Dynamics of Structures 2009-2010

summer home assignment

1 Dynamical Testing

A simply supported beam is loaded at mid-span with a harmonic load, with different frequencies and the same *nominal* load amplitude.

Due to the position of the load and the frequencies of excitation used (close to the natural frequency, as determined by preliminary tests), it is reasonable to assume that only the first mode of vibration of the beam is excited.

The steady-state response parameters (amplitude and phase difference) are measured for each loading frequency, and are here reported in table 1. Determine the characteristics of the first mode of vibration of the beam.

f [Hz]	p_0 [N]	Δ_{max} [mm]	θ [deg]
3.40	1200	5.811	6.879
3.80	1200	9.143	11.863
4.20	1200	21.296	32.012
4.60	1200	23.079	141.294
5.00	1200	8.450	165.476
5.40	1200	4.972	171.082

Table 1: load and response characteristics

Be warned that all the data reported in table 1 is affected by experimental errors, so that particular care should be used to get the best possible estimate of ω and ζ.

1.1 Solution

Denoting with the index i the parameters and measurements concerning test number i, $i = 1, \ldots, 6$, it is possible to write, for each test,

$$k - \omega_i^2 m = fracp_{0,i} \cos\theta_i \Delta_i, \qquad i = 1, \ldots, 6.$$

Having written 6 equations to determine k and m, there is no solution that satisfies all the equations. In these case, usually we choose the *best* solution as the solution that minimises the sum of the squares of the residuals,

$$\mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b}$$
$$|r^2| = \mathbf{r}^T\mathbf{r} = (\mathbf{x}^T\mathbf{A}^T\mathbf{b})(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$
$$\frac{d|r^2|}{dx} = 0 \Leftrightarrow \mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

A similar procedure can be used for estimating the damping coefficient and the damping ratio, as we can write

$$c = \frac{p_{0,i} \sin\theta_i}{\omega_i \Delta_i}, \qquad i = 1, \ldots, 6.$$

(note that in this case the output of the least squares procedure is equivalent to the mean value of the 6 estimates).

The table below shows the original data, as input yo the procedure that generated the data in Table 1, and the best estimates found with the least squares procedure

	Original	Estimates
Stiffness	492000	493178
Mass	637.91	637.58
Natural freq.	4.4200	4.4264
Damping ratio	3.2000	3.1867

2 Vibration Isolation

A rotating machine weights 75kN and during operation transmits to its rigid foundation a harmonic load of 2.4kN, at a frequency f=60Hz.

It is intended to reduce the transmitted force to 500N, suspending the machine over a system of elastic supports.

1. What should be the total stiffness of the elastic supports to achieve the required transitted force reduction?

2. What if the dynamic displacements that the machine will experience when mounted on the elastic support are too large for a correct operation? Is it possible to modify the support system so that the displacements are reduced and the harmonic force transmitted still is no greater than 500N?

2.1 Solution

The system mass is $75\,\mathrm{kN}/g = 7645.26\,\mathrm{kg}$, the excitation circular frequency is $2\pi\,60\,\mathrm{Hz} = 376.99\,\mathrm{rad\,s^{-1}}$.

For an undamped system the steady state displacement is

$$d_{ss} = \frac{p_0}{k - \omega^2 m}$$

and the steady state transmitted force is

$$f_{ss} = p_0 \frac{1}{1-\beta^2}, \qquad \text{where } \beta = \frac{\omega}{\omega_0},$$

in particular we know that there is a force reduction only for $\beta > 1$, so that to have a positive value on both sides of our equation we want to write

$$d_{ss} = \frac{p_0}{\omega^2 m - k} \quad \text{and} \quad f_{ss} = p_0 \frac{1}{\beta^2 - 1} = p_0 \frac{k}{\omega^2 m - k}.$$

The maximum value of k for which

$$f_{ss} = p_0 \frac{k}{\omega^2 m - k} < f_{max}$$

is

$$k_{max} = \frac{m\omega^2 f_{max}}{p_0 + f_{max}},$$

so that substituting our values give

$$k_{max} = \frac{7645.26\,\mathrm{kg} \cdot 142\,122.3\,\mathrm{rad^2\,s^{-2}} \cdot 500\,\mathrm{N}}{2900\,\mathrm{N}} = 187\,338.3\,\mathrm{kN\,m^{-1}}$$

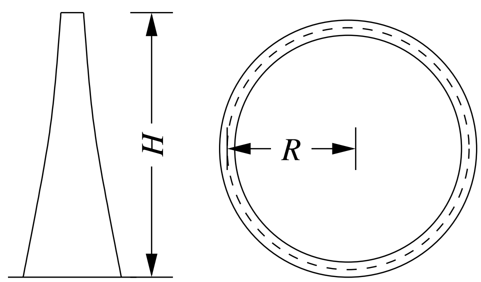
If it is required a lesser steady state dynamic displacement, given by

$$d_{ss} = \frac{p_0}{k - \omega^2 m},$$

it should be apparent that it is needed a larger denominator, and this is possible if a) we increase the system mass (e.g., adding some ballast) or b) we *decrease* the system stiffness.

3 Generalised Coordinates (flexible systems)

Estimate the natural frequency of vibration of a tower-like structure, characterised by an hollow, annular cross section with constant thickness $t = 0.32\,\mathrm{m}$, mean radius $R = R(x) = \exp(-0.28768\frac{x}{H})3.2\,\mathrm{m}$, height $H = 72\,\mathrm{m}$, Young modulus $E = 30\,\mathrm{GPa}$, density $\rho = 2500\,\mathrm{kg\,m^{-3}}$



3.1 Solution

A suitable shape function, that will be used in the following, is $\psi(x) = 1 - \cos\left(\frac{\pi x}{2H}\right)$.

The computations are presented in form of a tableau, the last two columns are the integrands in the estimates of $m^\star$ and $k^\star$, respectively, below these columns the summations according to the trapezoidal and Simpson's rules.

x	$R(x)$	$A(x)$	$J(x)$	$\psi(x)$	$\psi''(x)$	$\rho A \psi^2$	$EJ\psi''^2$
0	3.2000	6.4340	33.024	0.0000	$4.7596e-4$	$0.0000e+0$	$2.2444e+5$
9	3.0870	6.2067	29.653	0.0192	$4.6682e-4$	$5.7289e+0$	$1.9386e+5$
18	2.9779	5.9875	26.626	0.0761	$4.3973e-4$	$8.6734e+1$	$1.5445e+5$
27	2.8728	5.7760	23.908	0.1685	$3.9575e-4$	$4.1013e+2$	$1.1233e+5$
36	2.7713	5.5720	21.468	0.2928	$3.3656e-4$	$1.1950e+3$	$7.2951e+4$
45	2.6734	5.3752	19.277	0.4444	$2.6443e-4$	$2.6542e+3$	$4.0438e+4$
54	2.5790	5.1853	17.310	0.6173	$1.8214e-4$	$4.9401e+3$	$1.7229e+4$
63	2.4780	4.9822	15.360	0.8263	$8.2650e-5$	$8.5054e+3$	$3.1478e+3$
72	2.4000	4.8255	13.959	1.0000	$0.0000e+0$	$1.2064e+4$	$0.0000e+0$
					trap	$2.1446e+5$	$6.3597e+6$
					simp	$2.1243e+5$	$6.3384e+6$

The required natural frequency is, using trapezoidal rule, $f = 0.8667\,\mathrm{Hz}$ or, using Simpson's rule, $f = 0.8694\,\mathrm{Hz}$.

4 Numerical Integration

A single degree of freedom system, with a mass $m = 120\,\mathrm{kg}$, a stiffness $k = 125\,\mathrm{kN\,m^{-1}}$ and a damping ratio $\zeta = 0.12$ is at rest when it is subjected to an external force $p(t)$:

$$p(t) = \begin{cases} (4000(at)^3 - 1280(at)^2 + 96at)\mathrm{kN} & \text{for } 0.0 \leqslant t \leqslant 0.20\mathrm{s}, \\ 0.0 & \text{otherwise,} \end{cases}$$

where $a = 1\,\mathrm{s^{-1}}$.

1. Find the exact response in the time interval $0 \leqslant t \leqslant 0.5\,\mathrm{s}$, using superposition of the general integral and an appropriate particular solution.
2. Integrate the equation of motion numerically using the algorithm of constant acceleration, with two different integration steps $h_1 = 0.02\mathrm{s}$ and $h_2 = 0.005\mathrm{s}$, in the same time interval.
3. As above, this time using the linear acceleration algorithm.
4. Plot your results in a meaningful manner and comment your results.

4.1 Solution

The forcing function is

$$f(t) = f_0(at)^3 + f_1(at)^2 + f_2(at)^1 + f_3(at)^0,$$

we can write the particular integral in the form

$$\xi(t) = x_0(at)^3 + x_1(at)^2 + x_2(at)^1 + x_3(at)^0,$$

that, after substitution in the equation of motion, gives the equation

$$(kx_0)(at)^3 + (kx_1 + 3cx_0)(at)^2 + (kx_2 + 2cx_1 + 6mx_0)at + (kx_3 + cx_2 + 2mx_1) = f(t) = \cdots.$$

The above equation must hold for every t, $0 \leqslant t \leqslant 0.2\,\mathrm{s}$, so imposing that the coefficients of different powers of t are equal on both sides of the equation we have a system of linear equations that can be conveniently written as follows:

$$\begin{cases} x_0 = f_0/k \\ x_1 = (f_1 - 3c * x_0)/k \\ x_2 = (f_2 - 2c * x_1 - 6mx_0)/k \\ x_3 = (f_3 - cx_2 - 2mx_1)/k \end{cases}.$$

The natural frequency of our system is $\omega = \sqrt{k/m} = 32.275\,\mathrm{rad\,s^{-1}}$ and the damping coefficient is hence $c = 2\zeta\omega m = 929.516\,\mathrm{N\,s\,m^{-1}}$, that substituted in the above equations gives

$$\xi(t) = 32.0(at)^3 - 10.9539(at)^2 + 0.746589(at) + 0.0154797.$$

With $\omega_D = \omega\sqrt{1-\zeta^2} = 32.042\,\mathrm{rad\,s^{-1}}$, the displacement and the velocity can be written as

$$x(t) = \exp(-\zeta\omega t)\,(A\cos\omega_D t + B\sin\omega_D t) + \xi(t)$$
$$\dot{x}(t) = \exp(-\zeta\omega t)\,[\omega_D\,(B\cos\omega_D - A\sin\omega_D t) - \zeta\omega\,(A\cos\omega_D t + B\sin\omega_D t)] + \dot{\xi}(t)$$

Imposing initial rest conditions, it is

$$A = -\xi(0) = 0.015\,479\,7\,\mathrm{m}$$
$$B = \frac{\zeta\omega A - \dot{\xi}(0)}{\omega_D} = -0.025\,171\,66\,\mathrm{m}.$$

The displacement and velocity at the end of the forced response are

$$x(0.2) = -0.025\,883\,96\,\mathrm{m}$$
$$\dot{x}(0.20) = -0.102\,221\,6\,\mathrm{m\,s^{-1}}$$

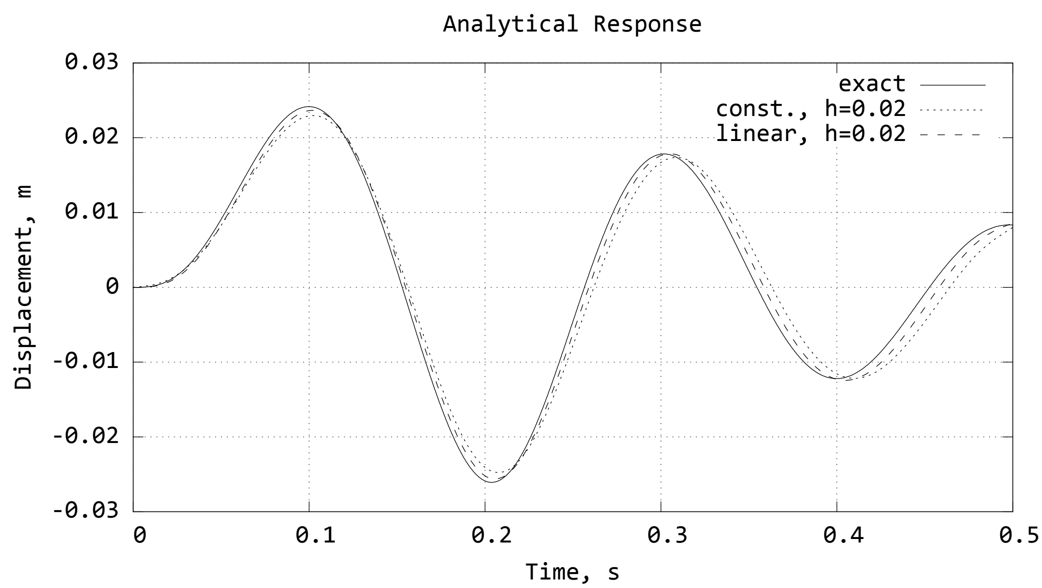
so that, using a shifted time coordinate $\tau = t - 0.20$ and imposing the above initial conditions the response in the free phase is

$$x(\tau) = \exp(-\zeta\omega\tau)\,(A\cos\omega_D\tau + B\sin\omega_D\tau)$$

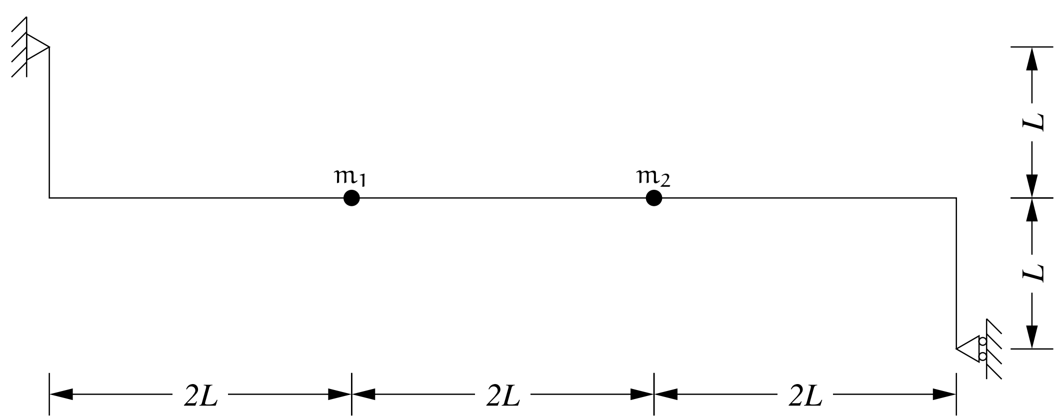
with

$$A = -0.025\,883\,96\,\mathrm{m}$$
$$B = -0.006\,318\,959\,\mathrm{m}$$

These results are summarised in the plot below



5 3 *DOF* System



An uniform, simply supported beam sustains two equal point masses, the beam mass being negligible with respect to the sustained mass. Consider negligible also the axial and shear deformations of the beam.

1. Write the structural matrices, find the eigenvalues solving the determinantal equation, find the eigenvectors of the system.

2. Examine the behaviour of the system when its supports are subjected to a horizontal earthquake excitation, $\ddot{u}_g = \ddot{u}_g(t)$.

5.1 Solution

First of all, the first and second degrees of freedom correspond to the vertical displacements of the masses, while the third degree of freedom is the common horizontal displacement of both masses.

We'll examine a short python program that can be used to compute the answers required.

The program begins with two `import` statements, the first imports all the names from the base part of the scipy *module*, the second imports the name `eigh` (that is a solver for generalised eigenproblems concerning hermitian matrices) from the linalg submodule.

```
from scipy import *
from scipy.linalg import eigh
```

then we define an utility function that given *a, b*, two polynomials (more on this later), and a length *l*, computes the definite integral $\int_0^l a(x) \times b(x)\,dx$

```
def integ(a, b, l):
    integral = polyint(a*b)
    return integral(l) -integral(0)
```

The flexibility matrix can be computed by integration, we observe that the bending moment, for each load condition, can be described by linear polynomials over 5 rectilinear pieces of the structure, top left, middle left, middle centre, middle right and bottom right.

Scipy has a function that defines polynomials of a single variable, `poly1d`, we abbreviate its name and then define a table as a list of three lists (one for each load condition), whose elements are five polynomials that describe the bending moments on each of the five segments of the structure. We define also a list `l` that contains the normalised lengths of each segment.

```
p = poly1d
m = [[p([1, 0]), p([ 1, -1]), p([0, 1]), p([0, 1]), p([1, 0])],
     [p([2, 0]), p([ 1, -2]), p([1, 0]), p([0, 2]), p([2, 0])],
     [p([0.5,0]), p([-0.5]), p([-0.5]), p([-0.5]), p([-0.5,0])]]
l = [1,2,2,2,1]
```

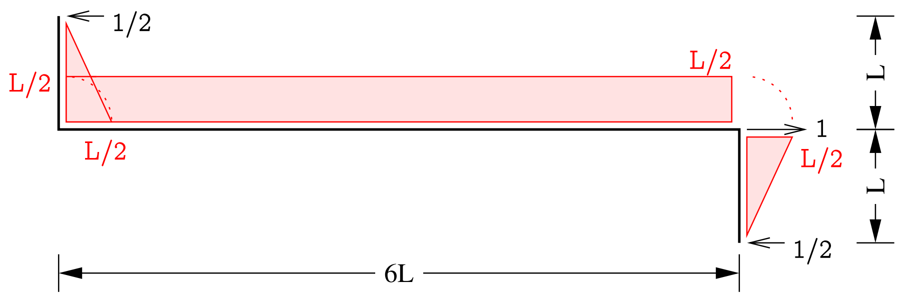
We proceed by creating a 3×3, void flexibility matrix, `flex`, and then we put into each position, `i,j` the sum of the five integrals, computed using `integ` and the moment for load case number `i` end the moment for load case `j`.

```
flex = matrix(zeros((3,3)))
for i in (0, 1, 2):
    for j in (0, 1, 2):
        flex[i,j] = sum(
          [integ(a, b, lg) for a, b, lg in zip(m[i], m[j], l)])
```

The result is

$$\mathbf{F} = \frac{L^3}{3EJ}\begin{bmatrix} 16 & 24 & -6 \\ 24 & 48 & -6 \\ -6 & -6 & 5 \end{bmatrix}$$

As a check, let's compute the $f_{3,3}$ term of the flexibility matrix. With reference to the following figure, where the bending moment due to application of a unit force directed as x_3 is depicted,



the work done by the unit force is

$$\mathbf{1} f_{3,3} = 2\int_0^L \frac{\mathbf{1}}{2} x \frac{x}{2EJ}\,dx + \int_0^{6L} \frac{\mathbf{1}L}{2}\frac{L}{2EJ}\,dx$$

simplifying,

$$f_{3,3} = 2\left[\frac{x^3}{12EJ}\right]_0^L + \left[\frac{L^2 x}{4EJ}\right]_0^{6L} = \left(\frac{2}{12}+\frac{6}{4}\right)\frac{L^3}{EJ} = \frac{2+18}{12}\frac{L^3}{EJ} = \frac{5L^3}{3EJ}.$$

Having the flexibility matrix the stiffness matrix is its inverse, while the mass matrix is a diagonal matrix with $m_3 = 2m$ because the motion in x_3 direction involves both the supported masses.

```
K = flex.I
M = matrix("1. 0. 0. ; 0. 1. 0.; 0. 0. 2.")
```

The stiffness matrix is

$$\mathbf{K} = \frac{EJ}{32L^3}\begin{bmatrix}+51 & -21 & +36\\ -21 & +11 & -12\\ +36 & -12 & +48\end{bmatrix}$$

The last part of the program computes the eigenvalues and the eigenvectors, the *starred* matrices and the modal load vector, the last one is computed taking into account that

$$\mathbf{p}(t) = \mathbf{M}\begin{Bmatrix}0\\0\\1\end{Bmatrix}\ddot{u}_g(t)$$

```
evals, evecs = eigh(K,M)
evecs = matrix(evecs)
M_star = evecs.T*M*evecs, ","
K_star = evecs.T*K*evecs, "."
L = evecs.T*M*matrix("0;0;1")
```

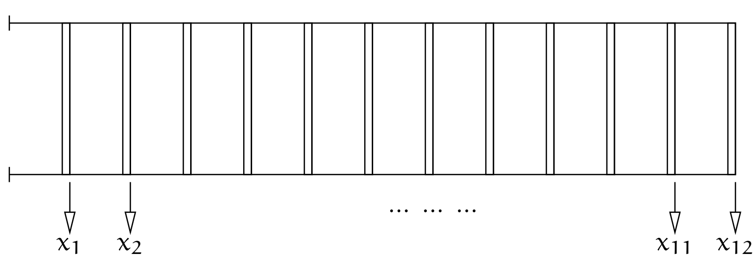
In the following, the results (starred matrices omitted, as the eigenvectors are orthonormal)

$$\omega_i^2 = \frac{EJ}{mL^3}\left\{0.0474 \quad 0.3198 \quad 2.3203\right\}$$

$$\boldsymbol{\Psi} = \begin{bmatrix}-0.4719 & 0.3319 & -0.8168\\ -0.8559 & -0.3945 & 0.3343\\ 0.1494 & -0.6059 & -0.3325\end{bmatrix}$$

$$\mathbf{p}^\star(t) = \begin{bmatrix}0.2988\\ -1.2118\\ -0.6650\end{bmatrix}\ddot{u}_g(t)$$

6 Rayleigh-Ritz & Subspace Iteration



The structure above can be analysed as a shear type building. The mass matrix is diagonal, with storey masses being all equal to m. The storey stiffnesses are linearly decreasing, $k_1 = 23k$, $k_2 = 22k$, ..., $k_{11} = 13k$ and $k_{12} = 12k$. For example, the stiffness matrix element $k_{9,9}$ is given by $k_{9,9} = k_9 + k_{10} = 29k$.

1. Find the lower four eigenvalues and eigenvectors of the structure[1] using the Rayleigh-Ritz procedure, denoting the Ritz base with $\hat{\mathbf{\Phi}}_0$ and the Ritz coordinates eigenvector matrix with $\mathbf{Z}$.

2. Do one subspace iteration, deriving a new set of Ritz base vectors,
$$\hat{\mathbf{\Phi}}_1 = \mathbf{K}^{-1}\mathbf{M}\mathbf{\Phi}\mathbf{Z}_0.$$

3. Find the lower four eigenvalues and eigenvectors of the structure using the Rayleigh-Ritz procedure with the Ritz base $\hat{\mathbf{\Phi}}_1$.

4. Discuss the two set of results.

6.1 Solution

The mass matrix is
$$\mathbf{M} = m\mathbf{I}$$
where $\mathbf{I}$ is a 12 by 12 unit matrix.

The stiffness matrix is given by
$$\mathbf{K} = k\begin{bmatrix}
45 & -22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-22 & 43 & -21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -21 & 41 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -20 & 39 & -19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -19 & 37 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -18 & 35 & -17 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -17 & 33 & -16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -16 & 31 & -15 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -15 & 29 & -14 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14 & 27 & -13 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -13 & 25 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 12
\end{bmatrix}$$

[1]The eigenvectors ψ_i of the structure are different from the eigenvectors in Ritz coordinates.

The first four eigenpairs have been numerically computed, within very good approximation, by a LINPACK subroutine, here the exact eigenvalues

$$0.3052 \quad 2.3834 \quad 6.3998 \quad 12.0934$$

and the corresponding exact eigenvalues, collected in the matrix $\mathbf{\Psi}$,

$$\mathbf{\Psi} = \begin{bmatrix} 0.0413 & -0.1171 & 0.1870 & -0.2465 \\ 0.0840 & -0.2267 & 0.3281 & -0.3687 \\ 0.1275 & -0.3159 & 0.3760 & -0.2844 \\ 0.1712 & -0.3719 & 0.3059 & -0.0239 \\ 0.2144 & -0.3842 & 0.1291 & 0.2655 \\ 0.2564 & -0.3463 & -0.1034 & 0.3926 \\ 0.2963 & -0.2576 & -0.3107 & 0.2479 \\ 0.3330 & -0.1250 & -0.4066 & -0.0932 \\ 0.3654 & 0.0363 & -0.3355 & -0.3819 \\ 0.3921 & 0.2029 & -0.1059 & -0.3613 \\ 0.4117 & 0.3452 & 0.1935 & -0.0030 \\ 0.4225 & 0.4308 & 0.4146 & 0.3882 \end{bmatrix} .$$

In the first Ritz iteration, I choose the following base vectors,

$$\mathbf{\Phi}_0 = \begin{bmatrix} 0.1000 & 0.1000 & 0.1000 & 0.1000 \\ 0.2000 & 0.2000 & 0.3000 & 0.3000 \\ 0.3000 & 0.3000 & 0.4000 & 0.1000 \\ 0.4000 & 0.4000 & 0.3000 & -0.1000 \\ 0.5000 & 0.4000 & 0.1000 & -0.3000 \\ 0.6000 & 0.3000 & -0.1000 & -0.3000 \\ 0.7000 & 0.2000 & -0.3000 & -0.1000 \\ 0.8000 & 0.1000 & -0.3000 & 0.1000 \\ 0.9000 & 0.0000 & -0.1000 & 0.3000 \\ 1.0000 & -0.1000 & 0.1000 & 0.1000 \\ 1.1000 & -0.2000 & 0.2000 & -0.1000 \\ 1.2000 & -0.3000 & 0.4000 & -0.4000 \end{bmatrix} .$$

while in two subsequent iterations, using the subspace method, the base vectors were given by

$$\Phi_1 = \begin{bmatrix} 0.1358 & -0.0478 & 0.0248 & 0.0195 \\ 0.2758 & -0.0930 & 0.0468 & 0.0323 \\ 0.4182 & -0.1307 & 0.0570 & 0.0222 \\ 0.5613 & -0.1557 & 0.0468 & -0.0016 \\ 0.7029 & -0.1604 & 0.0179 & -0.0253 \\ 0.8403 & -0.1421 & -0.0206 & -0.0315 \\ 0.9708 & -0.1041 & -0.0556 & -0.0153 \\ 1.0909 & -0.0495 & -0.0687 & 0.0123 \\ 1.1967 & 0.0163 & -0.0531 & 0.0347 \\ 1.2840 & 0.0851 & -0.0169 & 0.0276 \\ 1.3485 & 0.1458 & 0.0234 & 0.0021 \\ 1.3842 & 0.1852 & 0.0567 & -0.0257 \end{bmatrix}, \quad \Phi 2 = \begin{bmatrix} 0.1355 & 0.0489 & 0.0284 & 0.0212 \\ 0.2752 & 0.0949 & 0.0506 & 0.0320 \\ 0.4176 & 0.1324 & 0.0591 & 0.0236 \\ 0.5608 & 0.1561 & 0.0488 & 0.0002 \\ 0.7025 & 0.1613 & 0.0210 & -0.0241 \\ 0.8401 & 0.1453 & -0.0162 & -0.0327 \\ 0.9707 & 0.1079 & -0.0495 & -0.0185 \\ 1.0910 & 0.0524 & -0.0641 & 0.0097 \\ 1.1971 & -0.0151 & -0.0516 & 0.0318 \\ 1.2847 & -0.0849 & -0.0154 & 0.0274 \\ 1.3489 & -0.1448 & 0.0299 & -0.0007 \\ 1.3841 & -0.1811 & 0.0637 & -0.0302 \end{bmatrix}.$$

In the next display, the eigenvalues convergence, in the left column the exact results, in the rightmost the best estimates

exact	0	1	2
0.3052	0.3082	0.3052	0.3052
2.3834	2.5844	2.3855	2.3834
6.3998	7.2876	6.4485	6.4038
12.0934	13.9498	12.2310	12.1252

In the next display the first eigenvector convergence is shown, as you can see excellent approximation is rapidly achieved for this mode.

exact	0	1	2
0.0413	0.0450	0.0414	0.0414
0.0840	0.0885	0.0841	0.0840
0.1275	0.1274	0.1275	0.1275
0.1712	0.1733	0.1711	0.1712
0.2144	0.2156	0.2144	0.2144
0.2564	0.2549	0.2564	0.2564
0.2963	0.2982	0.2963	0.2963
0.3330	0.3344	0.3331	0.3330
0.3654	0.3637	0.3654	0.3654
0.3921	0.3848	0.3920	0.3921
0.4117	0.4095	0.4117	0.4117
0.4225	0.4286	0.4225	0.4225