

# SDOF linear oscillator

## Response to Periodic and Non-periodic Loadings

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## Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to  
Periodic Loading

Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings

## Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

## Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

# Response to Periodic Loading

SDOF linear oscillator

G. Boffi

## Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

We know the analytical expression of the steady-state response of a SDOF system, damped or undamped, for a harmonic loading characterized by its amplitude  $p_0$  and its circular frequency,  $\omega$ .

A periodic loading is characterized by the identity

$$P(t) = P(t + T)$$

where  $T$  is the *period* of the loading, and  $\omega_1 = \frac{2\pi}{T}$  is its *principal frequency*.

Periodic loadings can be expressed as an infinite series of harmonic functions using Fourier theorem. The steady-state response can be expressed as an infinite series, too.

Due to the asymptotic behaviour of  $D(\beta; \zeta)$  ( $D$  goes to zero for large, increasing  $\beta$ ) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

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Response to  
Periodic Loading

Introduction

Fourier Series  
Representation

Fourier Series of the  
Response

An example

Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings

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# Fourier Series

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms, and we can use the results deduced for harmonic loads to express the periodic response of a *SDOF* as a series of harmonic response terms.

Consider a loading of period  $T_p$ , its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j \omega_1 = j \frac{2\pi}{T_p}$$

When the loading function is known, the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t) dt,$$
$$a_j = \frac{2}{T_p} \int_0^{T_p} p(t) \cos \omega_j t dt, \quad b_j = \frac{2}{T_p} \int_0^{T_p} p(t) \sin \omega_j t dt.$$

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# Fourier Coefficients

If  $p(t)$  must be measured experimentally or computed numerically, we may assume that it is possible

(a) to divide the period in  $N$  equal parts  $\Delta t = T_p/N$ ,

(b) measure or compute  $p(t)$  at a discrete set of instants  $t_1, t_2, \dots, t_N$ , with  $t_m = m\Delta t$ ,

obtaining a discrete set of values  $p_m$ ,  $m = 1, \dots, N$  (note that  $p_0 = p_N$  by periodicity).

Using the trapezoidal rule of integration, with  $p_0 = p_N$  we can write, for example, the cosine-wave amplitude coefficients,

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$$\begin{aligned} a_j &\approx \frac{2\Delta t}{T_p} \sum_{m=1}^N p_m \cos \omega_j t_m \\ &= \frac{2}{N} \sum_{m=1}^N p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^N p_m \cos \frac{jm 2\pi}{N}. \end{aligned}$$

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It's worth to note that the discrete function  $\cos \frac{jm 2\pi}{N}$  is periodic with period  $N$ .

The Fourier series can be written in terms of the exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp i\omega_j t dt, \quad j = -\infty, \dots, +\infty.$$

For a sampled  $p_m$  we can write, using the trapezoidal integration rule and substituting  $t_m = m\Delta t = m T_p/N$ ,  $\omega_j = j 2\pi/T_p$ :

$$P_j \approx \frac{1}{N} \sum_{m=1}^N p_m \exp\left(-i \frac{2\pi j m}{N}\right),$$

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

# Exponential Form

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Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

We have seen that the steady-state response to the  $j$ th sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k} \left[ \frac{1}{1 - \beta_j^2} \right] \sin \omega_j t, \quad \beta_j = \omega_j / \omega_n,$$

analogously, for the  $j$ th cosine-wave harmonic,

$$x_j = \frac{a_j}{k} \left[ \frac{1}{1 - \beta_j^2} \right] \cos \omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k} \left\{ a_0 + \sum_{j=1}^{\infty} \left[ \frac{1}{1 - \beta_j^2} \right] (a_j \cos \omega_j t + b_j \sin \omega_j t) \right\}.$$

Response to  
Periodic Loading

Introduction

Fourier Series  
Representation

Fourier Series of the  
Response

An example

Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings



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Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

# Damped Response

In the case of a damped oscillator, we must substitute the steady state response for both the  $j$ th sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{j=1}^{\infty} \frac{2a_j \zeta \beta_j + b_j (1 - \beta_j^2)}{(1 - \beta_j^2)^2 + (2\zeta \beta_j)^2} \sin \omega_j t + \frac{1}{k} \sum_{j=1}^{\infty} \frac{-2b_j \zeta \beta_j + a_j (1 - \beta_j^2)}{(1 - \beta_j^2)^2 + (2\zeta \beta_j)^2} \cos \omega_j t.$$

As usual, the exponential notation is neater,

$$R(t) = \sum_{j=-\infty}^{\infty} P_j \frac{1}{(1 - \beta_j^2) + i(2\zeta \beta_j)} \exp i\omega_j t$$

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

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Response to  
Periodic Loading

Introduction

Fourier Series  
Representation

Fourier Series of the  
Response

An example

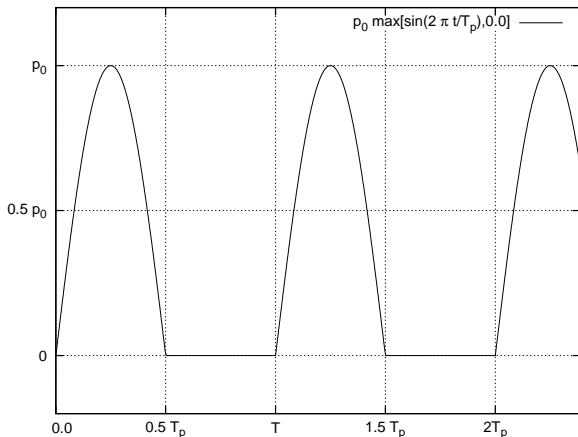
Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings

# Example

As an example, consider the loading

$$p(t) = \max\left\{p_0 \sin \frac{2\pi t}{T_p}, 0\right\}$$



Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

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$$a_j = \frac{2}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi t}{T_p} \cos \frac{2\pi jt}{T_p} dt = \begin{cases} 0 & \text{for } j \text{ odd} \\ \frac{p_0}{\pi} \left[ \frac{2}{1-j^2} \right] & \text{for } j \text{ even,} \end{cases}$$

$$b_j = \frac{2}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi t}{T_p} \sin \frac{2\pi jt}{T_p} dt = \begin{cases} \frac{p_0}{2} & \text{for } j = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

Response to  
Periodic Loading

Introduction

Fourier Series

Representation

Fourier Series of the  
Response

An example

Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings

## Example cont.

Assuming  $\beta_1 = 3/4$ , from

$$p = \frac{p_0}{\pi} \left( 1 + \frac{\pi}{2} \sin \omega_1 t - \frac{2}{3} \cos 2\omega_1 t - \frac{2}{15} \cos 4\omega_1 t - \dots \right)$$

with the dynamic amplification factors

$$D_1 = \frac{1}{1 - (1\frac{3}{4})^2} = \frac{16}{7},$$

$$D_2 = \frac{1}{1 - (2\frac{3}{4})^2} = -\frac{4}{5},$$

$$D_4 = \frac{1}{1 - (4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \dots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi} \left( 1 + \frac{8\pi}{7} \sin \omega_1 t + \frac{8}{15} \cos 2\omega_1 t + \frac{1}{60} \cos 4\omega_1 t + \dots \right)$$

Take note, these solutions are particular solutions! you must consider also the complementary solution to take into account initial conditions.

Response to  
Periodic Loading

Introduction

Fourier Series  
Representation

Fourier Series of the  
Response

An example

Response to  
Impulsive Loading

Response to  
General Dynamic  
Loadings

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# Response to Impulsive Loadings

SDOF linear  
oscillator

G. Boffi

Response to  
Periodic Loading

Response to  
Impulsive Loading

Introduction

Response to Half-Sine Wave  
Impulse

Response for Rectangular  
and Triangular Impulses

Shock or response spectra

Approximate Analysis of  
Response Peak

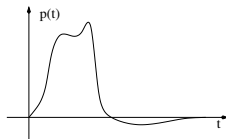
Response to  
General Dynamic  
Loadings



# Nature of Impulsive Loadings

An impulsive load is characterized

- ▶ by a single principal impulse, and
- ▶ by a relatively short duration.



- ▶ Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
- ▶ Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
- ▶ For this reason, in the following we'll consider only the undamped response to impulsive loads.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

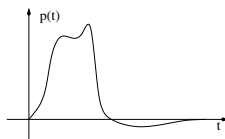
Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

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Response to  
Periodic Loading

Response to  
Impulsive Loading

Introduction

Response to Half-Sine Wave  
Impulse

Response for Rectangular  
and Triangular Impulses

Shock or response spectra

Approximate Analysis of  
Response Peak

Response to  
General Dynamic  
Loadings

# Definition of Maximum Response

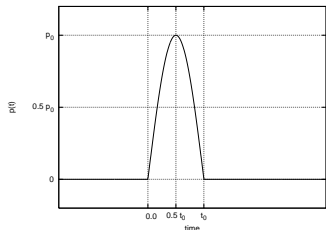
In general, when dealing with impulse response characterized by its duration  $t_0$  we are interested either in

- a* the maxima of the absolute values of the response ratio  $R(t)$  in  $0 < t < t_0$ , and in particular to the maximum value or,
- b* if no such maxima exist, in the maximum value of the free vibrations that are excited by the impulse.

# Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where  $\omega = \frac{2\pi}{2t_0}$  is the frequency associated with the load. Note that  $\omega t_0 = \pi$ .

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

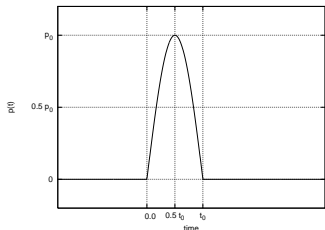
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

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Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

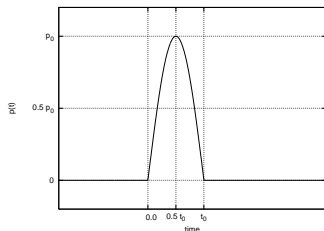
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

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Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

# Response to sine-wave impulse

Consider an undamped *SDOF* initially at rest, with natural circular frequency  $\omega_n$  and stiffness  $k$ . With reference to a half-sine impulse with duration  $t_0$ , the frequency ratio  $\beta$  is  $\omega/\omega_n = T_n/2t_0$ .

Its response ratio in the interval  $0 < t < t_0$  is

$$R(t) = \frac{1}{1 - \beta^2} \left( \sin \omega t - \beta \sin \frac{\omega t}{\beta} \right)$$

while for  $t > t_0$  the response ratio is

$$R(t) = \frac{-\beta}{1 - \beta^2} \left( \left( 1 + \cos \frac{\pi}{\beta} \right) \sin \left( \frac{\pi}{\beta} \frac{t - t_0}{t_0} \right) + \sin \frac{\pi}{\beta} \cos \left( \frac{\pi}{\beta} \frac{t - t_0}{t_0} \right) \right)$$

# Maximum response to sine impulse

(a) Since we are interested in the maximum response ratio during the excitation, we need to know when velocity is zero in the time interval  $0 \leq t \leq t_0$ ; from

$$\dot{R}(t) = \frac{\omega}{1 - \beta^2} (\cos \omega t - \cos \frac{\omega t}{\beta}) = 0.$$

we can see that the roots are

$\omega t = \mp \omega t / \beta + 2n\pi$ ,  $n = 0, \mp 1, \mp 2, \mp 3, \dots$ ; it is convenient to substitute  $\omega t = \pi\alpha$ , where  $\alpha = t/t_0$ ; substituting and solving for  $\alpha$  one has

$$\alpha = \frac{2n\beta}{\beta \mp 1}, \quad \text{with } n = 0, \mp 1, \mp 2, \dots, \text{ for } 0 < \alpha < 1.$$

The next slide regards the characteristics of these roots.

Response to  
Periodic Loading

Response to  
Impulsive Loading

Introduction

Response to Half-Sine Wave  
Impulse

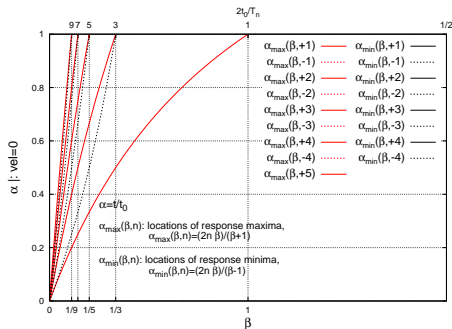
Response for Rectangular  
and Triangular Impulses

Shock or response spectra

Approximate Analysis of  
Response Peak

Response to  
General Dynamic  
Loadings





- ▶ No roots of type  $\alpha_{\min}$  for  $n > 0$ ;
- ▶ no roots of type  $\alpha_{\max}$  for  $n < 0$ ;
- ▶ no roots for  $\beta > 1$ , i.e., no roots for  $t_0 < \frac{T_n}{2}$ ;
- ▶ only one root of type  $\alpha_{\max}$  for  $\frac{1}{3} < \beta < 1$ , i.e.,  $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$ ;
- ▶ three roots, two maxima and one minimum, for  $\frac{1}{5} < \beta < \frac{1}{3}$ ;
- ▶ five roots, three maxima and two minima, for  $\frac{1}{7} < \beta < \frac{1}{5}$ ;
- ▶ etc etc.

In summary, to find the maximum of the response for an assigned  $\beta < 1$ , one has (a) to compute all  $\alpha_k = \frac{2k\beta}{\beta+1}$  until a root is greater than 1, (b) compute all the responses for  $t_k = \alpha_k t_0$ , (c) choose the maximum of the maxima.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

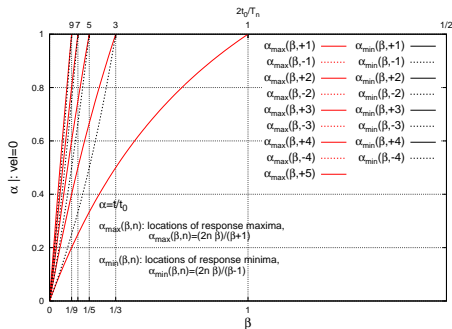
Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings



- ▶ No roots of type  $\alpha_{\min}$  for  $n > 0$ ;
- ▶ no roots of type  $\alpha_{\max}$  for  $n < 0$ ;
- ▶ no roots for  $\beta > 1$ , i.e., no roots for  $t_0 < \frac{T_n}{2}$ ;
- ▶ only one root of type  $\alpha_{\max}$  for  $\frac{1}{3} < \beta < 1$ , i.e.,  $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$ ;
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Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

# Maximum response for $\beta > 1$

For  $\beta > 1$ , the maximum response takes place for  $t > t_0$ , and its absolute value (see slide *Response to sine-wave impulse*) is

$$R_{\max} = \frac{\beta}{1 - \beta^2} \sqrt{\left(1 + \cos \frac{\pi}{\beta}\right)^2 + \sin^2 \frac{\pi}{\beta}},$$

using a simple trigonometric identity we can write

$$R_{\max} = \frac{\beta}{1 - \beta^2} \sqrt{2 + 2 \cos \frac{\pi}{\beta}}$$

but

$1 + \cos 2\phi = (\cos^2 \phi + \sin^2 \phi) + (\cos^2 \phi - \sin^2 \phi) = 2 \cos^2 \phi$ ,  
so that

$$R_{\max} = \frac{2\beta}{1 - \beta^2} \cos \frac{\pi}{2\beta}.$$

Response to  
Periodic Loading

Response to  
Impulsive Loading

Introduction

Response to Half-Sine Wave  
Impulse

Response for Rectangular  
and Triangular Impulses

Shock or response spectra

Approximate Analysis of  
Response Peak

Response to  
General Dynamic  
Loadings

Consider a rectangular impulse of duration  $t_0$ ,

$$p(t) = p_0 \begin{cases} 1 & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$

The response ratio and its time derivative are

$$R(t) = 1 - \cos \omega_n t, \quad \dot{R}(t) = \omega_n \sin \omega_n t,$$

and we recognize that we have maxima  $R_{\max} = 2$  for  $\omega_n t = n\pi$ , with the condition  $t \leq t_0$ . Hence we have no maximum during the loading phase for  $t_0 < T_n/2$ , and at least one maximum, of value  $2\Delta_{st}$ , if  $t_0 \geq T_n/2$ .

Response to  
Periodic Loading

Response to  
Impulsive Loading

Introduction

Response to Half-Sine Wave  
Impulse

Response for Rectangular  
and Triangular Impulses

Shock or response spectra

Approximate Analysis of  
Response Peak

Response to  
General Dynamic  
Loadings

## Rectangular Impulse (2)

For shorter impulses, the maximum response ratio is not attained during loading, so we have to compute the amplitude of the free vibrations after the end of loading (remember, as  $t_0 \leq T_n/2$  the velocity is positive at  $t = t_0$ !).

$$R(t) = (1 - \cos \omega_n t_0) \cos \omega_n (t - t_0) + (\sin \omega_n t_0) \sin \omega_n (t - t_0).$$

The amplitude of the response ratio is then

$$\begin{aligned} A &= \sqrt{(1 - \cos \omega_n t_0)^2 + \sin^2 \omega_n t_0} = \\ &= \sqrt{2(1 - \cos \omega_n t_0)} = 2 \sin \frac{\omega_n t_0}{2}. \end{aligned}$$

Let's consider the response of a *SDOF* to a triangular impulse,  $p(t) = p_0(1 - t/t_0)$  for  $0 < t < t_0$ . As usual, we must start finding the minimum duration that gives place to a maximum of the response in the loading phase, that is

$$R(t) = \frac{1}{\omega_n t_0} \sin \omega_n \frac{t}{t_0} - \cos \omega_n \frac{t}{t_0} + 1 - \frac{t}{t_0}, \quad 0 < t < t_0.$$

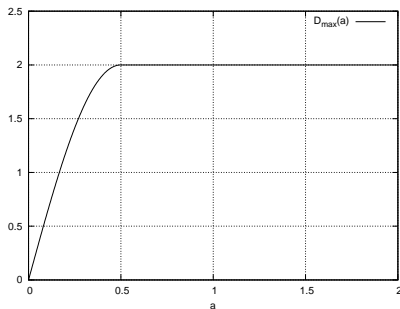
Taking the first derivative and setting it to zero, one can see that the first maximum occurs for  $t = t_0$  for  $t_0 = 0.37101 T_n$ , and substituting one can see that  $R_{\max} = 1$ .

# Triangular Impulse (2)

For load durations shorter than  $0.37101 T_n$ , the maximum occurs after loading and it's necessary to compute the displacement and velocity at the end of the load phase. For longer loads, the maxima are in the load phase, so that one has to find the all the roots of  $\dot{R}(t)$ , compute all the extreme values and finally sort out the absolute value maximum.

# Shock or response spectra

We have seen that the response ratio is determined by the ratio of the impulse duration to the natural period of the oscillator.



This plot is the response spectrum for a *rectangular* loading.

One can plot the maximum displacement ratio  $R_{\max}$  as a function of  $t_o/T_n$  for various forms of impulsive loads. Such plots are commonly known as displacement-response spectra, or simply as response spectra.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings



For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor  $\approx 1$

On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the loading system at the end of loading. The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

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# Approximate Analysis (2)

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)] dt.$$

When one notes that, for small  $t_0$ , the displacement is of the order of  $t_0^2$  while the velocity is in the order of  $t_0$ , it is apparent that the  $kx$  term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approx \int_0^{t_0} p(t) dt.$$

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

# Approximate Analysis (3)

Using the previous approximation, the velocity at time  $t_0$  is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t) dt,$$

and considering again a negligibly small displacement at the end of the loading,  $x(t_0) \cong 0$ , one has

$$x(t - t_0) \cong \frac{1}{m\omega_n} \int_0^{t_0} p(t) dt \sin \omega_n(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading (and will be discovered again in a few minutes).

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Please note that the above equation is exact for an infinitesimal impulse loading (and will be discovered again in a few minutes).

# Response to General Dynamic Loading

SDOF linear oscillator

G. Boffi

Response to Periodic Loading

Response to Impulsive Loading

Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

For an infinitesimal impulse, the impulse-momentum is exactly  $p(\tau) d\tau$  and the response is

$$dx(t - \tau) = \frac{p(\tau) d\tau}{m\omega_n} \sin \omega_n(t - \tau), \quad t > \tau,$$

and to evaluate the response at time  $t$  one has simply to sum all the infinitesimal contributions for  $\tau < t$ ,

$$x(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n(t - \tau) d\tau, \quad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

The derivation of the equation of motion for a generic load is analogous to what we have seen for undamped SDOF, the infinitesimal contribution to the response at time  $t$  of the load at time  $\tau$  is

$$dx(t) = \frac{p(\tau)}{m\omega_D} d\tau \sin \omega_D(t - \tau) \exp(-\zeta\omega_n(t - \tau)) \quad t \geq \tau$$

and integrating all infinitesimal contributions one has

$$x(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \sin \omega_D(t - \tau) \exp(-\zeta\omega_n(t - \tau)) d\tau, \quad t \geq 0.$$



# Evaluation of Duhamel integral, undamped

Using the trig identity

$$\sin(\omega_n t - \omega_n \tau) = \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau$$

the Duhamel integral is rewritten as

$$\begin{aligned} x(t) &= \frac{\int_0^t p(\tau) \cos \omega_n \tau d\tau}{m\omega_n} \sin \omega_n t - \frac{\int_0^t p(\tau) \sin \omega_n \tau d\tau}{m\omega_n} \cos \omega_n t \\ &= \mathcal{A}(t) \sin \omega_n t - \mathcal{B}(t) \cos \omega_n t \end{aligned}$$

where

$$\begin{cases} \mathcal{A}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \cos \omega_n \tau d\tau \\ \mathcal{B}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n \tau d\tau \end{cases}$$

# Numerical evaluation of Duhamel integral, undamped

Usual numerical procedures can be applied to the evaluation of  $\mathcal{A}$  and  $\mathcal{B}$ , e.g., using the trapezoidal rule, one can have, with  $\mathcal{A}_N = \mathcal{A}(N\Delta\tau)$  and  $y_N = p(N\Delta\tau) \cos(N\Delta\tau)$

$$\mathcal{A}_{N+1} = \mathcal{A}_N + \frac{\Delta\tau}{2m\omega_n} (y_N + y_{N+1}).$$

For a damped system, it can be shown that

$$x(t) = \mathcal{A}(t) \sin \omega_D t - \mathcal{B}(t) \cos \omega_D t$$

with

$$\mathcal{A}(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \frac{\exp \zeta \omega_n \tau}{\exp \zeta \omega_n t} \cos \omega_D \tau d\tau,$$
$$\mathcal{B}(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \frac{\exp \zeta \omega_n \tau}{\exp \zeta \omega_n t} \sin \omega_D \tau d\tau.$$

# Numerical evaluation of Duhamel integral, damped

Numerically, using e.g. Simpson integration rule and

$$y_N = p(N\Delta\tau) \cos\omega_D\tau,$$

$$\mathcal{A}_{N+2} = \mathcal{A}_N \exp(-2\zeta\omega_n\Delta\tau) + \frac{\Delta\tau}{3m\omega_D} [y_N \exp(-2\zeta\omega_n\Delta\tau) + 4y_{N+1} \exp(-\zeta\omega_n\Delta\tau) + y_{N+2}]$$

$$N = 0, 2, 4, \dots$$