An Introduction to Dynamics of Structures

Giacomo Boffi

Dipartimento di Ingegneria Strutturale, Politecnico di Milano

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Outline

Part I

Introduction

Our aim is to develop some analytical and numerical methods for the analysis of the stresses and deflections that the application of a time varying set of loads induces in a generic structure that moves in a neighborhood of a point of equilibrium.
We will see that these methods are extensions of the methods of standard static analysis, or to say it better, that static analysis is a special case of *dynamic analysis*.
If we restrict ourselves to analysis of *linear systems*, however, it is so convenient to use the principle of superposition to study the combined effects of static and dynamic loadings that different methods, of different character, are applied to these different loadings.

Definitions

Dynamic: something that varies over time

Dynamic Loading: a Loading that varies over time

Dynamic Response: the Response of a structural system to a dynamic loading, expressed in terms of stresses and/or deflections

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Types of Dynamic Analysis

Taking into account linear systems only, we must consider two different definitions of the loading to define two types of dynamic analysis.

Deterministic Analysis the time variation of the loading is fully known, and we can determine the complete time variation of all the response quantities that are required in our analysis.

Non-deterministic Analysis when the time variation of the loading is essentially random and is known *only* in terms of some *statistics*, also the structural response can be known only in terms of some statistics of the response quantities.

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Types of Dynamic Loadings

Dealing with deterministic loadings, we will study, in order of complexity,

- **Harmonic Load** the load is modulated by a harmonic function, characterized by a frequency and a phase, $p(t) = p_0 \sin(\omega t - \varphi)$
- **Periodic Load** the load repeat itself with a fixed period T, $p(t) = p(t + T)$
- **Non Periodic Load** here we see two sub-cases,

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- the load is described in terms of analytic functions, $p(t) = p_0 f(t)$,
- the load is experimentally measured, and is known only in a discrete set of instants; in this case, we say that we have a *time-history*.

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Characteristics of a Dynamical Problem

As both load and response vary over time, our methods of analysis have to provide the dynamical problem solution for every instant in the response.

More fundamentally, a dynamical problem is characterized by the relevance of *inertial forces*, arising from the motion of structural or serviced masses.

A dynamic analysis is *required* only when the inertial forces represent a significant portion of the total load, otherwise a static analysis will suffice, even if the loads are (slowly) varying over time.

Formulation of a Dynamical Problem

The inertial forces depend on deflections, the deflections depend also on inertial forces, we have a loop and our line of attack is of course to have a statement of the problem in terms of *differential equations*.

If the mass is distributed along the structure, also the inertial forces are distributed and the formulation of our problem must be in terms of partial differential equations, to take into account the spatial variations of both loading and response.

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Formulation of a Dynamical Problem

On the other hand, If we can assume, maybe at the price of some approximation in our model, that the masses are concentrated in a discrete set of *lumped masses*, the analytical problem is greatly simplified because

- the inertial forces are applied only at the lumped masses, and
- the deflections, velocities and accelerations need to be computed at these points only,

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Dynamic Degrees of Freedom

The *dynamic degrees of freedom* (DDOF) in a discretized system are the displacements components associated with the significant inertial forces, in correspondance with the lumped masses.
If the lumped mass can be considered dimensionless, then 3 DDOFs will suffice to represent the associated inertial force.
If the lumped mass must be considered with finite dimensions, then we have also inertial couples, and we need 6 DDOFs to represent the inertial force.
Of course, a continuous system has an infinite number of degrees of freedom.

Generalized Displacements

The lumped mass procedure that we have outlined is effective if a large proportion of the total mass is concentrated in a few points.
A primary example is a multistorey building, where one can consider a lumped mass in correspondence of each storey.
When the mass is distributed, we can simplify our problem using *generalized coordinates*. The deflections are expressed in terms of a linear combination of assigned functions of position, with the coefficients of the linear combination being the generalized coordinates. E.g., the deflectlions of a rectilinear beam can be expressed with a trigonometric series.

Generalized Displacements, cont.

To fully describe a displacement field, we need to combine an infinity of linearly indipendent *base functions*, but in practice a good approximation can be achieved using only a small number of functions and degrees of freedom.
Even if the method of generalized coordinates has its beauty, we must recognise that for each different problem we should derive an *ad hoc* formulation, without generality.

Finite Element Method

The *finite elements method* (FEM) combines aspects of lumped mass and generalized coordinates methods, providing a simple and reliable method of analysis, that can be easily programmed on a digital computer.
In the FEM, the structure is subdivided in a number of non overlapping pieces, called the *finite elements*, delimited by *nodes*.
The FEM uses *piecewise approximations* to the displacement field, derived from the displacements of the *nodes* that surround each particular element using *interpolating functions*, so that the displacement, deformation and stress field in each element can be expressed in terms of the unkown *nodal displacements*, that will be the dynamical DOFs.

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Finite Element Method, cont.

Main advanaeges of the FEM are

1. For each type of finite element, the procedure for deriving the displacement field is the same for every particular geometry and can be implemented in a computer program.
2. The desired level of approximation can be achieved by further subdiving the structure.
3. The resulting equations are only loosely coupled, leading to an easier computer solution.

Writing the eq. of motion

Introduction

Characteristics of a Dynamical Problem

Formulation of a Dynamical Problem

Formulation of the equation of motion

In a deterministic dynamic analysis, given a prescribed load, we want to evaluate the displacements in each instant of time.
In most cases, a limited number of DDOFs gives a sufficient accuracy, and in general the dynamical problem can be reduced to the determination of the time-histories of some selected component of displacements,
The mathematical expression that define the dynamic displacements is known as the *Equations of Motion* (EOM), a partial differential equation or a set of differential equations; the solution of the EOM gives the requested displacements.
The formulation of the EOM is the most important, often the most difficult part of our task of dynamic analysts.

Writing the EOM, cont.

We have a choice of techniques to help us in writing the EOM, namely:

- the D'Alembert Principle,
- the Principle of Virtual Displacements,
- the Variational Approach.

D'Alembert principle

By Newton's II law of motion, for any particle the rate of change of momentum is equal to the external force,

$$\vec{p}(t) = \frac{d}{dt}(m\frac{d\vec{u}}{dt}),$$

where $\vec{u}(t)$ is the particle displacent.
In structural dynamics, we may regard the mass as a constant, and thus write

$$\vec{p}(t) = m\ddot{\vec{u}},$$

where each operation of differentiation with respect to time is denoted with a dot.
If we write

$$\vec{p}(t) - m\ddot{\vec{u}} = 0$$

and interpret the term $-m\ddot{\vec{u}}$ as an *Inertial Force* that contrasts the acceleration of the particle, we have an equation of equilibrium for the particle.

D'Alembert principle, cont.

The concept that a mass developes an inertial force opposing its acceleration is known as the D'Alembert principle, and using this principle we can write the *eom* as a simple equation of equilibrium.
The term $\vec{p}(t)$ must comprise each different force acting on the particle, including the reactions of kinematic or elastic constraints opposing displacement, viscous forces and external, autonomous forces.
In many simple problems, the D'Alembert principle is the most direct and convenient method for the formulation of the *eom*.

Principle of virtual displacements

In a reasonably complex dynamic system (with articulated rigid bodies and external/internal constraints) the direct formulation of the *eom*, using D'Alembert principle, may result difficult.
However, in many cases the various forces acting on the system may be simply expressed in terms of the *ddof*, even if the equilibrium relationship between these forces may be difficult to express..
In these cases, application of the *Principle of Virtual Displacements* is very convenient.

Principle of Virtual Displacements, cont.

For example, considering an assemblage of rigid bodies, the *pvd* states that necessary and sufficient condition for equilibrium is that, for every *virtual displacement* (any infinitesimal displacement compatible with the restraints) the total work done by all the external forces is zero.
For an assemblage of rigid bodies, writing the *eom* requires

1. to identify all the external forces, comprising the inertial forces, and to express their values in terms of the *ddof*;
2. to compute the work done by these forces for different virtual displacements, one for each *ddof*;
3. to equate to zero all these work expressions.

The *pvd* is particularly convenient because we have only scalar equations, even if the forces and displacements are of vectorial nature.

Variational approach

Variational approaches do not consider directly the forces acting on the dynamic system, but rather are concerned with the variations of kinetic and potential energy, and lead, as well as the *pvd*, to a set of scalar equations.
The method to be used in a particular problem is mainly a matter of convenience and also of personal taste.

Part II

Single Degree of Freedom System

1 DOF System

Structural dynamics is all about a motion in the neighbourhood of a point of equilibrium.
We'll start by studying a generic single degree of freedom system, with *constant* mass m, subjected to a non-linear generic force $F = F(y, \dot{y})$, where y is the displacement and $\dot{y}$ the velocity of the particle. The equation of motion is

$$\ddot{y} = \frac{1}{m} F(y, \dot{y}) = f(y, \dot{y})$$

.
It is difficult to integrate the above equation in the general case, but it's easy when the motion occurs in a small neighbourhood of the equilibrium position.

1 DOF System, cont.

In a position of equilibrium, $y_{\text{eq.}}$, the velocity and the acceleration are zero, and hence $f(y_{\text{eq.}}, 0) = 0$.
The force can be linearized in a neighbourhood of $y_{\text{eq.}}$, 0:

$$f(y, \dot{y}) = f(y_{\text{eq.}}, 0) + \frac{\partial f}{\partial y}(y - y_{\text{eq.}}) + \frac{\partial f}{\partial \dot{y}}(\dot{y} - 0) + O(y, \dot{y}).$$

Assuming that $O(y, \dot{y})$ is small in a neighborhood of $y_{\text{eq.}}$, we can write the equation of motion

$$\ddot{x} + a\dot{x} + bx = 0$$

where $x = y - y_{\text{eq.}}$, $a = -\frac{\partial f}{\partial \dot{y}}$ and $b = -\frac{\partial f}{\partial y}$.
In an infinitesimal neighborhood of $y_{\text{eq.}}$, the equation of motion can be studied in terms of a linear differential equation of second order.

1 DOF System, cont.

A linear constant coefficient differential equation has the integral $x = A\exp(st)$, that substituted in the equation of motion gives

$$s^2 + as + b = 0$$

whose solutions are

$$s_{1,2} = -\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b}.$$

The general integral is

$$x(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t).$$

Given that for a free vibration problem $A_1,\ A_2$ are given by the initial conditions, the nature of the solution depends on the sign of the real part of s_1, s_2.

1 DOF System, cont.

If we write $s_i = r_i + \imath q_i$, then we have

$$\exp(s_i t) = \exp(\imath q_i t)\exp(r_i t).$$

If one of the $r_i > 0$, the response grows infinitely over time, even for an infinitesimal perturbation of the equilibrium, so that in this case we have an *unstable equilibrium*.
If both $r_i < 0$, the response decrease over time, so we have a *stable equilibrium*.
Finally, if both $r_i = 0$ the s's are imaginary, the response is harmonic with constant amplitude.

1 DOF System, cont.

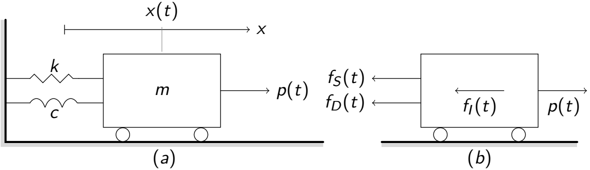
If $a > 0$ and $b > 0$, both roots are negative or complex conjugate with negative real part, the system is asympotically stable.
If $a = 0$ and $b > 0$, the roots are purely imaginary, the equilibrium is indifferent, the oscillations are harmonic.
If $a < 0$ or $b < 0$ at least one of the roots has a positive real part, and the system is unstable.

The basic dynamic system

A linear system is characterized by its mass distribution,, its elastic properties and its energy-loss mechanism.

In a single degree of freedom (*sdof*) system each property can be conveniently represented in a single physical element

- The entire mass, m, is concentrated in a rigid block, its position completely described by the coordinate $x(t)$.
- The elastic resistance to displacement is provided by a massless spring of stiffness k
- The energy loss is represented by a massless damper, its damping constant being c.
- Finally, the external loading is the time-varying force $p(t)$.



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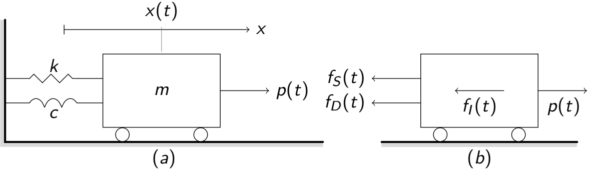

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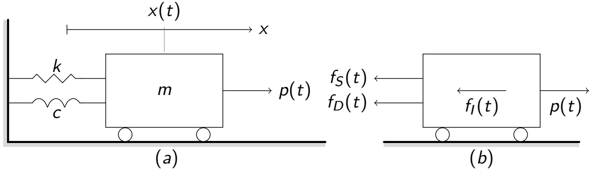

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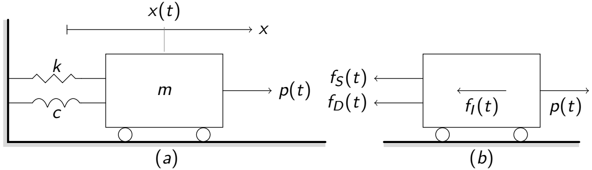

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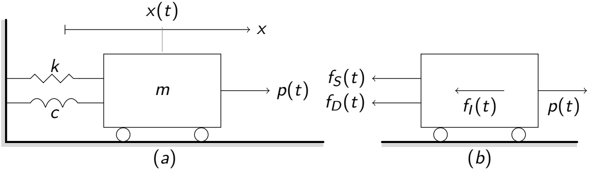

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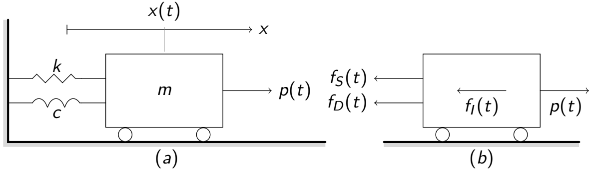
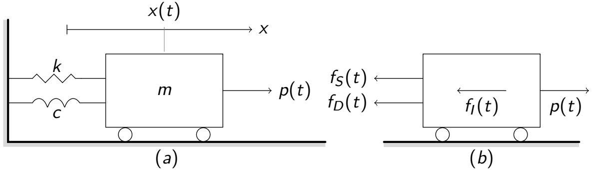


Equation of motion of the basic dynamic system



The equation of motion can be written using the D'Alembert Principle, expressing the equilibrium of all the forces acting on the mass including the inertial force.

The forces, positive if acting in the direction of the motion, are the external force, $p(t)$, and the resisting forces due to motion, i.e., the inertial force $f_\mathrm{I}(t)$, the damping force $f_\mathrm{D}(t)$ and the elastic force, $f_\mathrm{S}(t)$.

The equation of motion, merely expressing the equilibrium of these forces, is

$$f_\mathrm{I}(t) + f_\mathrm{D}(t) + f_\mathrm{S}(t) = p(t)$$

EOM of the basic dynamic system, cont.

The resisting forces in

$$f_{\mathrm{I}}(t) + f_{\mathrm{D}}(t) + f_{\mathrm{S}}(t) = p(t)$$

are functions of the displacement $x(t)$ or of one of its derivatives.
Note that the positive sense of these forces is opposite to the direction of motion.

EOM of the basic dynamic system, cont.

In accordance to D'Alembert principle, the inertial force is the product of the mass and acceleration

$$f_\mathrm{I}(t) = m\,\ddot{x}(t).$$

Assuming a viscous damping mechanism, the damping force is the product of the damping constant c and the velocity,

$$f_\mathrm{D}(t) = c\,\dot{x}(t).$$

Finally, the elastic force is the product of the elastic stiffness k and the displacement,

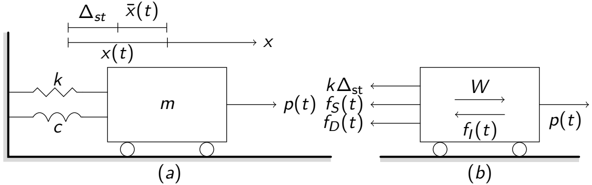
$$f_\mathrm{S}(t) = k\,x(t).$$

The differential equation of dynamic equilibrium

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t).$$

Note that this differential equation is a linear differential equation of the second order, with constant coefficients.

Influence of static forces



Considering the presence of a constant force, let's say W, the equation of motion is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t) + W,$$

but expressing the total displacement as the sum of a constant, static displacement and a dynamic displacement,

$$x(t) = \Delta_{\mathrm{st}} + \bar{x}(t),$$

substituting in we have

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,\Delta_{\mathrm{st}} + k\,\bar{x}(t) = p(t) + W.$$

Influence of static forces, cont.

Recognizing that $k\,\Delta_{\mathrm{st}} = W$ (so that the two terms, on opposite sides of the equal sign, cancel each other), that $\dot{x} = \dot{\bar{x}}$ and that $\ddot{x} = \ddot{\bar{x}}$ the *eom* is now

$$m\,\ddot{\bar{x}}(t) + c\,\dot{\bar{x}}(t) + k\,\bar{x}(t) = p(t)$$

The equation of motion expressed with reference to the static equilibrium position is not affected by static forces.
For this reasons, all displacements in further discussions will be referenced from the equibrium position and denoted, for simplicity, with $x(t)$.
Note that the *total* displacements, stresses. etc. *are influenced* by the static forces, and must be computed using the superposition of effects.

Influence of static forces, cont.

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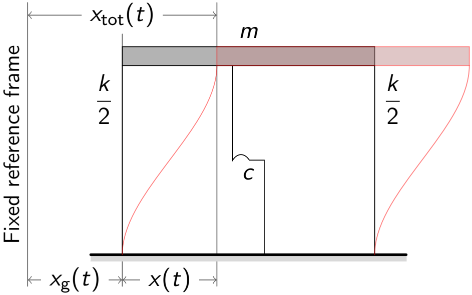
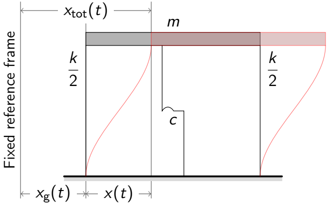
Influence of support motion



Displacements, deformations and stresses in a structure are induced also by a motion of its support.
Important examples of support motion are the motion of a building foundation due to earthquake and the motion of the base of a piece of equipment due to vibrations of the building in which it is housed.

Influence of support motion, cont.



Considering a support motion $x_g(t)$, defined with respect to a inertial frame of reference, the total displacement is

$$x_{tot}(t) = x_g(t) + x(t)$$

and the total acceleration is

$$\ddot{x}_{tot}(t) = \ddot{x}_g(t) + \ddot{x}(t).$$

While the elastic and damping forces are still proportional to *relative* displacements and velocities, the inertial force is proportional to the total acceleration,

$$f_I(t) = -m\ddot{x}_{tot}(t) = m\ddot{x}_g(t) + m\ddot{x}(t).$$

Writing the *eom* for a null external load, $p(t) = 0$, is hence

$$m\,\ddot{x}_{tot}(t) + c\,\dot{x}(t) + k\,x(t) = 0, \qquad \text{or,}$$
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Support motion is sufficient to excite a dynamic system: $p_{eff}(t) = -m\,\ddot{x}_g(t)$.

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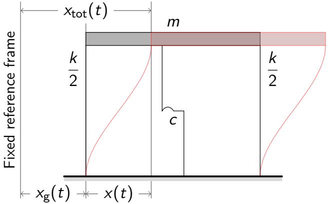

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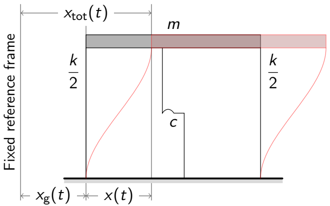

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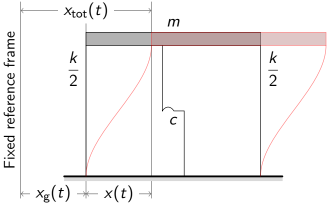

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Free vibrations

The equation of motion,

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t),$$

is a linear differential equation of the second order, with constant coefficients.
Prior to studying the response of a linear system to different types of dynamic loading, it is convenient to study its response in absence of a loading, $p(t) \equiv 0$, to find the so-called *homogeneous* or *complementary* solutions.

Undamped

An undamped system, where $c = 0$ and no energy dissipation takes place, is just an ideal notion, as it would be a realization of *motus perpetuum*. Nevertheless, it is an useful idealization.

In this case, the homogeneous equation of motion is

$$m\,\ddot{x}(t) + k\,x(t) = 0$$

which solution is of the form $\exp st$; substituting this solution in the above equation we have

$$(k + s^2 m)\,\exp st = 0$$

noting that $\exp st \neq 0$, we finally have

$$(k + s^2 m) = 0 \Rightarrow s = \pm\sqrt{-\frac{k}{m}}$$

As m and k are positive quantities, s must be imaginary.

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Undamped, cont.

Introducing the *natural* circular frequency ω_n

$$\omega_\mathrm{n}^2 = \frac{k}{m}$$

the solution of the algebraic equation in s is

$$s = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_\mathrm{n} \qquad \text{where } i = \sqrt{-1}$$

and the general integral of the homogeneous equation is

$$x(t) = G_1 \exp(i\omega_\mathrm{n} t) + G_2 \exp(-i\omega_\mathrm{n} t)$$

Undamped, cont.

We want a real solution, so we impose real initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0$$

Evaluating $x(t)$ for $t = 0$ and substituting in the general integral it is

$$\begin{cases} G_1 + G_2 & = x_0 \\ i\omega_\mathrm{n} G_1 - i\omega_\mathrm{n} G_2 & = \dot{x}_0 \end{cases}$$

Solving for G_1 and G_2, substituting these values in the homogeneous solution and collecting x_0 and $\dot{x}_0$, we finally find

$$x(t) = \frac{\exp(i\omega_\mathrm{n} t) + \exp(-i\omega_\mathrm{n} t)}{2}\, x_0 + \frac{\exp(i\omega_\mathrm{n} t) - \exp(-i\omega_\mathrm{n} t)}{2i}\, \frac{\dot{x}_0}{\omega_\mathrm{n}},$$

the Euler formulas tell us that above we have the cosine and the sine of $\omega_\mathrm{n} t$

$$x(t) = x_0 \cos(\omega_\mathrm{n} t) + (\dot{x}_o/\omega_\mathrm{n}) \sin(\omega_\mathrm{n} t).$$

Undamped, cont.

In effects, from the last last equation we derive our preferred manner of writing the homogeneous solution

$$x(t) = A\cos(\omega_\mathrm{n} t) + B\sin(\omega_\mathrm{n} t).$$

For the usual initial conditions, we have already seen that

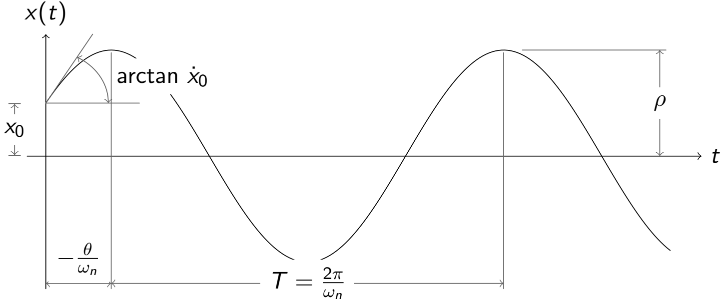
$$\begin{cases} A = x_0 \\ B = \dfrac{\dot{x}_0}{\omega_\mathrm{n}} \end{cases}$$

The homogeneous solution can also be expressed in terms of a single sine or cosine, introducing a phase difference φ, e.g.,

$$x(t) = C\cos(\omega_\mathrm{n} t - \varphi), \qquad \text{with} \begin{cases} C = \sqrt{A^2 + B^2} \\ \varphi = \arctan(B/A) \end{cases}$$

From a programmer's point of view, it is recommended to use the library function `atan2(Y,X)` that returns an angle in the range $(-\pi, +\pi]$.

The undamped free response



Dimensions must always be checked

It is worth noting that the coefficients A, B and C have the dimension of a lenght, the coefficient ω_n has the dimension of the reciprocal of time and that the coefficient φ is an angle, or in other terms is adimensional.

Damped Systems Behaviour

The viscous damping modifies the response of a *sdof* system introducing a decay in the amplitude of the response. Depending on the amount of damping, the response can be oscillatory or not. The amount of damping that separates the two behaviours is denoted as *critical damping*.

Solution of the characteristic equation

The equation of motion for a free vibrating damped system is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = 0,$$

substituting the solution $\exp st$ in the preceding equation and simplifying, we have that the parameter s must satisfy the equation

$$m\,s^2 + c\,s + k = 0$$

or, after dividing both members by m,

$$s^2 + \frac{c}{m}\,s + {\omega_n}^2 = 0$$

whose solutions are

$$s = -\frac{c}{2m} \mp \sqrt{\frac{c^2}{4m^2} - {\omega_n}^2}.$$

Critical Damping

The behaviour of the solution of the free vibration problem depends of course on the sign of the radicand $\frac{c^2}{4m^2} - {\omega_\mathrm{n}}^2$, and the value of c that make the radicand equal to zero is the value of the *critical damping*,

$$c_\mathrm{cr} = 2m\omega_\mathrm{n} = 2\sqrt{mk}.$$

A dynamical *sdof* is hence denoted as *critically* damped, *undercritically* damped or *overcritically* damped depending on the value of the damping coefficient with respect to the values of the mass and of the stiffness.

In the field of structural dynamics we may assume that every system is in effect an undercritically damped system, that is a system where the damping coefficient is smaller than the critical damping.

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Damping Ratio

If we introduce the ratio of the damping to the critical damping, or *critical damping ratio* ζ,

$$\zeta = \frac{c}{c_{\mathrm{cr}}}, \quad c = \zeta c_{\mathrm{cr}} = 2\zeta\omega_{\mathrm{n}} m$$

the solutions of the characteristic equation

$$s = -\frac{c}{2m} \mp \sqrt{\frac{c^2}{4m^2} - {\omega_{\mathrm{n}}}^2}.$$

can be rewritten as

$$s = -\zeta\omega_{\mathrm{n}} \mp \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}$$

and the equation of motion itself can be rewritten as

$$\ddot{x}(t) + 2\zeta\omega_{\mathrm{n}}\dot{x}(t) + {\omega_{\mathrm{n}}}^2 x(t) = 0$$

Undercritically damped SDOF

We start our study of the actual free vibration response of damped *sdof* from undercritically damped ones, as these represent the almost totality of cases in structural dynamics. We start by noting that $\zeta - 1 < 0$, hence it is

$$\omega_{\rm n}\sqrt{\zeta^2 - 1} = i\omega_{\rm n}\sqrt{1-\zeta^2} = i\omega_{\rm D},$$

where we have defined the *damped frequency*

$$\omega_{\rm D} = \omega_{\rm n}\sqrt{1-\zeta^2}.$$

The roots of the canonical equation can now be written

$$s = -\zeta\omega_{\rm n} \mp i\omega_{\rm D}$$

and the general integral of the equation of motion is, collecting the terms in $\exp(-\zeta\omega_{\rm n}t)$

$$x(t) = \exp(-\zeta\omega_{\rm n}t)\left[G_1\exp(-i\omega_{\rm D}t) + G_2\exp(+i\omega_{\rm D}t)\right]$$

Initial Conditions

By imposing the initial conditions, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, after a bit of algebra we can write the equation of motion for the given initial conditions, namely

$$x(t) = \exp(-\zeta\omega_\mathrm{n} t)\left[\frac{\exp(i\omega_\mathrm{D} t) + \exp(-i\omega_\mathrm{D} t)}{2} u_0 + \frac{\exp(i\omega_\mathrm{D} t) - \exp(-i\omega_\mathrm{D} t)}{2i}\,\frac{v_0 + \zeta\omega_\mathrm{n}\, u_0}{\omega_\mathrm{D}}\right].$$

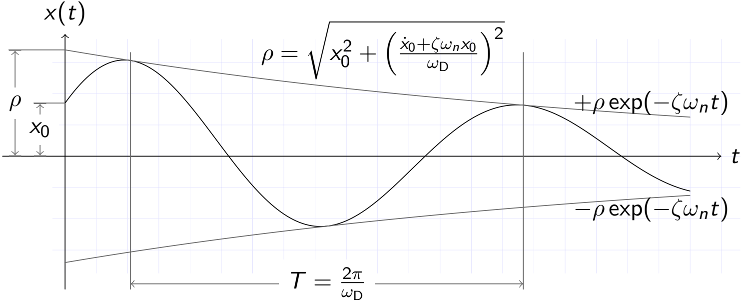
Applying the Euler formulas, we finally have the standard format of the general integral,

$$x(t) = \exp(-\zeta\omega_\mathrm{n} t)\left[A\cos(\omega_\mathrm{D} t) + B\sin(\omega_\mathrm{D} t)\right]$$

where

$$A = x_0, \qquad B = \frac{\dot{x}_0 + \zeta\omega_\mathrm{n}\, x_0}{\omega_\mathrm{D}}.$$

The damped free response



Critically damped *sdof*

In this case, $\zeta = 1$ and we have a double root for $s = -\omega_\mathrm{n}$, so that the general integral must be written in the form

$$x(t) = \exp(-\omega_\mathrm{n} t)(A + Bt).$$

The solution for given initial condition is

$$x(t) = \exp(-\omega_\mathrm{n} t)(u_0 + (v_0 - \omega_\mathrm{n}\, u_0)t),$$

note that, if $v_0 = 0$, the solution asymptotically approaches zero without crossing the zero axis.

Overcritically damped *sdof*

In this case, $\zeta > 1$ and

$$s = -\zeta\omega_\mathrm{n} \mp \omega_\mathrm{n}\sqrt{\zeta^2 - 1} = -\zeta\omega_\mathrm{n} \mp \hat{\omega}$$

where

$$\hat{\omega} = \omega_\mathrm{n}\sqrt{\zeta^2 - 1}$$

and, after some rearrangement, the general integral for the overdamped *sdof* can be written

$$x(t) = \exp(-\zeta\omega_\mathrm{n} t)\left(A\cosh(\hat{\omega}t) + B\sinh(\hat{\omega}t)\right)$$

Note that:

- as $\zeta\omega_\mathrm{n} > \hat{\omega}$, for increasing t the general integral goes to zero, and that
- as for increasing ζ we have that $\hat{\omega} \to \zeta\omega_\mathrm{n}$, the velocity with which the response approaches zero slows down for increasing ζ.

Measuring damping

The true damping characteristic of a typical structural system are complex and very difficult to define.
However, it is customary to express the damping of a real system in terms of equivalent viscous damping.
In practice, we assume in our *model of the real behaviour* a viscous damping dissipation mechanism and determine the amount of damping for which our model performs like the real structure during some kind of testing conditions.

Logarithmic decrement

One of the testing conditions that can be used to measure the equivalent viscous damping is simply a free vibration test. Consider a system in free vibration and two positive peaks, u_q and u_{q+r}, occuring at times $q\,(2\pi/\omega_\mathrm{D})$ and $(q+r)\,(2\pi/\omega_\mathrm{D})$, the ratio of these peaks is, using equation

$$\frac{u_q}{u_{q+r}} = \frac{\exp(-\zeta\omega_\mathrm{n} q 2\pi/\omega_\mathrm{D})}{\exp(-\zeta\omega_\mathrm{n}(q+r)2\pi/\omega_\mathrm{D})} = \exp(2r\pi\zeta\omega_\mathrm{n}/\omega_\mathrm{D})$$

Substituting $\omega_\mathrm{D} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$ and taking the logarithm of both members we obtain

$$\ln\left(\frac{u_q}{u_{q+r}}\right) = 2r\pi\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Logarithmic decrement, cont.

Solving for ζ, we finally get

$$\zeta = \ln\left(\frac{u_q}{u_{q+r}}\right)\left((2r\pi)^2 + \ln\left(\frac{u_q}{u_{q+r}}\right)^2\right)^{-\frac{1}{2}}$$

A simplified solution can be obtained from

$$\frac{u_q}{u_{q+r}} = \exp\frac{2r\pi\zeta}{\sqrt{1-\zeta^2}}$$

when $\zeta << 1$, as it is the case in structural dynamics:

$$\frac{u_q}{u_{q+r}} \approx \exp\frac{2r\pi\zeta}{1} \approx 1 + 2r\pi\zeta + \frac{(2r\pi\zeta)^2}{2!} + \ldots$$

$$\frac{u_q}{u_{q+r}} \approx 1 + 2r\pi\zeta \Rightarrow \zeta = \frac{x_q - x_{q+r}}{2r\pi\, x_{q+r}}$$

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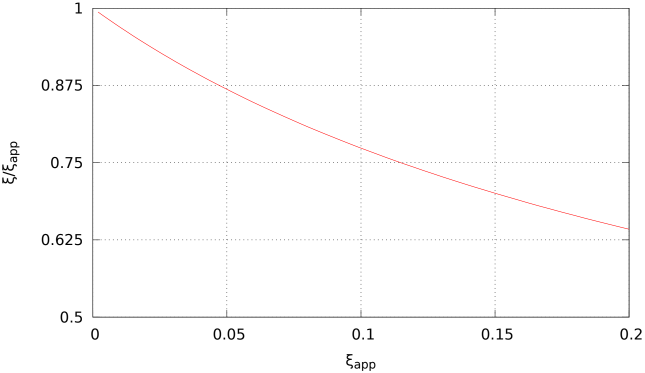
To get an idea of the approximation involved, have a look at the following graph



Home work: obtain a better approximate solution, taking into account that the calculators you find in potato chips do have the $\sqrt{\ }$ key...

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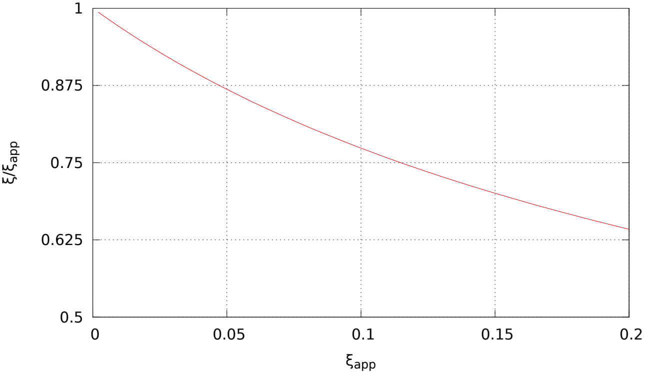

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