Fourier Transform
The Discrete Fourier Transform
The Fast Fourier Transform

SDOF linear oscillator

Frequency Domain Analysis

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Outline

Fourier Transform
- Extension of Fourier Series to non periodic functions
- Response in the Frequency Domain

The Discrete Fourier Transform
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- Aliasing

The Fast Fourier Transform
- The Fast Fourier Transform

Non periodic loadings

It is possible to extend the Fourier analysis to non periodic loading. Let's start from the Fourier series representation of the load $p(t)$,

$$p(t) = \sum_{-\infty}^{+\infty} P_r \exp(i\omega_r t), \quad \omega_r = r\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T_p},$$

introducing $P(i\omega_r) = P_r T_p$ and substituting,

$$p(t) = \frac{1}{T_p} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t) = \frac{\Delta\omega}{2\pi} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t).$$

Due to periodicity, we can modify the extremes of integration in the expression for the complex amplitudes,

$$P(i\omega_r) = \int_{-T_p/2}^{+T_p/2} p(t) \exp(-i\omega_r t)\, dt.$$

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Non periodic loadings (2)

If the loading period is extended to infinity to represent the non-periodicity of the loading ($T_p \to \infty$) then (a) the frequency increment becomes infinitesimal ($\Delta\omega = \frac{2\pi}{T_p} \to d\omega$) and ($b$) the discrete frequency ω_r becomes a continuous variable, ω.
In the limit, for $T_p \to \infty$ we can then write

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(i\omega) \exp(i\omega t)\, d\omega$$

$$P(i\omega) = \int_{-\infty}^{+\infty} p(t) \exp(-i\omega t)\, dt,$$

which are known as the inverse and the direct Fourier Transforms, respectively, and are collectively known as the Fourier transform pair.

SDOF Response

In analogy to what we have seen for periodic loads, the response of a damped SDOF system can be written in terms of $H(i\omega)$, the complex frequency response function,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(i\omega)\, P(i\omega) \exp i\omega t\, dt, \quad \text{where}$$

$$H(i\omega) = \frac{1}{k}\left[\frac{1}{(1-\beta^2)+i(2\zeta\beta)}\right] = \frac{1}{k}\left[\frac{(1-\beta^2)-i(2\zeta\beta)}{(1-\beta^2)^2+(2\zeta\beta)^2}\right], \quad \beta = \frac{\omega}{\omega_n}.$$

To obtain the response *through frequency domain*, you should evaluate the above integral, but analytical integration is not always possible, and when it is possible, it is usually very difficult, implying contour integration in the complex plane (for an example, see Example **E6-3** in Clough Penzien).

Discrete Fourier Transform

To overcome the analytical difficulties associated with the inverse Fourier transform, one can use appropriate numerical methods, leading to good approximations.
Consider a loading of finite period T_p, divided into N equal intervals $\Delta t = T_p/N$, and the set of values $p_s = p(t_s) = p(s\Delta t)$. We can approximate the complex amplitude coefficients with a sum,

$$P_r = \frac{1}{T_p}\int_0^{T_p} p(t)\exp(-i\omega_r t)\,dt, \quad \text{that, by trapezoidal rule, is}$$

$$\approx \frac{1}{N\Delta t}\left(\Delta t\sum_{s=0}^{N-1} p_s \exp(-i\omega_r t_s)\right) = \frac{1}{N}\sum_{s=0}^{N-1} p_s \exp(-i\frac{2\pi rs}{N}).$$

Discrete Fourier Transform (2)

In the last two passages we have used the relations

$$p_N = p_0, \quad \exp(i\omega_r t_N) = \exp(ir\Delta\omega T_p) = \exp(ir2\pi) = \exp(i0)$$

$$\omega_r\, t_s = r\Delta\omega\, s\Delta t = rs\,\frac{2\pi}{T_p}\,\frac{T_p}{N} = \frac{2\pi\, rs}{N}.$$

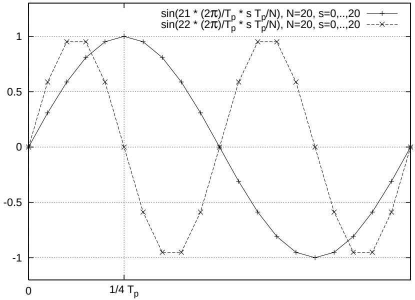
Take note that the discrete function $\exp(-i\frac{2\pi rs}{N})$, defined for integer r, s is periodic with period N, implying that the complex amplitude coefficients are themselves periodic with period N.

$$P_{r+N} = P_r$$

Starting in the time domain with N distinct complex numbers, p_s, we have found that in the frequency domain our load is described by N distinct complex numbers, P_r, so that we can say that our function is described by the same amount of information in both domains.

Aliasing

Only $N/2$ distinct frequencies ($\sum_0^{N-1} = \sum_{-N/2}^{+N/2}$) contribute to the load representation, what if the *frequency content* of the loading has contributions from frequencies higher than $\omega_{N/2}$? What happens is *aliasing*, i.e., the upper frequencies contributions are mapped to contributions of lesser frequency.



See the plot above: the contributions from the high frequency sines, *when sampled*, are indistinguishable from the contributions from lower frequency components, i.e., are *aliased* to lower frequencies!

Aliasing (2)

- The maximum frequency that can be described in the DFT is called the Nyquist frequency, $\omega_{Ny} = \frac{1}{2}\frac{2\pi}{\Delta t}$.
- It is usual in signal analysis to remove the signal's higher frequency components preprocessing the signal with a *filter* or a *digital filter*.
- It is worth noting that the *resolution* of the DFT in the frequency domain for a given sampling rate is proportional to the number of samples, i.e., to the duration of the sample.

Fast Fourier Transform (1)

The algorithm is the same for bot direct and inverse DFT, so let us consider the direct transform,

$$A_r = P_r N = \sum_{s=0}^{N-1} p_s \exp(-i\frac{2\pi\, rs}{N}), \qquad r = 0, 1, 2, \ldots, N-1$$

A straightforward implementation requires about N^2 complex products, becoming prohibitive for even moderately large N's. The FFT is based on the decomposition of N in a product of its factors, but the algorithm was developed and is simpler to understand and implement if $N = 2^\gamma$. In this case each r, s in the interval 0, $N-1$ can be expressed in terms of binary (i.e., 0 or 1) coefficients

$$r = 2^{\gamma-1}\mathrm{r}_{\gamma-1} + 2^{\gamma-2}\mathrm{r}_{\gamma-2} + \cdots + 2^0\mathrm{r}_0$$
$$s = 2^{\gamma-1}\mathrm{s}_{\gamma-1} + 2^{\gamma-2}\mathrm{s}_{\gamma-2} + \cdots + 2^0\mathrm{s}_0$$

Fast Fourier Transform (2)

With $W_N = \exp(-i2\pi/N)$ we write

$$A(\mathrm{r}_{\gamma-1}, \mathrm{r}_{\gamma-2}, \ldots, \mathrm{r}_0) = \sum_{\mathrm{s}_0=0}^{1} \sum_{\mathrm{s}_1=0}^{1} \cdots \sum_{\mathrm{s}_{\gamma-2}=0}^{1} \sum_{\mathrm{s}_{\gamma-1}=0}^{1} p_0(\mathrm{s}_{\gamma-1}, \mathrm{s}_{\gamma-2}, \ldots, \mathrm{s}_0) W_N^{rs}$$

The subscript $_0$ is added to the p coefficients for a reason that will become apparent as the algorithm develops.

Fast Fourier Transform (3)

Fourier Transform
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The Fast Fourier Transform

In the previous slide we wrote W_N^{rs} , but we have to use the binary representation of r and s,

$$W_N^{rs} = W_N^{(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)(2^{\gamma-1}\mathrm{s}_{\gamma-1}+2^{\gamma-2}\mathrm{s}_{\gamma-2}+\cdots+2^0\mathrm{s}_0)}$$

.

As $W_N^{a+b} = W_N^a\, W_N^b$, we can expand the equation above to separate the contributions of the different binary indices in the binary representation of s

$$\begin{aligned} W_N^{rs} = {} & W_N^{(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)(2^{\gamma-1}\mathrm{s}_{\gamma-1})} \\ & \times W_N^{(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)(2^{\gamma-2}\mathrm{s}_{\gamma-2})} \\ & \times \cdots \times W_N^{(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)(2^0\mathrm{s}_0)} \end{aligned}$$

In the next slide we are going to rewrite each term in the above expression, to obtain a surprising simplification.

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Fast Fourier Transform (4)

The first term on the right in previous slide was

$$W_N^{(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\dots+2^0r_0)(2^{\gamma-1}s_{\gamma-1})}$$

expanding the exponent

$$\begin{aligned}&= W_N^{2^\gamma(2^{\gamma-2}r_{\gamma-1}s_{\gamma-1})} \times W_N^{2^\gamma(2^{\gamma-3}r_{\gamma-2}s_{\gamma-1})} \times \cdots \\ &\qquad \cdots \times W_N^{2^\gamma(2^1 r_1 s_{\gamma-1})} \times W_N^{2^{\gamma-1}(r_0 s_{\gamma-1})} \\ &= W_N^{2^{\gamma-1}(2^0 r_0)s_{\gamma-1}}\end{aligned}$$

because $W_N^{2^\gamma(\text{integer})} = 1$.
In a similar manner, we have

$$W_N^{(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\dots+2^0r_0)(2^{\gamma-2}s_{\gamma-2})} = W_N^{2^{\gamma-2}(2^1r_1+2^0r_0)s_{\gamma-2}}$$

$$W_N^{(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\dots+2^0r_0)(2^{\gamma-3}s_{\gamma-3})} = W_N^{2^{\gamma-3}(2^2r_2+2^1r_1+2^0r_0)s_{\gamma-3}}$$

$$\cdots$$

$$W_N^{(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\dots+2^0r_0)(2^0s_0)} = W_N^{2^0(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\dots+2^0r_0)s_0}$$

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&= W_N^{2^\gamma(2^{\gamma-2}r_{\gamma-1}s_{\gamma-1})} \times W_N^{2^\gamma(2^{\gamma-3}r_{\gamma-2}s_{\gamma-1})} \times \cdots \\
&\qquad \cdots \times W_N^{2^\gamma(2^1r_1s_{\gamma-1})} \times W_N^{2^{\gamma-1}(r_0s_{\gamma-1})} \\
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&\cdots \\
W_N^{(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\cdots+2^0r_0)(2^0s_0)} &= W_N^{2^0(2^{\gamma-1}r_{\gamma-1}+2^{\gamma-2}r_{\gamma-2}+\cdots+2^0r_0)s_0}
\end{aligned}$$

Fast Fourier Transform (5)

Substituting all W_N terms, in their reduced form,

$$A(\mathrm{r}_{\gamma-1},\mathrm{r}_{\gamma-2},\ldots,\mathrm{r}_0)=$$

$$\sum_{\mathrm{s}_0=0}^{1}\sum_{\mathrm{s}_1=0}^{1}\cdots\sum_{\mathrm{s}_{\gamma-2}=0}^{1}\sum_{\mathrm{s}_{\gamma-1}=0}^{1} p_0(\mathrm{s}_{\gamma-1},\mathrm{s}_{\gamma-2},\ldots,\mathrm{s}_0)\times W_N^{2^{\gamma-1}(2^0\mathrm{r}_0)\mathrm{s}_{\gamma-1}}$$

$$\times W_N^{2^{\gamma-2}(2^1\mathrm{r}_1+2^0\mathrm{r}_0)\mathrm{s}_{\gamma-2}}\times\cdots\times W_N^{2^0(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)\mathrm{s}_0}$$

Carrying out all summations in succession, we have

$$\sum_{\mathrm{s}_{\gamma-1}=0}^{1} p_0(\mathrm{s}_{\gamma-1},\mathrm{s}_{\gamma-2},\ldots,\mathrm{s}_0)\times W_N^{2^{\gamma-1}(2^0\mathrm{r}_0)\mathrm{s}_{\gamma-1}}\equiv p_1(\mathrm{r}_0,\mathrm{s}_{\gamma-2},\ldots,\mathrm{s}_0)$$

$$\sum_{\mathrm{s}_{\gamma-2}=0}^{1} p_1(\mathrm{r}_0,\mathrm{s}_{\gamma-2},\ldots,\mathrm{s}_0)\times W_N^{2^{\gamma-2}(2^1\mathrm{r}_1+2^0\mathrm{r}_0)\mathrm{s}_{\gamma-2}}\equiv p_2(\mathrm{r}_0,\mathrm{r}_1,\mathrm{s}_{\gamma-3},\ldots,\mathrm{s}_0)$$

Fast Fourier Transform (6)

Proceding with the summations, we arrive at the last one, that gives us the coefficients A_n

$$\sum_{\mathrm{s}_0=0}^{1} p_{\gamma-1}(\mathrm{r}_0, \mathrm{r}_1, \ldots \mathrm{r}_{\gamma-2}, \mathrm{s}_0) \times W_N^{2^0(2^{\gamma-1}\mathrm{r}_{\gamma-1}+2^{\gamma-2}\mathrm{r}_{\gamma-2}+\cdots+2^0\mathrm{r}_0)\mathrm{s}_0}$$

$$\equiv p_\gamma(\mathrm{r}_0, \mathrm{r}_1, \ldots, \mathrm{r}_{\gamma-2}, \mathrm{r}_{\gamma-1})$$

$$\equiv A(\mathrm{r}_{\gamma-1}, \mathrm{r}_{\gamma-2}, \ldots, \mathrm{r}_0)$$

The coefficients are computed in an order different from what intended (we have the so called *bit reversal*), but it's simple to reorder them.
Our last remark, the number of multiplications using the Cooley-Tukey algorithm is in the order of $N\log(N)$, the savings even for moderately large N with respect to N^2 are well worth the complication of the FFT algorithm.

FFT looks complex? Here it is an example of a simple, standard implementation of the FFT.

Alternative Algorithm

Decimation in Time algorithm by Tukey and Cooley (1965), assume N is even, and divide the DFT summation to consider even and odd indices s

$$X_r = \sum_{s=0}^{N-1} x_s e^{-\frac{2\pi i}{N} sr}, \qquad r = 0, \dots, N-1$$

$$= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N}(2q)r} + \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N}(2q+1)r}$$

collecting $e^{-\frac{2\pi i}{N} r}$ in the second term and letting $2q/N = 2/(N/2)$

$$= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N/2} qr} + e^{-\frac{2\pi i}{N} r} \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N/2} qr}$$

We have two DFT's of length $N/2$, the operations count is hence $2(N/2)^2 = N^2/2$, but we have to combine these two halves in the full DFT.

Alternative 2

Say that

$$X_r = E_r + e^{-\frac{2\pi i}{N} r} O_r$$

where E_r and O_r are the even and odd half-DFT's, of which we computed only coefficients from 0 to $N/2 - 1$.
To get the full sequence we have to note that

1. the E and O DFT's are periodic with period $N/2$, and
2. $\exp(-2\pi i(r+N/2)/N) = e^{-\pi i}\exp(-2\pi ir/N) = -\exp(-2\pi ir/N)$,

so that we can write

$$X_r = \begin{cases} E_r + \exp(-2\pi ir/N) O_r & \text{if } r < N/2, \\ E_{r-N/2} - \exp(-2\pi ir/N) O_{r-N/2} & \text{if } r \geq N/2. \end{cases}$$

The algorithm that was outlined can be applied to the computation of each of the half-DFT's when $N/2$ were even, so that the operation count goes to $N^2/4$. If $N/4$ were even ...

Pseudocode for CT algorithm


```
def fft2(X, N):
  if N = 1 then
     Y = X
  else
     Y0 = fft2(X0, N/2)
     Y1 = fft2(X1, N/2)
     for k = 0 to N/2-1
       Y_k        = Y0_k + exp(2 pi i k/N) Y1_k
       Y_(k+N/2) = Y0_k - exp(2 pi i k/N) Y1_k
     endfor
  endif
return Y
```

Dynamic Response (1)

To evaluate the dynamic response of a linear SDOF system in the frequency domain, use the inverse DFT,

$$x_s = \sum_{r=0}^{N-1} V_r \exp(i\frac{2\pi\, rs}{N}), \quad s = 0, 1, \ldots, N-1$$

where $V_r = H_r\, P_r$. P_r are the discrete complex amplitude coefficients computed using the direct DFT, and H_r is the discretization of the complex frequency response function, that for viscous damping is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2)+i(2\zeta\beta_r)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2)-i(2\zeta\beta_r)}{(1-\beta_r^2)^2+(2\zeta\beta_r)^2}\right], \quad \beta_r = \frac{\omega_r}{\omega_n}.$$

while for hysteretic damping is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2)+i(2\zeta)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2)-i(2\zeta)}{(1-\beta_r^2)^2+(2\zeta)^2}\right].$$

Transfer Functions

Fourier Transform

The Discrete Fourier Transform

The Fast Fourier Transform

The Fast Fourier Transform

The response of a linear SDOF system to arbitrary loading can be evaluated by a convolution integral in the time domain,

$$x(t) = \int_0^t p(t)\, h(t-\tau)\, d\tau,$$

with the unit impulse response function $h(t) = \frac{1}{m\omega_D} \exp(-\zeta\omega_n t) \sin(\omega_D t)$, or through the frequency domain using the Fourier integral

$$x(t) = \int_{-\infty}^{+\infty} H(\omega) P(\omega) \exp(i\omega t)\, d\omega,$$

where $H(\omega)$ is the complex frequency response function.

Transfer Functions

These response functions, or *transfer* functions, are connected by the direct and inverse Fourier transforms:

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\, \exp(-i\omega t)\, dt,$$
$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega)\, \exp(i\omega t)\, d\omega.$$

Relationship of transfer functions

We write the response and its Fourier transform:

$$x(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau$$

$$X(\omega) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau\right] \exp(-i\omega t)\,dt$$

the lower limit of integration in the first equation was changed from 0 to $-\infty$ because $p(\tau) = 0$ for $\tau < 0$, and since $h(t-\tau) = 0$ for $\tau > t$, the upper limit of the second integral in the second equation can be changed from t to $+\infty$,

$$X(\omega) = \lim_{s\to\infty} \int_{-s}^{+s} \int_{-s}^{+s} p(\tau)h(t-\tau)\exp(-i\omega t)\,dt\,d\tau$$

Relationship of transfer functions

Introducing a new variable $\theta = t - \tau$ we have

$$X(\omega) = \lim_{s\to\infty} \int_{-s}^{+s} p(\tau)\exp(-i\omega\tau)\,d\tau \int_{-s-\tau}^{+s-\tau} h(\theta)\exp(-i\omega\theta)\,d\theta$$

with $\lim_{s\to\infty} s - \tau = \infty$, we finally have

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{+\infty} p(\tau)\exp(-i\omega\tau)\,d\tau \int_{-\infty}^{+\infty} h(\theta)\exp(-i\omega\theta)\,d\theta \\ &= P(\omega)\int_{-\infty}^{+\infty} h(\theta)\exp(-i\omega\theta)\,d\theta \end{aligned}$$

where we have recognized that the first integral is the Fourier transform of $p(t)$.

Relationship of transfer functions

Our last relation was

$$X(\omega) = P(\omega) \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta$$

but $X(\omega) = H(\omega)P(\omega)$, so that, noting that in the above equation the last integral is just the Fourier transform of $h(\theta)$, we may conclude that, effectively, $H(\omega)$ and $h(t)$ form a Fourier transform pair.