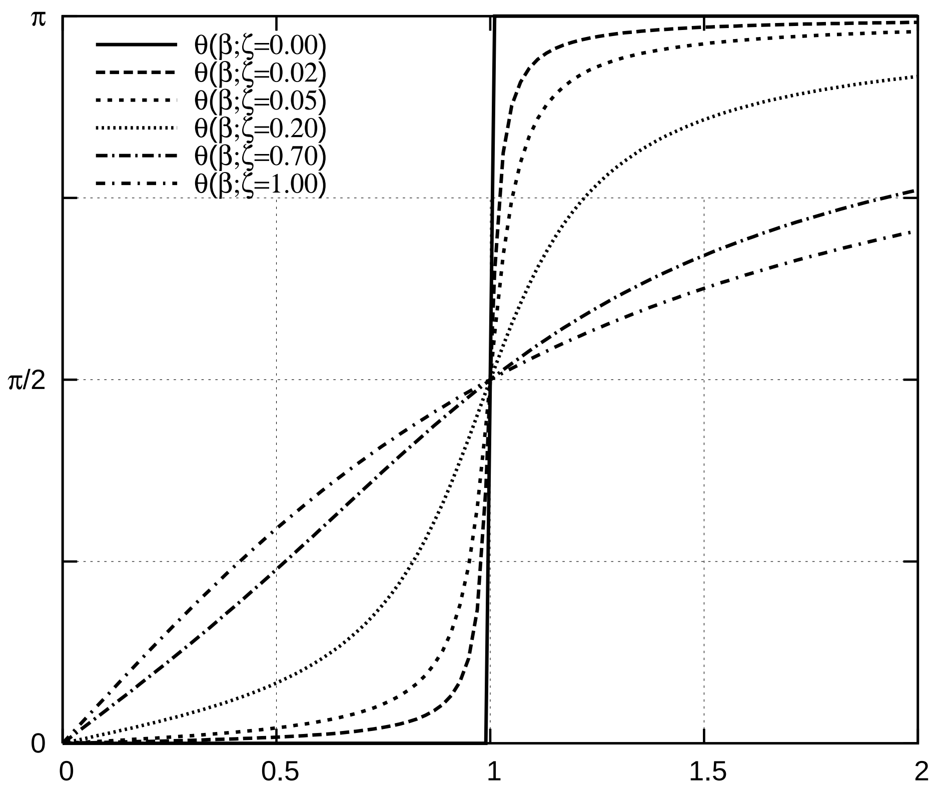

$\theta(\beta,\zeta)$ has a sharper variation around $\beta = 1$ for decreasing values of ζ, but it is apparent that, in the case of slightly damped structures, the response is approximately in phase for low frequencies of excitation, and in opposition for high frequencies. It is worth mentioning that for $\beta = 1$ we have that the response is in perfect quadrature with the load: this is very important to detect resonant response in dynamic tests of structures.

Dynamic Magnification Ratio

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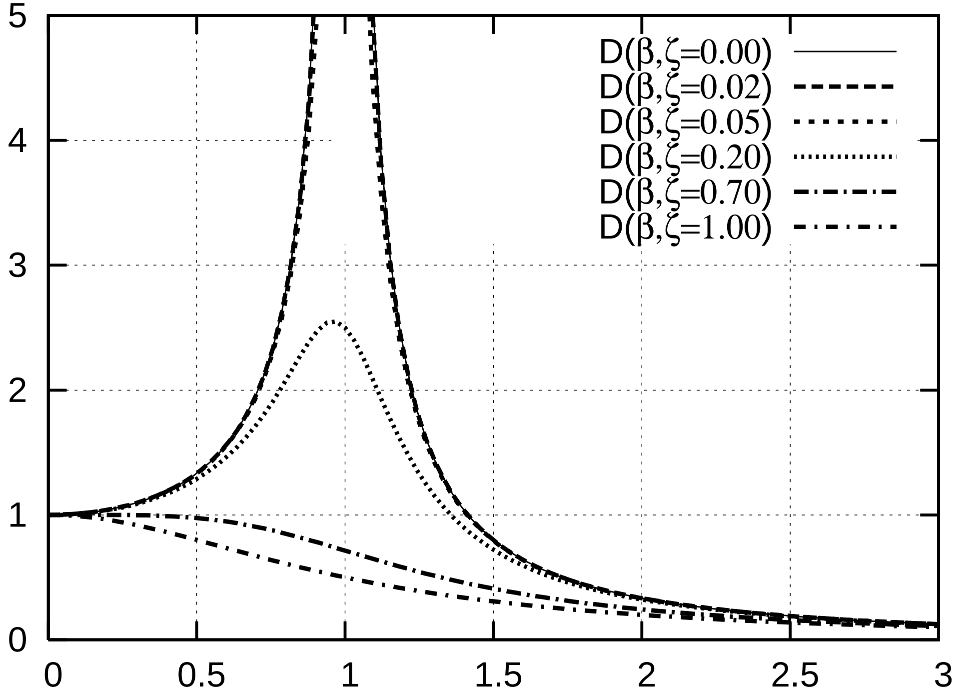

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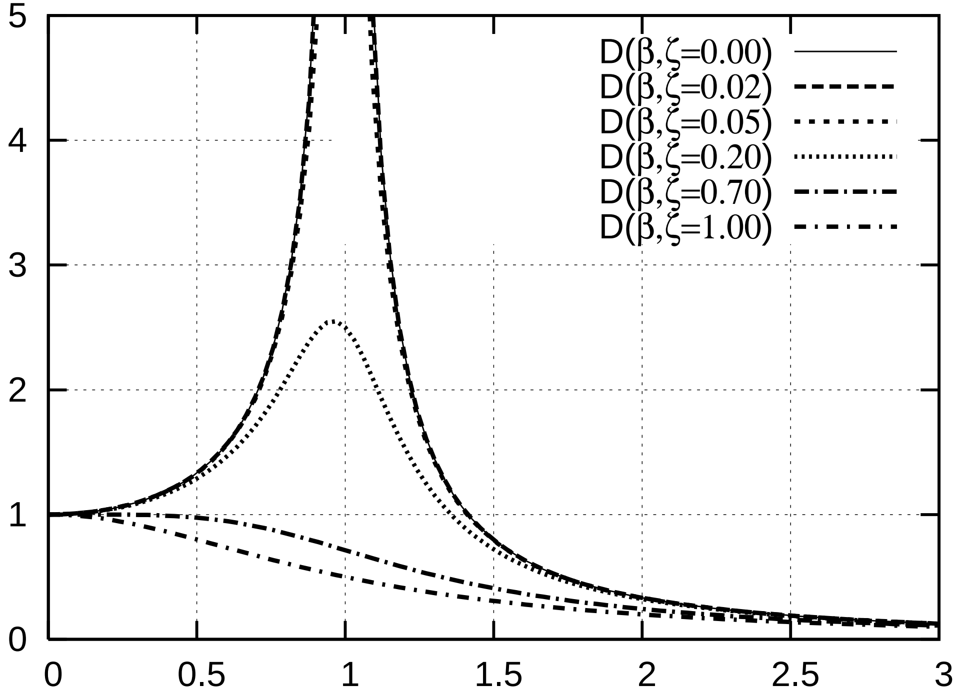

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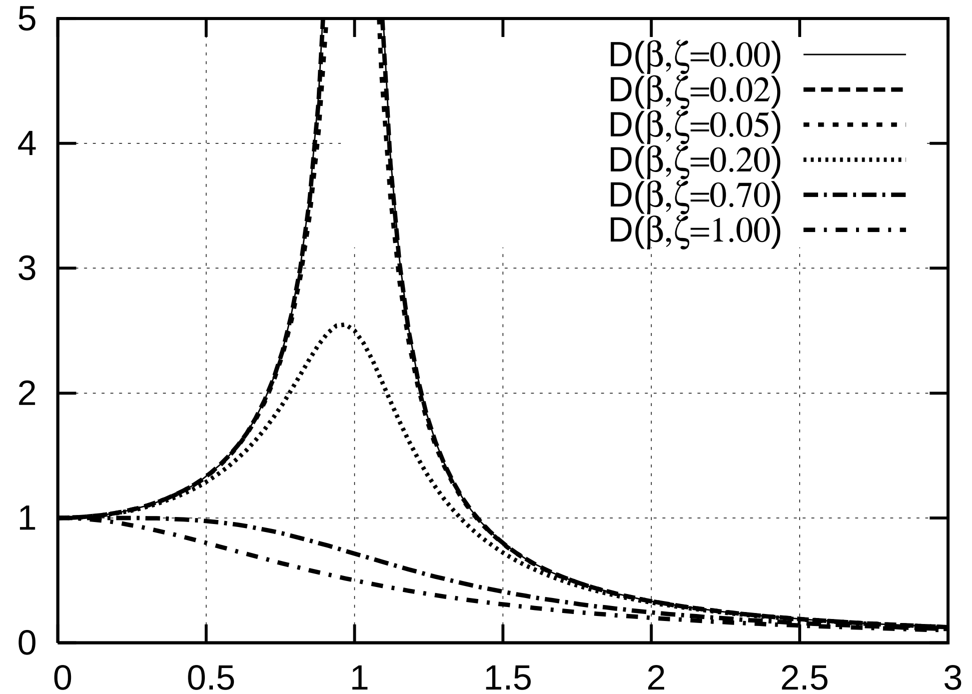

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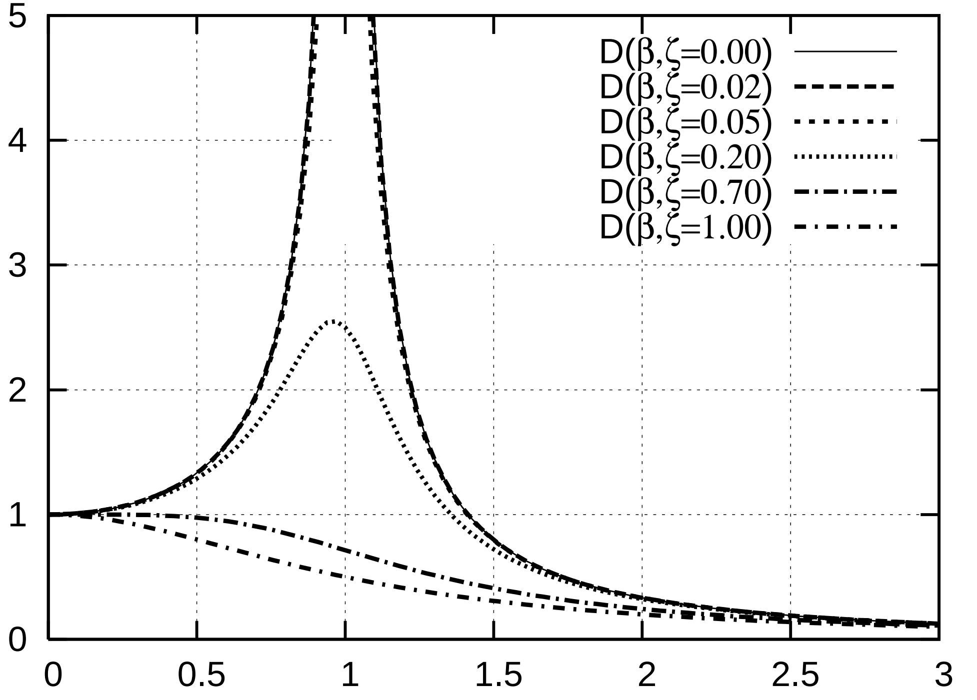

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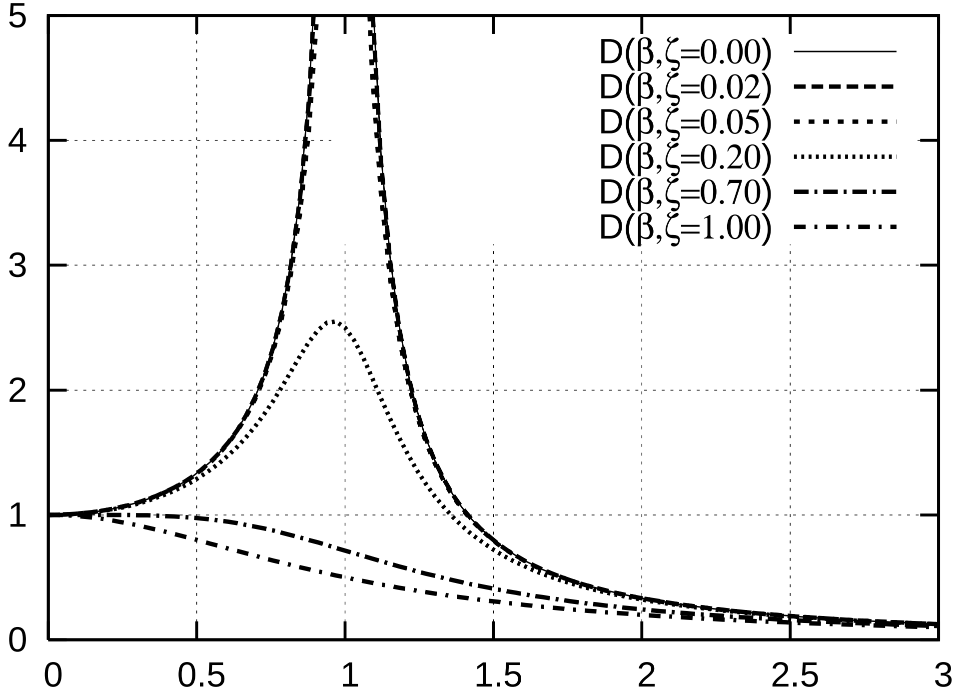

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- for dampings $\zeta > \frac{1}{\sqrt{2}}$ we have no peaks.

Dynamic Magnification Ratio (2)

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The location of the response peak is given by the equation

$$\frac{d\,D(\beta,\zeta)}{d\,\beta} = 0,$$

that solved gives the root

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As we are interested in non negative β, we are restricted to $0 < \zeta \le \frac{1}{\sqrt{2}}$. In this interval, substituting β_0 in the expression of the response ratio, we have

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$$D_{max} = \frac{1}{2\zeta} \frac{1}{\sqrt{1 - \zeta^2}}.$$

Note that, for a relatively large damping ratio, $\zeta = 20\%$, the error of $1/2\zeta$ with respect to D_{max} is in order of 2%.

Harmonic Exponential Loading

Consider:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{p_0}{m}\exp(i(\omega t - \phi))$$

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$$\xi = G\exp(i\omega t), \quad \dot{\xi} = i\omega G\exp(i\omega t), \quad \ddot{\xi} = -\omega^2 G\exp(i\omega t),$$

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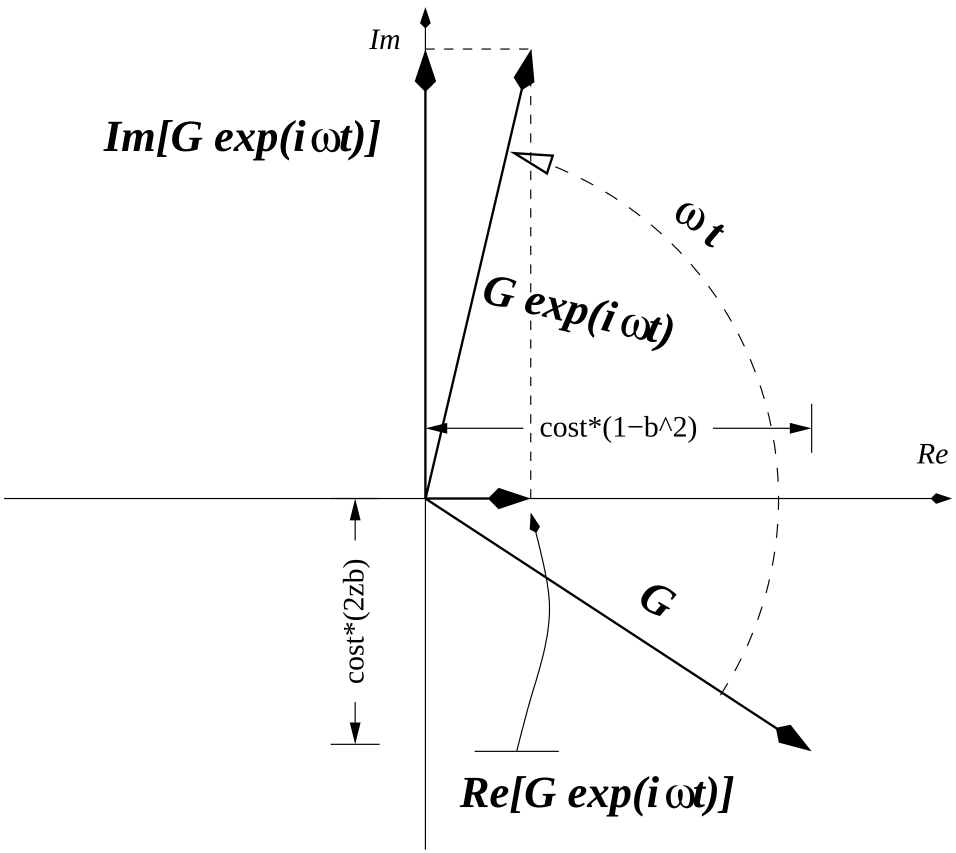
where G is a complex constant. Substituting, removing the dependency on $\exp(i\omega t)$ and solving for G yields

$$G = \Delta_{st}\left[\frac{1}{(1-\beta^2) + i(2\zeta\beta)}\right] = \Delta_{st}\left[\frac{(1-\beta^2) - i(2\zeta\beta)}{(1-\beta^2)^2 + (2\zeta\beta)^2}\right].$$



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Response in the complex plane

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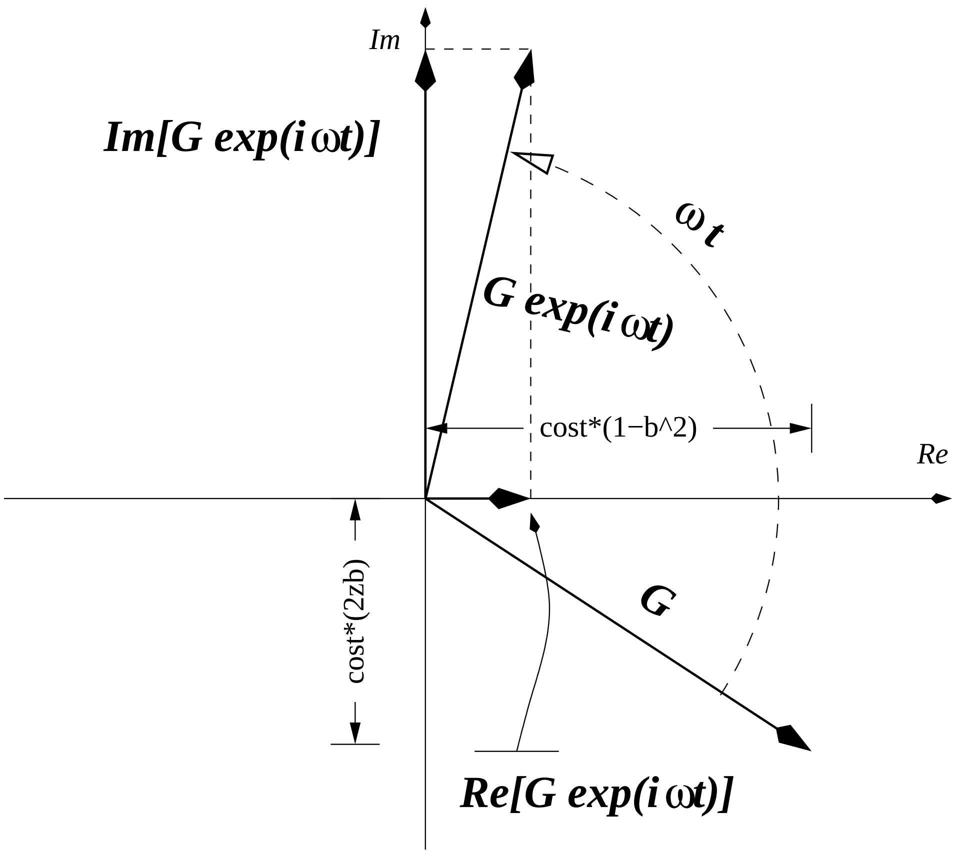

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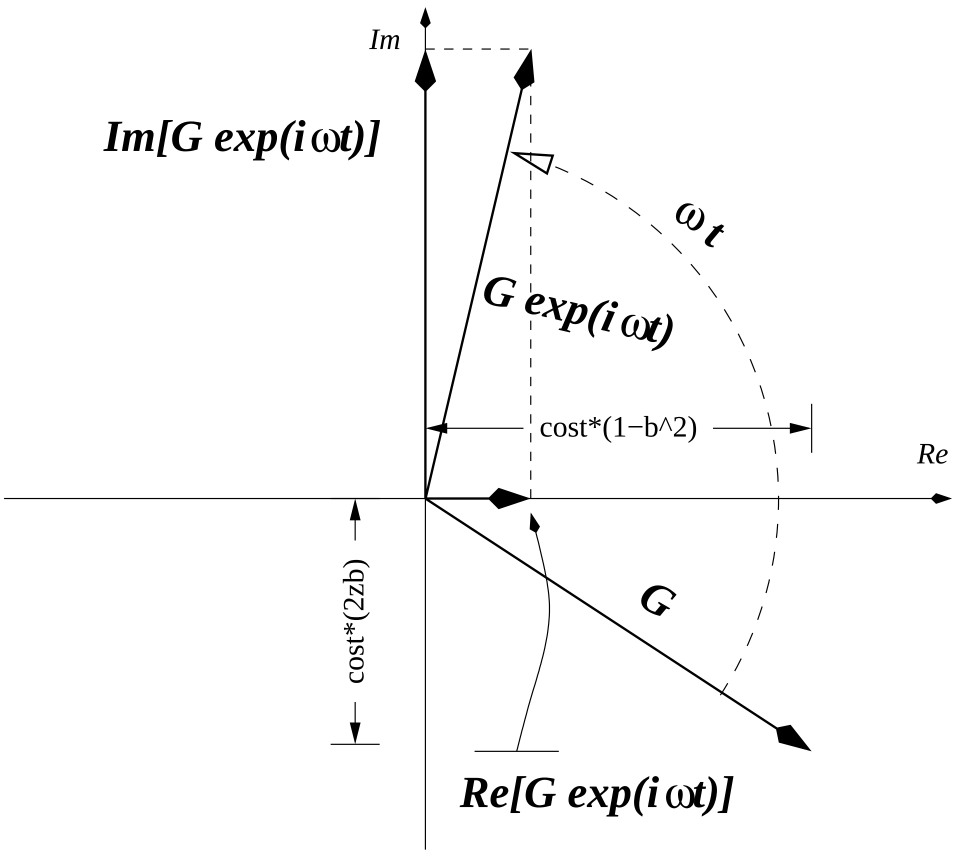

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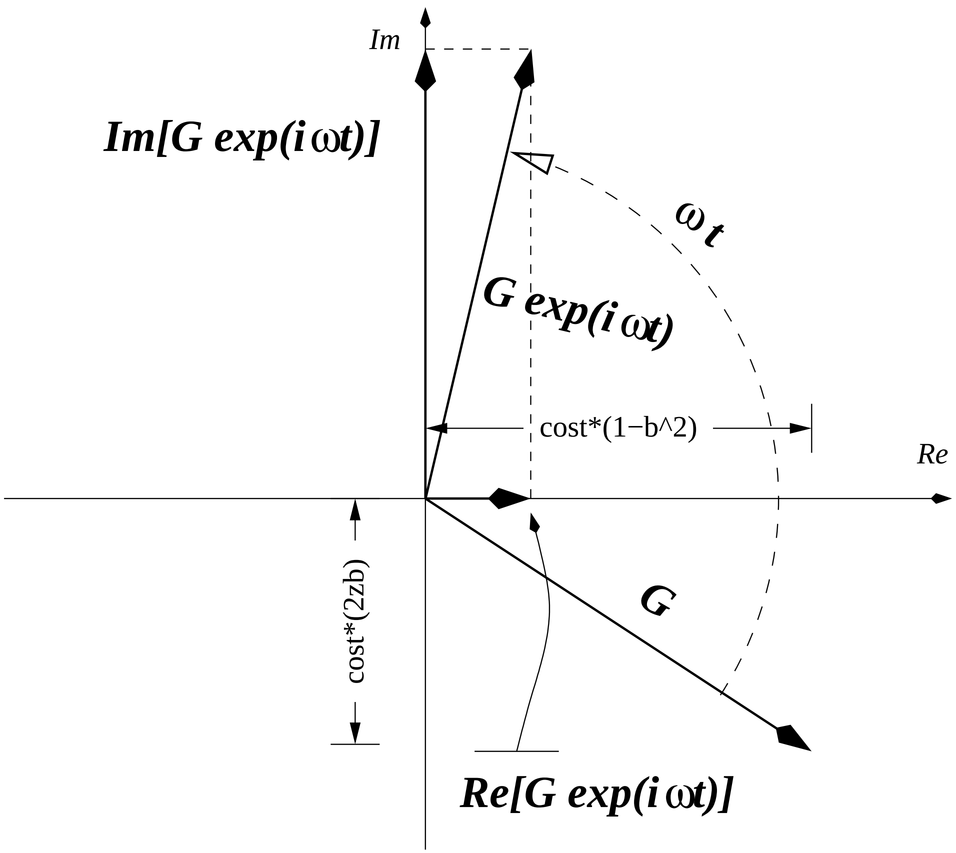

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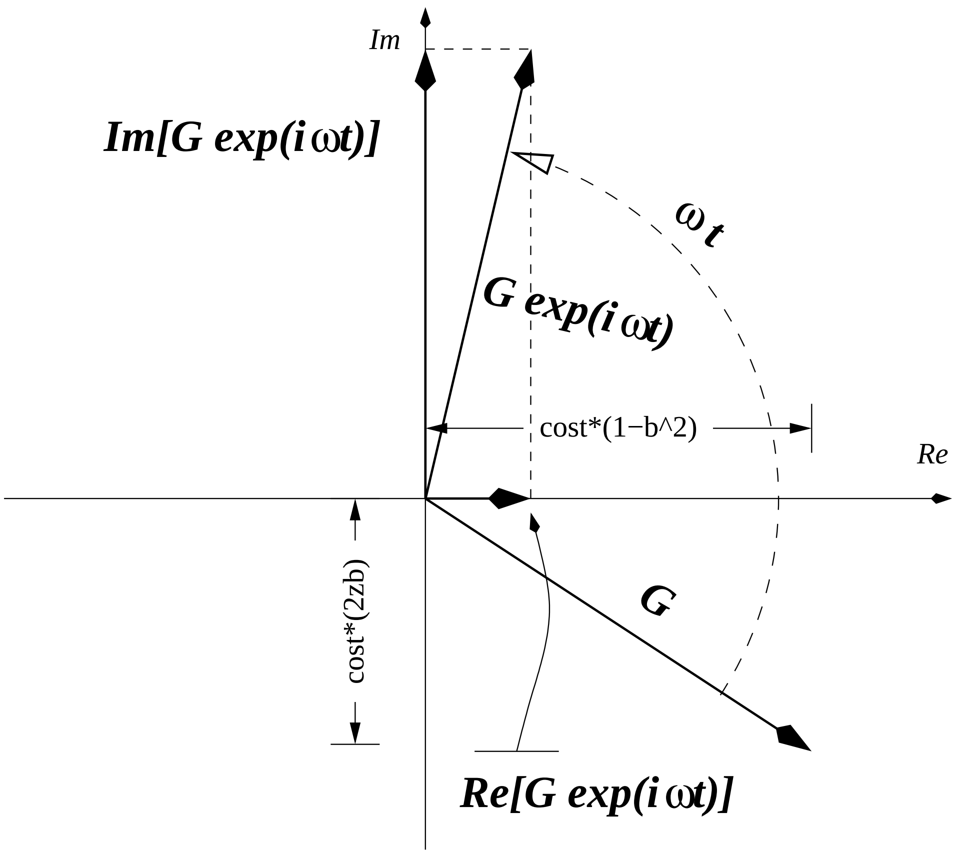

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- projecting the resulting vector on the axes, we have the real and imaginary part of the response,
- these two vector are rotated 90 degrees with respect to the response to real harmonic load, $p_0 \sin \omega t$.

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If we take into account that D is nearly equal to 1 in the range $0.0 < \beta < 0.6$ for $\zeta = 0.7$, we see that the displacements will be proportional to the accelerations of the support for applied frequencies up to about six-tenths of the natural frequency of the instrument, if the damping ratio is $\zeta \approxeq 0.7$.

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Consider now a harmonic *displacement* of the support, $u_g(t) = u_g \sin \omega t$. The support acceleration (disregarding the sign) is $a_g(t) = \omega^2 u_g \sin \omega t$.

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- With the equation of motion: $\ddot{x} + 2\zeta\beta\omega_n\dot{x} + \omega_n^2 x = -\omega^2 u_g \sin \omega t$,
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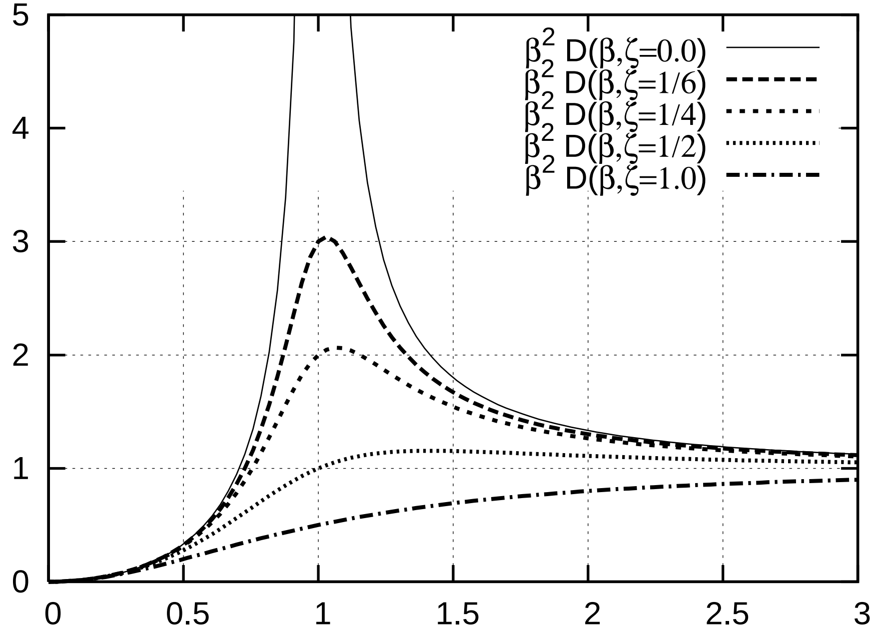

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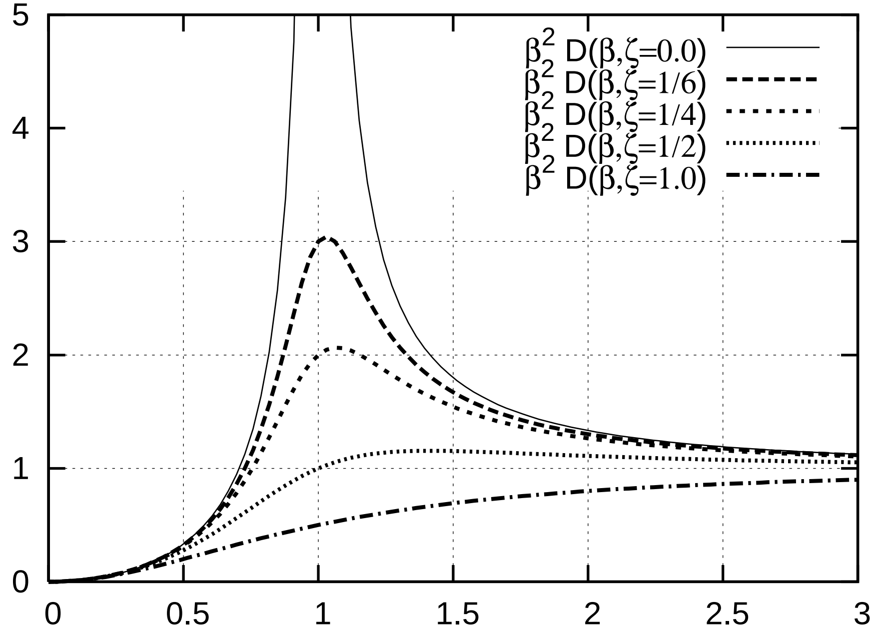

We see that the displacement of the instrument is approximately equal to the support displacement for all the excitation frequencies greater than the natural frequency of the instrument, for a damping ratio $\zeta \approxeq .5$.

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- prevention of harmful vibrations in supporting structures due to oscillatory forces produced by operating equipment,
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Force Isolation

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A rotating machine produces an oscillatory force $p_0 \sin \omega t$ due to unbalance in its rotating part, has a mass m and is mounted on a SDOF spring-damper support.

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$$\begin{aligned} f_S &= k\, x_{\mathsf{ss}} = p_0\, D \sin(\omega t - \theta), \\ f_D &= c\, \dot{x}_{\mathsf{ss}} = \frac{c p_0\, D\, \omega}{k} \cos(\omega t - \theta) = 2\, \zeta\, \beta\, p_0\, D \cos(\omega t - \theta). \end{aligned}$$

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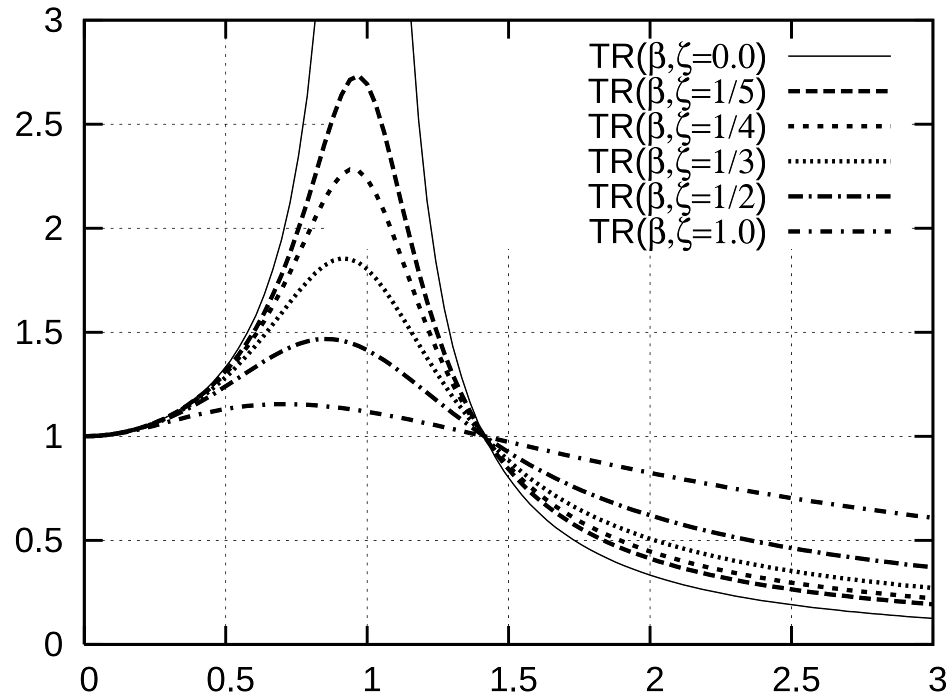

Displacement Isolation

POLITECNICO DI MILANO

Another problem concerns the harmonic support motion $u_g(t) = u_{g_0} \exp i\omega t$ forcing a steady-state relative displacement of some supported (spring+damper) equipment of mass m (using exp notation) $x_{\text{ss}} = u_{g_0} \beta^2 D \exp i\omega t$, and the mass total displacement is given by

$$\begin{aligned} x_{\text{Tot}} &= x_{\text{ss}} + u_g(t) = u_{g_0} \left(\frac{\beta^2}{(1-\beta^2) + 2\, i\, \zeta\beta} + 1 \right) \exp i\omega t \\ &= u_{g_0} (1 + 2i\zeta\beta) \frac{1}{(1-\beta^2) + 2\, i\, \zeta\beta} \exp i\omega t \\ &= u_{g_0} \sqrt{1 + (2\zeta\beta)^2}\, D \exp i\,(\omega t - \varphi). \end{aligned}$$

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If we define the transmissibility ratio TR as the ratio of the maximum total response to the support displacement amplitude, we find that, as in the previous case,

$$\text{TR} = D\,\sqrt{1+(2\zeta\beta)^2}.$$

Isolation Effectiveness

Define the isolation effectiveness,

$$\text{IE} = 1 - \text{TR},$$

IE=1 means complete isolation, i.e., $\beta = \infty$, while IE=0 is no isolation, and takes place for $\beta = \sqrt{2}$.

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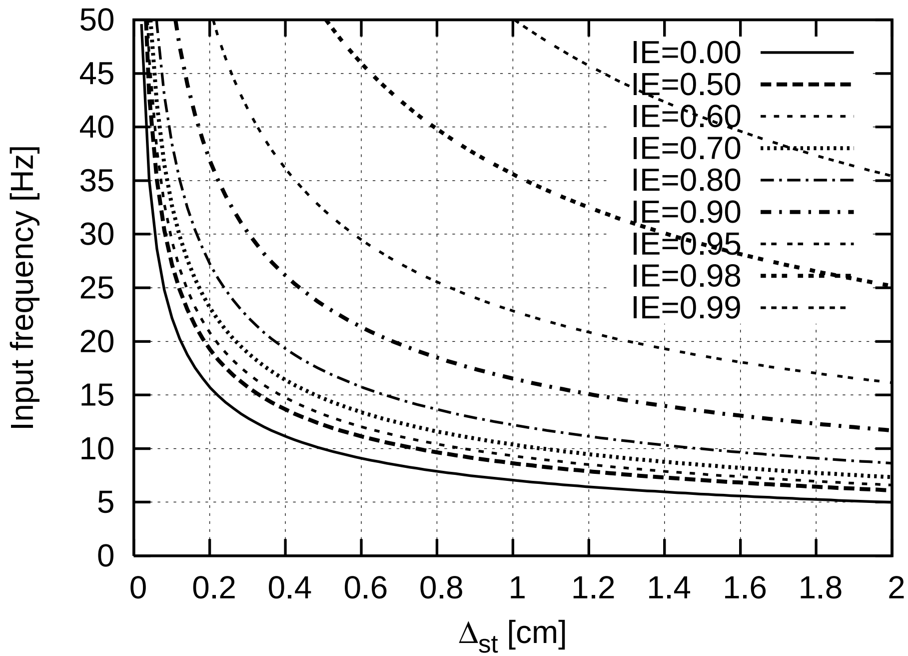
can be plotted f vs Δ_{st} for different values of IE, obtaining a design chart.



Knowing the frequency of excitation and the required level of vibration isolation efficiency (IE), one can determine the minimum static deflection (proportional to the spring flexibility) required to achieve the required IE. It is apparent that any isolation system must be very flexible to be effective.

Evaluation of viscous-damping ratio

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The mass and stiffness of phisycal systems of interest are usually evaluated easily, but this is not feasible for damping, as the energy is dissipated by different mechanisms, some one not fully understood... it is even possible that dissipation cannot be described in term of viscous-damping, But it generally is possible to measure an equivalent viscous-damping ratio by experimental methods:

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We already have discussed the free-vibration decay method,

$$\zeta = \frac{\delta_m}{2\pi\, m\, (\omega_n/\omega_D)}$$

with $\delta_m = \ln \frac{x_n}{x_{n+m}}$, *logarithmic decrement*. The method is simple and its requirements are minimal, but some care must be taken in the interpretation of free-vibration tests, because the damping ratio decreases with decreasing amplitudes of the response,

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The most problematic aspect here is getting a good estimate of Δ_{st}, if the results of a static test aren't available.

Half power

The adimensional frequencies where the response is $1/\sqrt{2}$ times the peak value can be computed from the equation

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For small ζ we can use the binomial approximation and write

$$\beta_{1,2}=\left(1-2\zeta^2\mp 2\zeta\sqrt{1-\zeta^2}\right)^{\frac{1}{2}}\approxeq 1-\zeta^2\mp\zeta\sqrt{1-\zeta^2}$$

Half power (2)

POLITECNICO DI MILANO

From the approximate expressions for the difference of the half power frequency ratios,

$$\beta_2 - \beta_1 = 2\zeta\sqrt{1-\zeta^2} \approxeq 2\zeta$$

and their sum

$$\beta_2 + \beta_1 = 2(1-\zeta^2) \approxeq 2$$

we can deduce that

$$\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1} = \frac{f_2 - f_1}{f_2 + f_1} \approxeq \frac{2\zeta\sqrt{1-\zeta^2}}{2(1-\zeta^2)} \approxeq \zeta, \text{ or } \zeta \approxeq \frac{f_2 - f_1}{f_2 + f_1}$$

where $f_1,\ f_2$ are the frequencies at which the steady state amplitudes equal $1/\sqrt{2}$ times the peak value, frequencies that can be determined from a dynamic test where detailed test data is available.

Resonance Cyclic Energy Loss

POLITECNICO DI MILANO

If it is possible to determine the phase of the s-s response, it is possible to measure ζ from the amplitude ρ of the resonant response, because the external force is equilibrated *only* by the viscous force, as both spring and inertia are in quadrature with the excitation:

$$p_0 = c\,\dot{x} = 2\zeta\omega_n m\left(\omega_n\rho\right)$$

so that we have

$$\zeta = \frac{p_0}{2m\omega_n^2\rho}$$