Multi Degrees of Freedom Systems

MDOF's

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Introductory Remarks

The Homogeneous Problem

Modal Analysis

Examples

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- Properties of Structural Matrices
- An example

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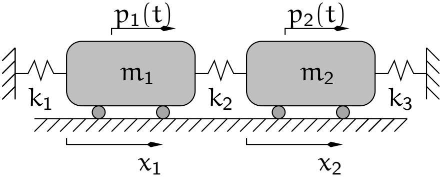
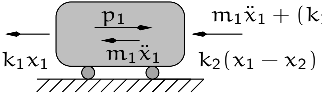
Introductory Remarks

Consider an undamped system with two masses and two degrees of freedom,

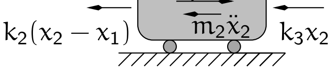


write the equation of equilibrium, using the D'Alembert principle, for each mass:



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t)$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t)$$



The equation of motion of a 2DOF system

With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_3x_2 & = p_1(t) \\ m_1\ddot{x}_1 - k_2x_1 + (k_2 + k_3)x_2 & = p_2(t). \end{cases}$$

The equation of motion of a 2DOF system

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Introducing the loading vector $\mathbf{p}$, the vector of inertial forces $\mathbf{f}_I$ and the vector of elastic forces $\mathbf{f}_S$,

$$\mathbf{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \mathbf{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\mathbf{f}_I + \mathbf{f}_S = \mathbf{p}(t).$$

$\mathbf{f}_S = \mathbf{K}\,\boldsymbol{x}$

It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements

$$\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix},$$

in terms of a matrix product

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \boldsymbol{x}$$

or, introducing the stiffness matrix $\mathbf{K}$,

$$\mathbf{K} = \begin{bmatrix} k - 1 + k_2 & -k_2 \\ -k_2 & k_2 + k + 3 \end{bmatrix},$$

we can write

$$\mathbf{f}_S = \mathbf{K}\,\boldsymbol{x}$$

$\mathbf{f}_\mathrm{I} = \mathbf{M}\,\ddot{\boldsymbol{x}}$

Analogously, introducing the mass matrix $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\mathbf{f}_\mathrm{I} = \mathbf{M}\,\ddot{\boldsymbol{x}}.$$

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Matrix Equation

Finally it is possible to write the equation of motion in matricial format:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t).$$

In the following we will see how it is possible to consider the effects of damping introducing a *damping matrix* $\mathbf{C}$ and writing

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t),$$

however it is now more productive fixing our attention on undamped systems.

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Properties of $\mathbf{K}$

- if $\mathbf{K}$ were symmetrical, the force on mass j due to an unit displacement of mass i would be equal to the force on mass i due to an unit displacement of mass j; as this is true because the two masses are joined by the same spring, we have that $\mathbf{K}$ is symmetrical.
- The strain energy V for a discrete system can be written

$$V = \frac{1}{2}\mathbf{x}^T \mathbf{f}_S = \frac{1}{2}\mathbf{x}^T \mathbf{K}\,\mathbf{x},$$

because the strain energy is positive it follows that $\mathbf{K}$ is a positive definite matrix.

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Properties of $\mathbf{M}$

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive, as well as the stiffness matrix is symmetrical and definite positive.

En passant, take note that the kinetic energy for a discrete system is

$$\mathrm{T} = \frac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\mathbf{M}\,\dot{\boldsymbol{x}}.$$

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Properties of $\boldsymbol{M}$

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Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with one exception.

For a general structural system, $\mathbf{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy could be zero.

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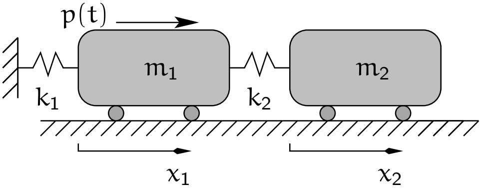
The problem



$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

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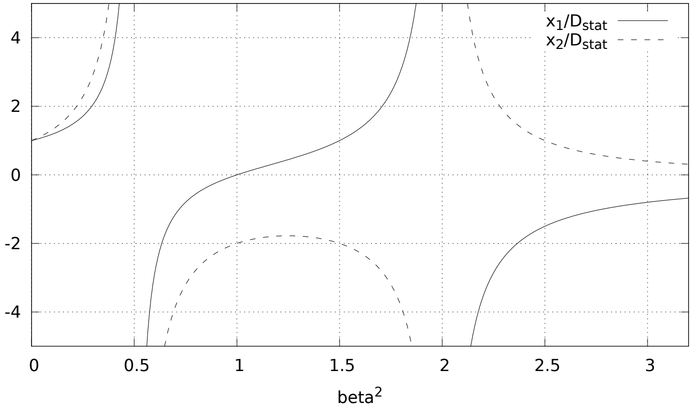
The solution

...

The solution, graphically



Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we start writing the homogeneous equation of motion,

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = 0,$$

and use the technique of separation of variables

$$\mathbf{x}(t) = \boldsymbol{\psi}(A \sin \omega t + B \cos \omega t)$$

where $\boldsymbol{\psi}$ is a fixed, unknown vector, named a *shape vector*.
Substituting in the equation of motion, we have

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\boldsymbol{\psi}(A \sin \omega t + B \cos \omega t) = 0$$

Homogeneous equation of motion

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Eigenvalues

The previous equation must hold for every value of t, so it can be reduced to

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \boldsymbol{\psi} = 0$$

We have a homogeneous linear equation, with unknowns ψ_i and the matrix of coefficients that depends on the parameter ω^2.

The trivial solution being

$$\boldsymbol{\psi} = 0,$$

different solutions are available when

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation is verified.

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Eigenvectors

For a system with N degrees of freedom the expansion of $\det\left(\mathbf{K}-\omega^2\mathbf{M}\right)$ is an algebraic polynomial of degree N in ω^2, whose roots, ω_i^2, $i = 1, \ldots, N$ are all real and greater than zero.
Substituting one of the roots ω_i^2 in the characteristic equation,

$$\left(\mathbf{K}-\omega_i^2\mathbf{M}\right)\boldsymbol{\psi}_i = 0$$

each one of the N *eigenvectors* $\boldsymbol{\psi}_i$ can be computed, except for an undetermined common scale factor.
A common choice for the normalisation of the eigenvectors is *normalisation with respect to the mass matrix*,

$$\boldsymbol{\psi}_i^T\mathbf{M}\,\boldsymbol{\psi}_i = 1$$

Initial Conditions

The most general expression (*general integral*) for the displacement of an homogeneous system is

$$\boldsymbol{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t)$$

In the general integral there are 2N unknown *constants of integration*, that must be determined in terms of the initial conditions, usually expressed in terms of initial displacements and initial velocities,

$$\begin{cases} \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \dot{\boldsymbol{x}}(0) = \dot{\boldsymbol{x}}_0 \end{cases} \Rightarrow \begin{cases} x_{i,0} = \sum_{j=1}^{N} \psi_{ij} B_j \\ \dot{x}_{i,0} = \sum_{j=1}^{N} \omega_j \psi_{ij} A_j \end{cases} \text{for } i = 1, \ldots, N,$$

where ψ_{ij} is the i-nth component of $\boldsymbol{\psi}_j$.

OOOPS!

i forgot to link the pdf with the last week lesson... i will put the link in place tonight, but if you don't trust me, after the class come here with your USB key, you'll be welcome!

Orthogonality - 1

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\mathbf{K}\,\psi_r = \omega_r^2 \mathbf{M}\,\psi_r$$
$$\mathbf{K}\,\psi_s = \omega_s^2 \mathbf{M}\,\psi_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\psi_s^T \mathbf{K}\,\psi_r = \omega_r^2 \psi_s^T \mathbf{M}\,\psi_r$$
$$\psi_r^T \mathbf{K}\,\psi_s = \omega_s^2 \psi_r^T \mathbf{M}\,\psi_s$$

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Orthogonality - 2

The term $\psi_s^T \mathbf{K} \psi_r$ is a scalar, hence

$$\psi_s^T \mathbf{K} \psi_r = \left(\psi_s^T \mathbf{K} \psi_r\right)^T = \psi_r^T \mathbf{K}^T \psi_s$$

but $\mathbf{K}$ is symmetrical, $\mathbf{K}^T = \mathbf{K}$ and we have

$$\psi_s^T \mathbf{K} \psi_r = \psi_r^T \mathbf{K} \psi_s.$$

By a similar derivation

$$\psi_s^T \mathbf{M} \psi_r = \psi_r^T \mathbf{M} \psi_s.$$

Orthogonality - 3

Substituting our last identities in the previous equations, we have

$$\psi_r^T K \psi_s = \omega_r^2 \psi_r^T M \psi_s$$
$$\psi_r^T K \psi_s = \omega_s^2 \psi_r^T M \psi_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2)\, \psi_r^T M \psi_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\psi_r^T M \psi_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 3

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$$\psi_r^T \mathbf{M}\, \psi_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M} \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i$$

and

$$\boldsymbol{\psi}_i^T \mathbf{K} \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

Orthogonality - 4

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and

$$\psi_i^T \mathbf{K} \psi_i = \omega_i^2 M_i.$$

Eigenvectors are a base

The eigenvector are linearly independent, so for every vector $\boldsymbol{x}$ we can write

$$\boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j, \quad \text{with } q_j = \frac{\boldsymbol{\psi}_j^T \boldsymbol{M} \boldsymbol{x}}{M_j}$$

because of orthogonality and, generalising,

$$\boldsymbol{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\boldsymbol{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t),$$

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}\, \boldsymbol{q}(t), \qquad \ddot{\boldsymbol{x}}(t) = \boldsymbol{\Psi}\, \ddot{\boldsymbol{q}}(t).$$

where $\boldsymbol{q}(t)$ is the vector of *modal coordinates* and $\boldsymbol{\Psi}$, whose columns are the eigenvectors, is the *eigenvector matrix*.

Generalized SDOF's

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EoM in Modal Coordinates...

Substitute in the equation of motion,

$$\mathbf{M}\,\mathbf{\Psi}\,\ddot{\mathbf{q}} + \mathbf{K}\,\mathbf{\Psi}\,\mathbf{q} = \mathbf{p}(t)$$

premultiply by $\mathbf{\Psi}^T$

$$\mathbf{\Psi}^T\mathbf{M}\,\mathbf{\Psi}\,\ddot{\mathbf{q}} + \mathbf{\Psi}^T\mathbf{K}\,\mathbf{\Psi}\,\mathbf{q} = \mathbf{\Psi}^T\mathbf{p}(t)$$

with obvious definitions

$$\mathbf{M}^\star\ddot{\mathbf{q}} + \mathbf{K}^\star\,\mathbf{q} = \mathbf{p}^\star(t)$$

... are N independent equations!

By the preceding obvious definitions we have that the generic element of the *starred* matrices can be expressed in terms of single eigenvectors,

$$M^\star_{ij} = \boldsymbol{\psi}_i^T \mathbf{M}\,\boldsymbol{\psi}_j = \quad \delta_{ij} M_i,$$
$$K^\star_{ij} = \boldsymbol{\psi}_i^T \mathbf{K}\,\boldsymbol{\psi}_j = \omega_i^2 \delta_{ij} M_i.$$

where δ_{ij} is the *Kroneker* symbol,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p^\star_i = \boldsymbol{\psi}_i^T \mathbf{p}(t)$ we have a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p^\star_i(t), \qquad i = 1, \ldots, N$$

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Initial Conditions Revisited

The initial conditions

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases}$$

Consider, e.g., the initial displacements: we can write

$$x_0 = \Psi\, q_0$$

premultiplying both members by $\Psi^T M$,

$$\Psi^T M\, x_0 = \Psi^T M\, \Psi\, q_0 = M^\star q_0$$

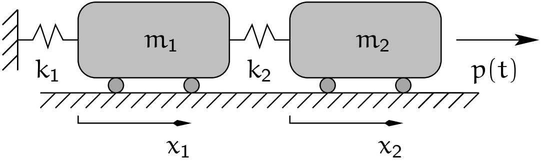
premultiplying by the inverse of $M^\star$ and taking into account that $M^\star$ is diagonal,

$$q_0 = (M^\star)^{-1}\, \Psi^T M\, x_0 \quad \Rightarrow \quad q_{i0} = \frac{\psi_i^T M\, x_0}{M_i}$$

analogously

$$\dot{q}_{i0} = \frac{\psi_i{}^T M\, \dot{x}_0}{M_i}$$

2 DOF System



$$k_1 = k, \quad k_2 = 2k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\mathbf{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = k \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

Characteristic Equation

The characteristic equation is

$$\left\|\mathbf{K}-\omega^2\mathbf{M}\right\| = \begin{Vmatrix} 3k-2\omega^2 m & -2k \\ -2k & 2k-\omega^2 m \end{Vmatrix} = 0.$$

Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m}\frac{7-\sqrt{33}}{4} \qquad \omega_2^2 = \frac{k}{m}\frac{7+\sqrt{33}}{4}$$

$$\omega_1^2 = 0.31386\frac{k}{m} \qquad \omega_2^2 = 3.18614\frac{k}{m}$$

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Eigenvectors

The first of the characteristic equation, substituting ω_1^2, gives

$$k\,(3 - 2 \times 0.31386)\psi_{11} - 2k\psi_{21} = 0$$

while substituting ω_2^2 gives

$$k\,(3 - 2 \times 3.18614)\psi_{12} - 2k\psi_{22} = 0$$

solving with $\psi_{21} = \psi_{22} = 1$ gives

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} +0.84307 \\ +1.00000 \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -0.59307 \\ +1.00000 \end{Bmatrix},$$

the *unnormalized* eigenvectors.

Normalization

We compute first M_1 and M_2,

$$M_1 = \boldsymbol{\psi}_1^T \mathbf{M} \boldsymbol{\psi}_1$$
$$= \begin{Bmatrix} 0.84307, & 1 \end{Bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix}$$
$$= \begin{Bmatrix} 1.68614m, & m \end{Bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} = 2.42153m$$

$$M_2 = 1.70346m$$

the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{2.42153}, \qquad \alpha_2 = \sqrt{1.70346}.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.54177 & -0.45440 \\ +0.64262 & +0.76618 \end{bmatrix}$$

Modal Loadings

The modal loading is

$$\begin{aligned}\mathbf{p}^\star(t) &= \mathbf{\Psi}^T\,\mathbf{p}(t)\\ &= p_0 \begin{bmatrix}+0.54177 & +0.64262\\ -0.45440 & +0.76618\end{bmatrix} \begin{Bmatrix}0\\1\end{Bmatrix} \sin\omega t\\ &= p_0 \begin{Bmatrix}+0.64262\\ +0.76618\end{Bmatrix} \sin\omega t\end{aligned}$$

Modal EoM

Substituting its modal expansion for $\boldsymbol{x}$ into the equation of motion and premultiplying by $\boldsymbol{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} m\ddot{q}_1 + 0.31386k\, q_1 = +0.64262\, p_0 \sin\omega t \\ m\ddot{q}_2 + 3.18614k\, q_2 = +0.76618\, p_0 \sin\omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by m both equations, we have

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = +0.64262\, \dfrac{p_0}{m} \sin\omega t \\ \ddot{q}_2 + \omega_2^2 q_2 = +0.76618\, \dfrac{p_0}{m} \sin\omega t \end{cases}$$

Particular Integral

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 \left(\omega_1^2 - \omega^2\right) \sin \omega t = \frac{p_1^\star}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{p_1^\star}{m} \frac{1}{\omega_1^2 - \omega^2}$$

with $\omega_1^2 = K_1/m \;\Rightarrow\; m = K_1/\omega_1^2$:

$$C_1 = \frac{p_1^\star}{K_1} \frac{\omega_1^2}{\omega_1^2 - \omega^2} = \Delta_{st}^{(1)} \frac{1}{1 - \beta_1^2} \quad \text{with } \Delta_{st}^{(1)} = \frac{p_1^\star}{K_1} = 2.047 \frac{p_0}{k} \text{ and } \beta_1 = \frac{\omega}{\omega_1}$$

of course

$$C_2 = \Delta_{st}^{(2)} \frac{1}{1 - \beta_2^2} \quad \text{with } \Delta_{st}^{(2)} = \frac{p_2^\star}{K_2} = 0.2404 \frac{p_0}{k} \text{ and } \beta_2 = \frac{\omega}{\omega_2}$$

Integrals

The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin\omega_1 t + B_1 \cos\omega_1 t + \Delta_{st}^{(1)} \dfrac{\sin\omega t}{1-\beta_1^2} \\ q_2(t) = A_2 \sin\omega_2 t + B_2 \cos\omega_2 t + \Delta_{st}^{(2)} \dfrac{\sin\omega t}{1-\beta_2^2} \end{cases}$$

for a system initially at rest

$$\begin{cases} q_1(t) = \Delta_{st}^{(1)} \dfrac{1}{1-\beta_1^2} (\sin\omega t - \beta_1 \sin\omega_1 t) \\ q_2(t) = \Delta_{st}^{(2)} \dfrac{1}{1-\beta_2^2} (\sin\omega t - \beta_2 \sin\omega_2 t) \end{cases}$$

we are interested in structural degrees of freedom, too... disregarding transient

$$\begin{cases} x_1(t) = \left(\psi_{11}\dfrac{\Delta_{st}^{(1)}}{1-\beta_1^2} + \psi_{12}\dfrac{\Delta_{st}^{(2)}}{1-\beta_2^2}\right)\sin\omega t = \left(\dfrac{1.10926}{1-\beta_1^2} - \dfrac{0.109271}{1-\beta_2^2}\right)\dfrac{p_0}{k}\sin\omega t \\ x_2(t) = \left(\psi_{21}\dfrac{\Delta_{st}^{(1)}}{1-\beta_1^2} + \psi_{22}\dfrac{\Delta_{st}^{(2)}}{1-\beta_2^2}\right)\sin\omega t = \left(\dfrac{1.31575}{1-\beta_1^2} + \dfrac{0.184245}{1-\beta_2^2}\right)\dfrac{p_0}{k}\sin\omega t \end{cases}$$