Generalized Single Degree of Freedom Systems

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Outline

Introductory Remarks

Assemblage of Rigid Bodies

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Vibration Analysis by Rayleigh's Method

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Refinement of Rayleigh's Estimates

Introductory Remarks

Until now we assumed that our systems were composed by a single mass, connected to a fixed reference by means of a spring and a damper.
This is an unnatural restriction, and we will see that many different systems can be studied as *SDOF* systems, either exactly or under some simplifying assumption.

1. *SDOF* rigid body assemblages, where flexibility is concentrated in a number of springs and dampers, can be studied, e.g., using the Principle of Virtual Displacements and the D'Alembert Principle.
2. simple structural systems can be studied, in an approximate manner, assuming a fixed pattern of displacements, whose amplitude (the single degree of freedom) varies with time.

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Further Remarks on Rigid Assemblages

In rigid body assemblages the limitation to a single shape of displacement is a consequence of the configuration of the system, i.e., the disposition of supports and internal hinges. When the equation of motion is written in terms of a single parameter and its time derivatives, we recognise that the terms that figure as coefficients in the equation of motion can be regarded as the *generalised* properties of the assemblage: generalised mass, damping and stiffness on left hand, generalised loading on right hand.

$$m^\star\ddot{x} + c^\star\dot{x} + k^\star x = p^\star(t)$$

Further Remarks on Flexible Systems

In flexible systems an infinite variety of deformation patterns is possible.
It is by assumption that we restrict the motion to a single shape, but under this assumption the system configuration is mathematically described by a single parameter, so that

- we can compute the *generalised* mass, damping, stiffness properties of the *SDOF* system,
- our *model* can be analysed in exactly the same way as a strict *SDOF* system.

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Final Remarks on Generalised *SDOF* Systems

From the previous comments, it should be apparent that everything we have seen regarding the behaviour and the integration of the equation of motion of proper *SDOF* systems applies to rigid body assemblages and to *SDOF* models of flexible systems, provided that we have the means for determining the *generalised* properties of the complex dynamical system under investigation.

Assemblages of Rigid Bodies

- planar rigid bodies, that is Bi-dimensional bodies constrained to move in a plane,
- flexibility is *concentrated* in discrete elements, that is springs and dampers,
- rigid bodies are connected to a fixed reference and to each other by means of springs, dampers and smooth, bilateral constraints (read hinges, double pendulums and rollers),
- inertial forces are distributed on all the surface of each rigid body, but we can consider only their resultant
  - a force applied to the centre of mass of the body, proportional to acceleration and total mass $M = \int dm$
  - a couple, proportional to angular acceleration and the moment of inertia J of the rigid body, $J = \int (x^2 + y^2) dm$.

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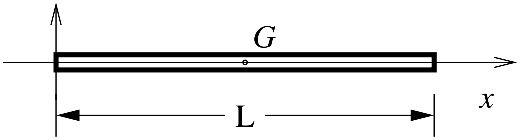
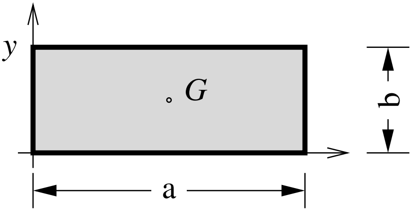
Rigid Bar



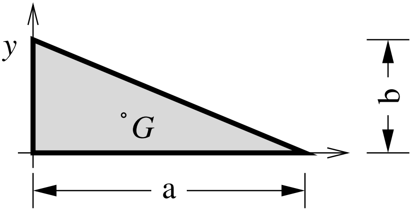
Unit mass	$\bar{m} = \text{constant},$
Length	$L,$
Centre of Mass	$x_G = L/2,$
Total Mass	$m = \bar{m}L,$
Moment of Inertia	$J = m\frac{L^2}{12} = \bar{m}\frac{L^3}{12}$

Rigid Rectangle



Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/2, \quad y_G = b/2$
Total Mass	$m = \gamma a b,$
Moment of Inertia	$J = m\frac{a^2+b^2}{12} = \gamma\frac{a^3b+ab^3}{12}$

Rigid Triangle

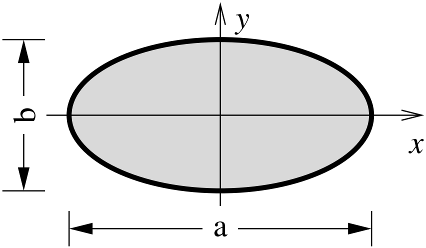


For a right triangle.

Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/3, \quad y_G = b/3$
Total Mass	$m = \gamma ab/2,$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{18} = \gamma\dfrac{a^3b+ab^3}{36}$

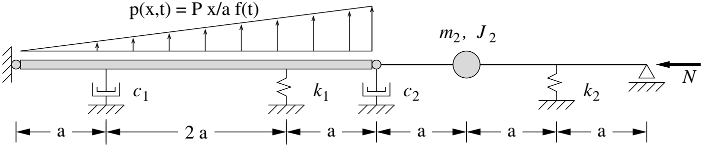
Rigid Oval

When $a = b = D = 2R$ the oval is a circle.



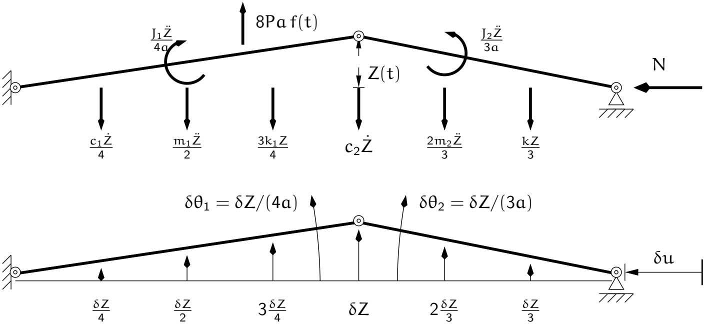
Unit mass	$\gamma = \text{constant},$
Axes	$a,\ b$
Centre of Mass	$x_G = y_G = 0$
Total Mass	$m = \gamma \dfrac{\pi a b}{4},$
Moment of Inertia	$J = m\dfrac{L^2}{12} = \bar{m}\dfrac{L^3}{12}$

trabacolo1



The mass of the left bar is $m_1 = \bar{m}\, 4a$ and its moment of inertia is $J_1 = m_1 \frac{(4a)^2}{12} = 4a^2 m_1/3$.
The maximum value of the external load is
$P_{max} = P\, 4a/a = 4P$ and the resultant of triangular load is
$R = 4P \times 4a/2 = 8Pa$

Forces and Virtual Displacements



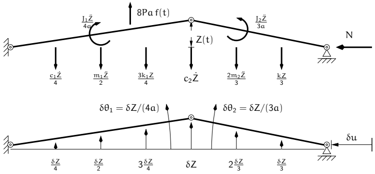
$$u = 7a - 4a\cos\theta_1 - 3a\cos\theta_2, \quad \delta u = 4a\sin\theta_1\delta\theta_1 + 3a\sin\theta_2\delta\theta_2$$

$$\delta\theta_1 = \delta Z/(4a), \quad \delta\theta_2 = \delta Z/(3a)$$

$$\sin\theta_1 \approx Z/(4a), \quad \sin\theta_2 \approx Z/(3a)$$

$$\delta u = \left(\frac{1}{4a} + \frac{1}{3a}\right) Z\,\delta Z = \frac{7}{12a} Z\,\delta Z$$

Principle of Virtual Displacements



$$\begin{aligned}\delta W_I &= -m_1\frac{\ddot Z}{2}\frac{\delta Z}{2} - J_1\frac{\ddot Z}{4a}\frac{\delta Z}{4a} - m_2\frac{2\ddot Z}{3}\frac{2\delta Z}{3} - J_2\frac{\ddot Z}{3a}\frac{\delta Z}{3a}\\ &= -\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot Z\,\delta Z\end{aligned}$$

$$\delta W_D = -c_1\frac{\dot Z}{4}\frac{\delta Z}{4} - -c_2 Z\,\delta Z = -\left(c_2 + c_1/16\right)\dot Z\,\delta Z$$

$$\delta W_S = -k_1\frac{3Z}{4}\frac{3\delta Z}{4} - k_2\frac{Z}{3}\frac{\delta Z}{3} = -\left(\frac{9k_1}{16} + \frac{k_2}{9}\right)Z\,\delta Z$$

$$\delta W_{Ext} = 8Pa\,f(t)\frac{2\delta Z}{3} + N\frac{7}{12a}Z\,\delta Z$$

Principle of Virtual Displacements

For a rigid body in condition of equilibrium the total virtual work must be equal to zero

$$\delta W_\mathsf{I} + \delta W_\mathsf{D} + \delta W_\mathsf{S} + \delta W_\mathsf{Ext} = 0$$

Substituting our expressions of the virtual work contributions and simplifying δZ, the equation of equilibrium is

$$\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z} + \\ + \left(c_2 + c_1/16\right)\dot{Z} + \left(\frac{9k_1}{16} + \frac{k_2}{9}\right)Z = \\ 8Pa\,f(t)\frac{2}{3} + N\frac{7}{12a}Z$$

Introducing the generalised properties $m^\star$, etc

$$m^\star\ddot{Z} + c^\star\dot{Z} + k^\star Z = p^\star f(t)$$

It is worth writing down the expression of $k^\star$:

$$k^\star = \frac{9k_1}{16} + \frac{k_2}{9} - \frac{7}{12a}N$$

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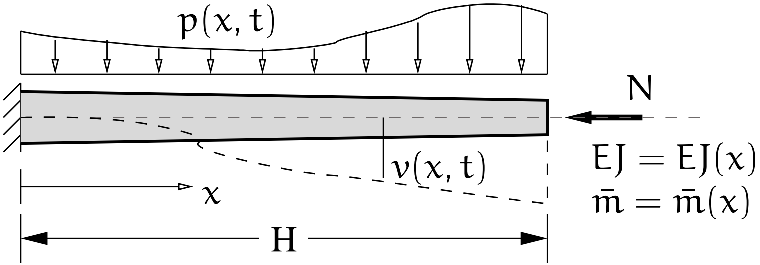
Let's start with an example...

Consider a cantilever, with varying properties $\bar{m}$ and EJ, subjected to a load that is function of both time t and position x, also the transverse displacements v will be function of time and position,

$$v = v(x, t)$$



... and an hypothesis

To study the previous problem, we introduce an *approximate model* by the following hypothesis,

$$v(x,t) = \Psi(x)\,Z(t),$$

that is, the hypothesis of *separation of variables*

Note that $\Psi(x)$, the *shape function*, is adimensional, while $Z(t)$ is dimensionally a generalised displacement, usually chosen to characterise the structural behaviour.

In our example we can use the displacement of the tip of the chimney, thus implying that $\Psi(H) = 1$ because

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Principle of Virtual Displacements

For a flexible system, the PoVD states that, at equilibrium,

$$\delta W_{\mathsf{E}} = \delta W_{\mathsf{I}}.$$

The virtual work of external forces can be easily computed, the virtual work of internal forces is usually approximated by the virtual work done by bending moments, that is

$$\delta W_{\mathsf{I}} \approx \int M\,\delta\chi$$

where χ is the curvature and $\delta\chi$ the virtual increment of curvature.

δW_E

The external forces are $p(x,t)$, N and the forces of inertia f_I; we have, by separation of variables, that $\delta v = \Psi(x)\delta Z$ and we can write

$$\delta W_p = \int_0^H p(x,t)\delta v \, dx = \left[\int_0^H p(x,t)\Psi(x)\, dx\right] \delta Z = p^\star(t)\,\delta Z$$

$$\begin{aligned}\delta W_{\text{Inertia}} &= \int_0^H -\bar{m}(x)\ddot{v}\delta v \, dx = \int_0^H -\bar{m}(x)\Psi(x)\ddot{Z}\Psi(x)\, dx\;\delta Z \\ &= \left[\int_0^H -\bar{m}(x)\Psi^2(x)\, dx\right] \ddot{Z}(t)\,\delta Z = m^\star \ddot{Z}\,\delta Z.\end{aligned}$$

The virtual work done by the axial force deserves a separate treatment...

δW_N

The virtual work of N is $\delta W_N = N\delta u$ where δu is the variation of the vertical displacement of the top of the chimney.
We start computing the vertical displacement of the top of the chimney in terms of the rotation of the axis line, $\phi \approx \Psi'(x)Z(t)$,

$$u(t) = H - \int_0^H \cos\phi\, dx = \int_0^H (1-\cos\phi)\, dx,$$

substituting the well known approximation $\cos\phi \approx 1 - \frac{\phi^2}{2}$ in the above equation we have

$$u(t) = \int_0^H \frac{\phi^2}{2}\, dx = \int_0^H \frac{\Psi'^2(x)Z^2(t)}{2}\, dx$$

hence

$$\delta u = \int_0^H \Psi'^2(x)Z(t)\delta Z\, dx = \int_0^H \Psi'^2(x)\, dx\; Z\delta Z$$

and

$$\delta W_N = \left[\int_0^H \Psi'^2(x)\, dx\; N\right] Z\,\delta Z = k_G^\star\, Z\,\delta Z$$

δW_{Int}

Approximating the internal work with the work done by bending moments, for an infinitesimal slice of beam we write

$$dW_{\text{Int}} = \frac{1}{2} M v''(x,t)\,dx = \frac{1}{2} M \Psi''(x) Z(t)\,dx$$

with $M = EJ(x)v''(x)$

$$\delta(\,dW_{\text{Int}}) = EJ(x)\Psi''^{\,2}(x)Z(t)\delta Z\,dx$$

integrating

$$\delta W_{\text{Int}} = \left[\int_0^H EJ(x)\Psi''^{\,2}(x)\,dx\right] Z\delta Z = k^\star\, Z\,\delta Z$$

Remarks

- the shape function *must* respect the geometrical boundary conditions of the problem, i.e., both

$$\Psi_1 = x^2 \qquad \text{and} \qquad \Psi_2 = 1 - \cos\frac{\pi x}{2H}$$

are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$

- better results are obtained when the second derivative of the shape function at least *resembles* the typical distribution of bending moments in our problem, so that between

$$\Psi_1'' = \text{constant} \qquad \text{and} \qquad \Psi_2'' = \frac{\pi^2}{4H^2}\cos\frac{\pi x}{2H}$$

the second choice is preferable.

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Example

Using $\Psi(x) = 1 - \cos\frac{\pi x}{2H}$, with $\bar{m}$ = constant and EJ = constant, with a load characteristic of seismic excitation, $p(t) = -\bar{m}\ddot{v}_g(t)$,

$$m^\star = \bar{m}\int_0^H (1 - \cos\frac{\pi x}{2H})^2\,dx = \bar{m}(\frac{3}{2} - \frac{4}{\pi})H$$

$$k^\star = EJ\frac{\pi^4}{16H^4}\int_0^H \cos^2\frac{\pi x}{2H}\,dx = \frac{\pi^4}{32}\frac{EJ}{H^3}$$

$$k_G^\star = N\frac{\pi^2}{4H^2}\int_0^H \sin^2\frac{\pi x}{2H}\,dx = \frac{\pi^2}{8H}N$$

$$p_g^\star = -\bar{m}\ddot{v}_g(t)\int_0^H 1 - \cos\frac{\pi x}{2H}\,dx = -\left(1 - \frac{2}{\pi}\right)\bar{m}H\,\ddot{v}_g(t)$$

Vibration Analysis

- The process of estimating the vibration characteristics of a complex system is known as *vibration analysis*.
- We can use our previous results for flexible systems, based on the *SDOF* model, to give an estimate of the natural frequency $\omega^2 = k^\star/m^\star$
- A different approach, proposed by Lord Rayleigh, starts from different premises to give the same results but the *Rayleigh's Quotient* method is important because it offers a better understanding of the vibrational behaviour, eventually leading to successive refinements of the first estimate of ω^2.

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Rayleigh's Quotient Method

Our focus will be on the *free vibration* of a flexible, undamped system.

- inspired by the free vibrations of a proper *SDOF* we write

$$Z(t) = Z_0 \sin \omega t \text{ and } v(x,t) = Z_0 \Psi(x) \sin \omega t,$$

- the displacement and the velocity are in quadrature: when v is at its maximum $\dot{v} = 0$ (hence $V = V_{\max}$, $T = 0$) and when $v = 0$ $\dot{v}$ is at its maximum (hence $V = 0$, $T = T_{\max}$,
- disregarding damping, the energy of the system is constant during free vibrations,

$$V_{\max} + 0 = 0 + T_{\max}$$

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Rayleigh' s Quotient Method

Now we write the expressions for V_{max} and T_{max},

$$V_{max} = \frac{1}{2} Z_0^2 \int_S EJ(x) \Psi''^2(x)\,dx,$$
$$T_{max} = \frac{1}{2} \omega^2 Z_0^2 \int_S \bar{m}(x) \Psi^2(x)\,dx,$$

equating the two expressions and solving for ω^2 we have

$$\omega^2 = \frac{\int_S EJ(x)\Psi''^2(x)\,dx}{\int_S \bar{m}(x)\Psi^2(x)\,dx}.$$

Recognizing the expressions we found for $k^\star$ and $m^\star$ we could question the utility of Rayleigh's Quotient...

Rayleigh's Quotient Method

- In Rayleigh's method we know the specific time dependency of the structure's free vibrations and hence that, at least for constant unit mass structures, the *shape* of the inertial forces

$$f_I = -\bar{m}\ddot{v} = \bar{m}\omega^2 Z_0 \Psi(x) \sin \omega t$$

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- Thinking backwards we can say that, if Ψ is the real shape assumed by the structure in free vibrations, the displacements v due to a loading $f_I = \bar{m}(x)\Psi(x)$ are proportional to $\Psi(x)$ through a constant factor, and equilibrium is respected all over the structure during free vibrations.
- Having deduced a new shape function Ψ_1 from the displacements due to a loading $f_I = \bar{m}(x)\Psi_0(x)$, we speculate that the new shape function is a better approximation of the true mode shape. We will demonstrate the correctness of this hypothesis.

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Selection of mode shapes

Given different shape functions Ψ_i and considering the true shape of free vibration Ψ, in the former cases equilibrium is not respected by the structure itself.

To keep inertia induced deformation proportional to Ψ_i we must consider the presence of additional elastic constraints.

This leads to the following considerations

- the frequency of vibration of a structure with additional constraints is higher than the true natural frequency,
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Selection of mode shapes 2

In general the selection of trial shapes goes through two steps,

1. the analyst considers the flexibilities of different parts of the structure and the presence of symmetries to devise an approximate shape,
2. the structure is loaded with constant loads directed as the assumed displacements, the displacements are computed and used as the shape function,

of course a little practice helps a lot in the the choice of a proper pattern of loading...

Refinement R_{00}

Choose a trial function $\Psi^{(0)}(x)$ and write

$$v^{(0)} = \Psi^{(0)}(x) Z^{(0)} \sin \omega t$$

$$V_{\max} = \frac{1}{2} Z^{(0)2} \int EJ\Psi^{(0)\prime\prime 2}\, dx$$

$$T_{\max} = \frac{1}{2} \omega^2 Z^{(0)2} \int \bar{m} \Psi^{(0)2}\, dx$$

our first estimate R_{00} of ω^2 is

$$\omega^2 = \frac{\int EJ\Psi^{(0)\prime\prime 2}\, dx}{\int \bar{m}\Psi^{(0)2}\, dx}.$$

Refinement R_{01}

We try to give a better estimate of V_{max} computing the external work done by the inertial forces,

$$p^{(0)} = \omega^2 \bar{m}(x) v^{(0)} = Z^{(0)} \omega^2 \Psi^{(0)}(x)$$

the deflections due to $p^{(0)}$ are

$$v^{(1)} = \omega^2 \frac{v^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \frac{Z^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \bar{Z}^{(1)},$$

where we write $\bar{Z}^{(1)}$ because we need to keep the unknown ω^2 in evidence. The maximum strain energy is

$$V_{max} = \frac{1}{2} \int p^{(0)} v^{(1)} \, dx = \frac{1}{2} \omega^4 Z^{(0)} \bar{Z}^{(1)} \int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, dx$$

Equating to our previus estimate of T_{max} we find the R_{01} estimate

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x) \Psi^{(0)} \Psi^{(0)} \, dx}{\int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, dx}$$

Refinement R_{11}

With little additional effort it is possible to compute T_{max} from $v^{(1)}$:

$$T_{max} = \frac{1}{2}\omega^2 \int \bar{m}(x) v^{(1)2}\, dx = \frac{1}{2}\omega^6 \bar{Z}^{(1)2} \int \bar{m}(x)\Psi^{(1)2}\, dx$$

equating to our last approximation for V_{max} we have the R_{11} approximation to the frequency of vibration,

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x)\Psi^{(0)}\Psi^{(1)}\, dx}{\int \bar{m}(x)\Psi^{(1)}\Psi^{(1)}\, dx}.$$

Of course the procedure can be extended to compute better and better estimates of ω^2 but usually the refinements are not extended beyond R_{11}, because it would be contradictory with the quick estimate nature of the Rayleigh's Quotient method and also because R_{11} estimates are usually very good ones.

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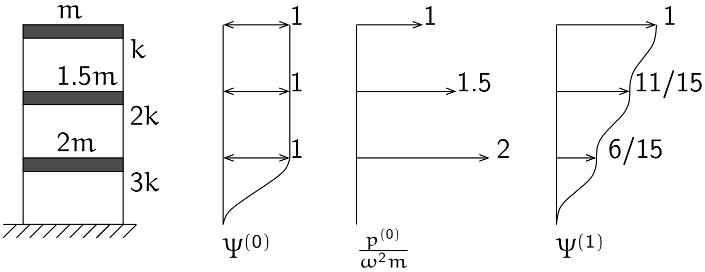
Refinement Example



$$v^{(1)} = \frac{15}{4}\frac{m}{k}\omega^2\Psi^{(1)}$$

$$\bar{Z}^{(1)} = \frac{15}{4}\frac{m}{k}$$