SDOF linear oscillator

Response to Harmonic Loading

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March 16, 2011

Outline of parts 1 and 2

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Part I

Response of an Undamped Oscillator to Harmonic Load

The Equation of Motion

The *SDOF* equation of motion for a harmonic loading is:

$$m\,\ddot{x} + k\,x = p_0 \sin \omega t.$$

We seek a particular solution to this equation, in terms of a harmonic function with the same circular frequency, ω,

$$\xi(t) = C \sin \omega t, \quad \ddot{\xi}(t) = -\omega^2\, C \sin \omega t.$$

Substituting x with ξ and simplifying, we get

$$C\,(k - \omega^2 m) = p_0.$$

The Particular Integral

Starting from our last equation, $C\,(k - \omega^2 m) = p_0$,:

- solving for C we get $C = \frac{p_0}{k - \omega^2 m}$,
- collecting k in the right member divisor: $C = \frac{p_0}{k} \frac{1}{1-\omega^2 \frac{m}{k}}$
- but $k/m = \omega_n^2$, so that, with $\beta = \omega/\omega_n$, we get: $C = \frac{p_0}{k} \frac{1}{1-\beta^2}$.

We can now write the particular solution, with the dependencies on β singled out in the second term:

$$\xi(t) = \frac{p_0}{k} \frac{1}{1-\beta^2} \sin \omega t.$$

The general integral for $p(t) = p_0 \sin \omega t$ is hence

$$x(t) = A \sin \omega_n t + B \cos \omega_n t + \frac{p_0}{k} \frac{1}{1-\beta^2} \sin \omega t.$$

Response Ratio and Dynamic Amplification Factor

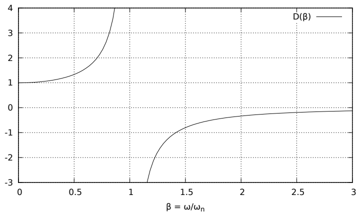
Defining the static deformation, $\Delta_{st} = p_0/k$, we may write the particular solution in terms of Δ_{st} and the *Response Ratio*, $R(t;\ \beta)$, whose amplitude depends *only* on the *frequency ratio* $\beta = \frac{\omega}{\omega_n}$,

$$\xi(t) = \Delta_{st}\, R(t;\ \beta).$$

The *dynamic amplification factor* $D(\beta)$ can be defined as follows:

$$R(t;\ \beta) = \frac{1}{1-\beta^2}\ \sin\omega t = D(\beta)\ \sin\omega t.$$

$D(\beta)$ is stationary and almost equal to 1 when $\omega << \omega_n$ (this is a *quasi*-static behaviour), it grows out of bound when $\beta \Rightarrow 1$ (resonance), it is negative for $\beta > 1$ and goes to 0 when $\omega >> \omega_n$ (high-frequency loading).



Dynamic Amplification Factor, the plot

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Undamped Response
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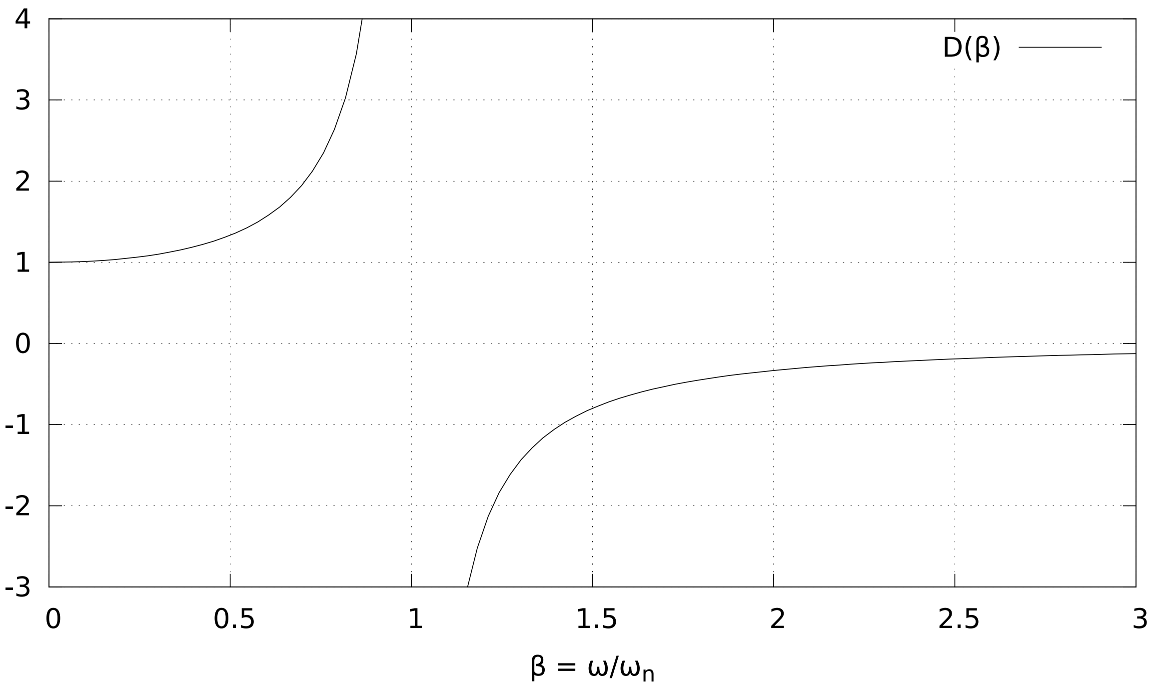
Resonant Response



Response from Rest Conditions

Starting from rest conditions means that $x(0) = \dot{x}(0) = 0$.
Let's start with $x(t)$, then evaluate $x(0)$ and finally equate this last expression to 0:

$$x(t) = A \sin \omega_n t + B \cos \omega_n t + \Delta_{st} D(\beta) \sin \omega t,$$
$$x(0) = B = 0.$$

We do as above for the velocity:

$$\dot{x}(t) = \omega_n (A \cos \omega_n t - B \sin \omega_n t) + \Delta_{st} D(\beta) \omega \cos \omega t,$$
$$\dot{x}(0) = \omega_n A + \omega \Delta_{st} D(\beta) = 0 \Rightarrow$$
$$\Rightarrow A = -\Delta_{st} \frac{\omega}{\omega_n} D(\beta) = -\Delta_{st} \beta D(\beta)$$

Substituting, A and B in $x(t)$ above, collecting Δ_{st} and $D(\beta)$ we have, for $p(t) = p_0 \sin \omega t$, the response from rest:

$$x(t) = \Delta_{st} D(\beta) (\sin \omega t - \beta \sin \omega_n t).$$

Resonant Response from Rest Conditions

We have seen that the response to harmonic loading with zero initial conditions is

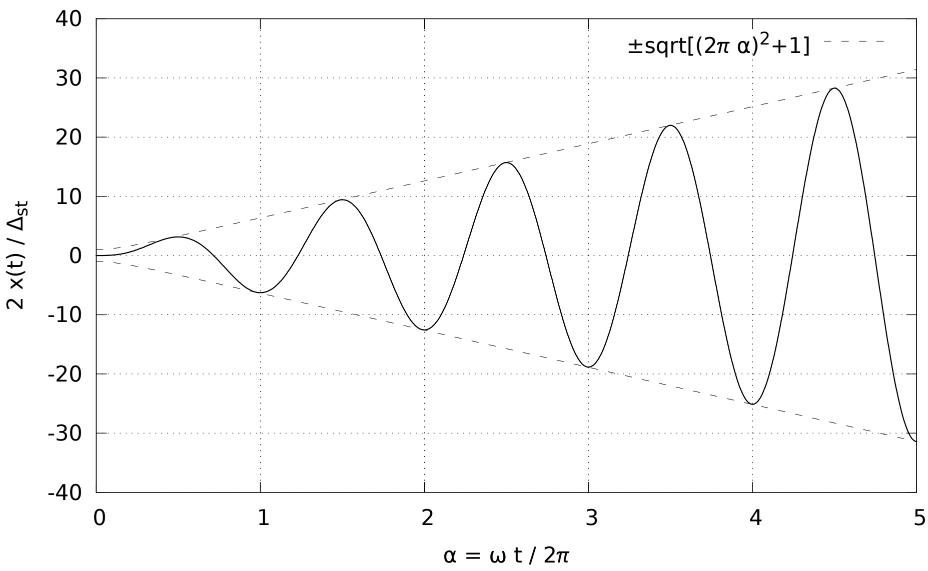
$$x(t; \beta) = \Delta_{st} \frac{(\sin \omega t - \beta \sin \omega_n t)}{1 - \beta^2}.$$

To determine resonant response, we compute the limit for $\beta \to 1$ using the *de l'Hôpital* rule (first, we write $\beta\omega_n$ in place of ω, finally we substitute $\omega_n = \omega$ as $\beta = 1$):

$$\lim_{\beta \to 1} x(t; \beta) = \lim_{\beta \to 1} \Delta_{st} \frac{\partial(\sin \beta\omega_n t - \beta \sin \omega_n t)/\partial\beta}{\partial(1 - \beta^2)/\partial\beta}$$
$$= \frac{\Delta_{st}}{2} (\sin \omega t - \omega t \cos \omega t).$$

As you can see, there is a term in quadrature with the loading, whose amplitude grows linearly and without bounds.

Resonant Response, the plot



$$\frac{2}{\Delta_{st}} x(t) = \sin \omega t - \omega t \cos \omega t = \sin 2\pi\alpha - 2\pi\alpha \cos 2\pi\alpha.$$

note that the amplitude $\mathcal{A}$ of the *normalized* envelope, with respect to the normalized abscissa $\alpha = \omega t / 2\pi$, is $\mathcal{A} = \sqrt{1 + (2\pi\alpha)^2} \xrightarrow{\text{for large } \alpha} 2\pi\alpha$, as the two components of the response are in *quadrature*.

home work

1. Find the response from rest initial conditions for an undamped system, with $p(t) = p_0 \cos \omega t$.
2. Derive the expression for the resonant response with $p(t) = p_0 \cos \omega t$, $\omega = \omega_n$.

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Resonant Response

Part II

Response of the Damped Oscillator to Harmonic Loading

The Equation of Motion for a Damped Oscillator

The *SDOF* equation of motion for a harmonic loading is:

$$m\,\ddot{x} + c\,\dot{x} + k\,x = p_0 \sin \omega t.$$

A particular solution to this equation is a harmonic function not in phase with the input: $x(t) = G \sin(\omega t - \theta)$; it is however equivalent and convenient to write :

$$\xi(t) = G_1 \sin \omega t + G_2 \cos \omega t,$$

where we have simply a different formulation, no more in terms of amplitude and phase but in terms of the amplitudes of two harmonics in quadrature, as in any case the particular integral depends on two free parameters.

The Particular Integral

Substituting $x(t)$ with $\xi(t)$, dividing by m it is

$$\ddot{\xi}(t) + 2\zeta\omega_n\dot{\xi}(t) + \omega_n^2\xi(t) = \frac{p_0}{k}\frac{k}{m}\sin\omega t,$$

Substituting the most general expressions for the particular integral and its time derivatives

$$\begin{aligned}\xi(t) &= G_1 \sin\omega t + G_2 \cos\omega t,\\ \dot{\xi}(t) &= \omega\,(G_1 \cos\omega t - G_2 \sin\omega t),\\ \ddot{\xi}(t) &= -\omega^2\,(G_1 \sin\omega t + G_2 \cos\omega t).\end{aligned}$$

in the above equation it is

$$-\omega^2\,(G_1 \sin\omega t + G_2 \cos\omega t) + 2\zeta\omega_n\omega\,(G_1 \cos\omega t - G_2 \sin\omega t) +$$
$$+\omega_n^2(G_1 \sin\omega t + G_2 \cos\omega t) = \Delta_{st}\omega_n^2 \sin\omega t$$

The particular integral, 2

Dividing our last equation by ω_n^2 and collecting $\sin\omega t$ and $\cos\omega t$ we obtain

$$\left(G_1(1-\beta^2) - G_2 2\beta\zeta\right)\sin\omega t +$$
$$+\left(G_1 2\beta\zeta + G_2(1-\beta^2)\right)\cos\omega t = \Delta_{st}\sin\omega t.$$

Evaluating the eq. above for $t = \frac{\pi}{2\omega}$ and $t = 0$ we obtain a linear system of two equations in G_1 and G_2:

$$\begin{aligned}G_1(1-\beta^2) - G_2 2\zeta\beta &= \Delta_{st}.\\ G_1 2\zeta\beta + G_2(1-\beta^2) &= 0.\end{aligned}$$

The determinant of the linear system is

$$\det = (1-\beta^2)^2 + (2\zeta\beta)^2$$

and its solution is

$$G_1 = +\Delta_{st}\frac{(1-\beta^2)}{\det}, \qquad G_2 = -\Delta_{st}\frac{2\zeta\beta}{\det}.$$

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Stationary Response
Phase Angle
Dynamic Magnification
Exponential Load

Accelerometre, etc

The Particular Integral, 3

Substituting G_1 and G_2 in our expression of the particular integral it is

$$\xi(t) = \frac{\Delta_{st}}{\det}\left((1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t\right).$$

The general integral for $p(t) = p_0 \sin\omega t$ is hence

$$x(t) = \exp(-\zeta\omega_n t)\,(A\,\sin\omega_D t + B\,\cos\omega_D t) + \\ + \Delta_{st}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

For $p(t) = p_{sin}\sin\omega t + p_{cos}\cos\omega t$, $\Delta_{sin} = p_{sin}/k$, $\Delta_{cos} = p_{cos}/k$ it is

$$x(t) = \exp(-\zeta\omega_n t)\,(A\,\sin\omega_D t + B\,\cos\omega_D t) + \\ + \Delta_{sin}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det} + \\ + \Delta_{cos}\frac{(1-\beta^2)\cos\omega t + 2\beta\zeta\sin\omega t}{\det}.$$

Stationary Response

Examination of the general integral

$$x(t) = \exp(-\zeta\omega_n t)\,(A\,\sin\omega_D t + B\,\cos\omega_D t) + \\ + \Delta_{st}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

shows that we have a *transient response*, that depends on the initial conditions and damps out for large values of the argument of the real exponential, and a so called *steady-state response*, corresponding to the particular integral, $x_{s\text{-}s}(t) \equiv \xi(t)$, that remains constant in amplitude and phase as long as the external loading is being applied.
From an engineering point of view, we have a specific interest in the steady-state response, as it is the long term component of the response.

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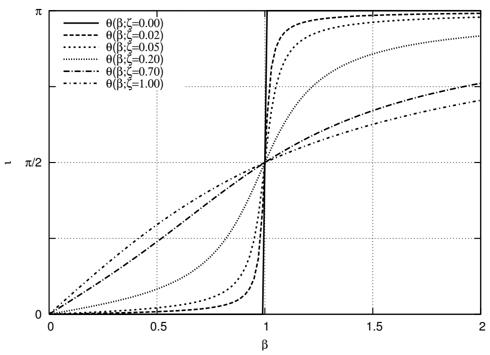
The Angle of Phase

To write the *stationary response* in terms of a *dynamic amplification factor*, it is convenient to reintroduce the amplitude and the phase difference θ and write:

$$\xi(t) = \Delta_{st}\, R(t;\ \beta, \zeta), \quad R = D(\beta, \zeta) \sin(\omega t - \theta).$$

Let's start analyzing the phase difference $\theta(\beta, \zeta)$. Its expression is:

$$\theta(\beta, \zeta) = \arctan \frac{2\zeta\beta}{1-\beta^2}.$$

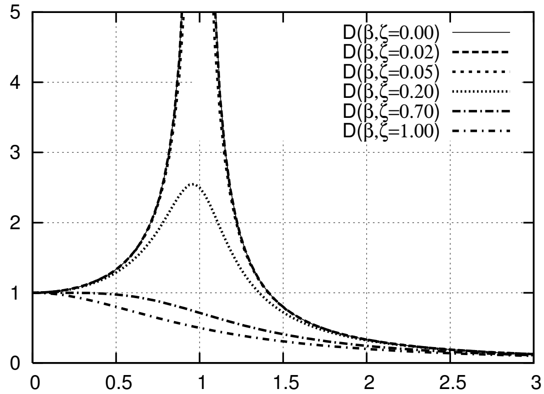


$\theta(\beta, \zeta)$ has a sharper variation around $\beta = 1$ for decreasing values of ζ, but it is apparent that, in the case of slightly damped structures, the response is approximately in phase for low frequencies of excitation, and in opposition for high frequencies. It is worth mentioning that for $\beta = 1$ we have that the response is in perfect quadrature with the load: this is very important to detect resonant response in dynamic tests of structures.

Dynamic Magnification Ratio

The dynamic magnification factor, $D = D(\beta, \zeta)$, is the amplitude of the stationary response normalized with respect to Δ_{st}:

$$D(\beta, \zeta) = \frac{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}{(1-\beta^2)^2 + (2\beta\zeta)^2} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}$$



- $D(\beta, \zeta)$ has larger peak values for decreasing values of ζ,
- the approximate value of the peak, very good for a slightly damped structure, is $1/2\zeta$,
- for larger damping, peaks move toward the origin, until for $\zeta = \frac{1}{\sqrt{2}}$ the peak is in the origin,
- for dampings $\zeta > \frac{1}{\sqrt{2}}$ we have no peaks.

Dynamic Magnification Ratio (2)

The location of the response peak is given by the equation

$$\frac{d\,D(\beta,\zeta)}{d\,\beta} = 0, \quad \Rightarrow \quad \beta^3 + 2\beta^2 - \beta = 0$$

the 3 roots are

$$\beta_i = 0, \pm\sqrt{1-2\zeta^2}.$$

We are interested in a real, positive root, so we are restricted to $0 < \zeta \leqslant \frac{1}{\sqrt{2}}$. In this interval, substituting $\beta = \sqrt{1-2\zeta^2}$ in the expression of the response ratio, we have

$$D_{\max} = \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}}.$$

For $\zeta = \frac{1}{\sqrt{2}}$ there is a maximum corresponding to $\beta = 0$.
Note that, for a relatively large damping ratio, $\zeta = 20\%$, the error of $1/2\zeta$ with respect to $D_{\max}$ is in the order of 2%.

Harmonic Exponential Load

Consider the *EOM* for a load modulated by an exponential of imaginary argument:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \Delta_{st}\omega_n^2 \exp(i(\omega t - \phi)).$$

Note that the phase can be disregarded as we can represent its effects with a constant factor, as it is
$\exp(i(\omega t - \phi)) = \exp(i\omega t)/\exp(i\phi)$.
The particular solution and its derivatives are

$$\xi = G\exp(i\omega t), \quad \dot{\xi} = i\omega G\exp(i\omega t), \quad \ddot{\xi} = -\omega^2 G\exp(i\omega t),$$

where G is a complex constant.
Substituting, dividing by ω_n^2, removing the dependency on $\exp(i\omega t)$ and solving for G yields

$$G = \Delta_{st}\left[\frac{1}{(1-\beta^2) + i(2\zeta\beta)}\right] = \Delta_{st}\left[\frac{(1-\beta^2) - i(2\zeta\beta)}{(1-\beta^2)^2 + (2\zeta\beta)^2}\right].$$

Note how simpler it is to represent the stationary response of a damped oscillator using the complex exponential representation.

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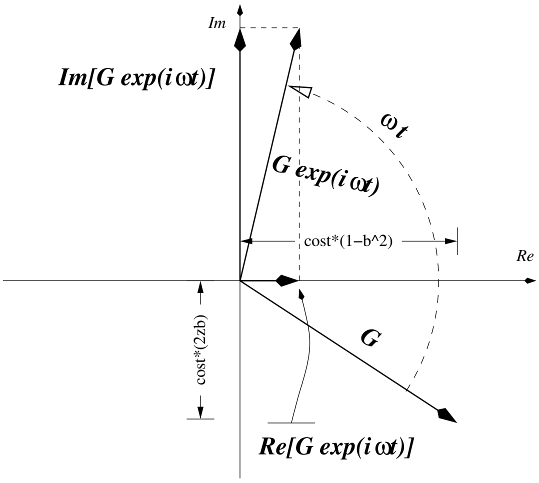
Response in the Complex Plane



The *stationary response* is

$$\xi(t) = \Delta_{st} \frac{(1-\beta^2) - i(2\zeta\beta)}{(1-\beta^2)^2 + (2\zeta\beta)^2} \exp(i\omega t)$$

- we plot G in the complex plane,
- we multiply G by $\exp(i\omega t)$, that is equivalent to rotate G by the angle ωt,
- projecting the resulting vector on the axes, we have the real and imaginary part of the response,
- these two vectors are rotated 90 degrees with respect to the response to the real harmonic load, $p_0 \sin \omega t$ that we have studied,
- what if $p(t) = p_0 \cos \omega t$?

Measuring Support Accelerations

We have seen that in seismic analysis the loading is proportional to the ground acceleration.
A simple oscillator, when properly damped, may serve the scope of measuring support accelerations.

Measuring Support Accelerations, 2

With the equation of motion valid for a harmonic support acceleration:

$$\ddot{x} + 2\zeta\beta\omega_n\dot{x} + \omega_n^2 x = -a_g \sin\omega t,$$

the stationary response is $\xi = \frac{m\,a_g}{k}\,D(\beta,\zeta)\,\sin(\omega t - \theta)$.
If the damping ratio of the oscillator is $\zeta \approxeq 0.7$, then the Dynamic Amplification $D(\beta) \approxeq 1$ for $0.0 < \beta < 0.6$!
Oscillator's displacements will be proportional to the accelerations of the support for applied frequencies up to about six-tenths of the natural frequency of the instrument.
As it is possible to record the oscillator displacements by means of electro-mechanical or electronic devices, it is hence possible to measure, within an almost constant scale factor, the ground accelerations component up to a frequency of the order of 60% of the natural frequency of the oscillator.

Measuring Displacements

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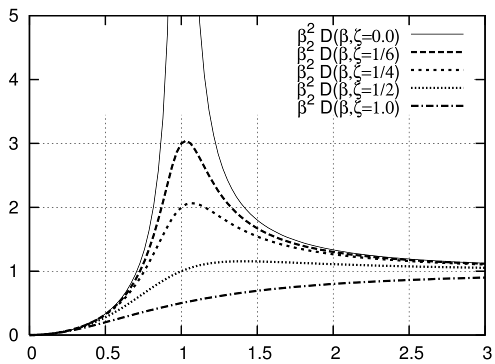
Measuring Displacements

Consider now a harmonic *displacement* of the support, $u_g(t) = u_g \sin\omega t$.
The support acceleration (disregarding the sign) is $a_g(t) = \omega^2 u_g \sin\omega t$.

With the equation of motion: $\ddot{x} + 2\zeta\beta\omega_n\dot{x} + \omega_n^2 x = -\omega^2 u_g \sin\omega t$, the stationary response is $\xi = u_g\,\beta^2\,D(\beta,\zeta)\,\sin(\omega t - \theta)$.

Let's see a graph of the dynamic amplification factor derived above.

We see that the displacement of the instrument is approximately equal to the support displacement for all the excitation frequencies greater than the natural frequency of the instrument, for a damping ratio $\zeta \approxeq .5$.



It is possible to measure the support displacement measuring the deflection of the oscillator, within an almost constant scale factor, for excitation frequencies larger than ω_n.

Part III

Vibration Isolation

Vibration Isolation

Vibration isolation is a subject too broad to be treated in detail, we'll present the basic principles involved in two problems,

1. prevention of harmful vibrations in supporting structures due to oscillatory forces produced by operating equipment,
2. prevention of harmful vibrations in sensitive instruments due to vibrations of their supporting structures.

Force Isolation

Consider a rotating machine that produces an oscillatory force $p_0 \sin \omega t$ due to unbalance in its rotating part, that has a total mass m and is mounted on a spring-damper support. Its steady-state relative displacement is given by

$$x_{\text{s-s}} = \frac{p_0}{k} D \sin(\omega t - \theta).$$

This result depend on the assumption that the supporting structure deflections are negligible respect to the relative system motion.
The steady-state spring and damper forces are

$$f_S = k\, x_{ss} = p_0\, D \sin(\omega t - \theta),$$
$$f_D = c\, \dot{x}_{ss} = \frac{c p_0\, D\, \omega}{k} \cos(\omega t - \theta) = 2\, \zeta\, \beta\, p_0\, D \cos(\omega t - \theta).$$

Transmitted force

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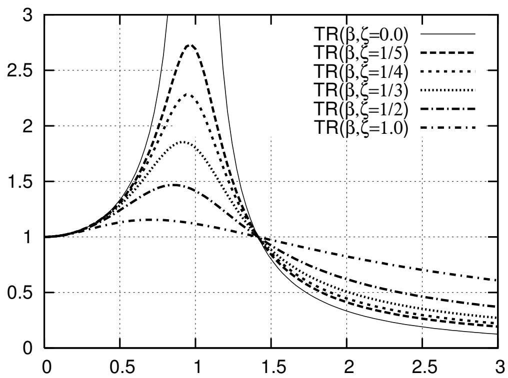
The spring and damper forces are in quadrature, so the amplitude of the steady-state reaction force is given by

$$f_{\text{max}} = p_0\, D \sqrt{1 + (2\zeta\beta)^2}$$

The ratio of the maximum transmitted force to the amplitude of the applied force is the *transmissibility ratio* (TR),

$$TR = \frac{f_{\text{max}}}{p_0} = D \sqrt{1 + (2\zeta\beta)^2}.$$



1. For $\beta < \sqrt{2}$, TR is always greater than 1: the transmitted force is amplified. 2. For $\beta > \sqrt{2}$, TR is always smaller than 1 and for the same β TR decreases with ζ.

Displacement Isolation

Another problem concerns the harmonic support motion $u_g(t) = u_{g_0} \exp i\omega t$ forcing a steady-state relative displacement of some supported (spring+damper) equipment of mass m (using exp notation) $x_{ss} = u_{g_0}\, \beta^2 D \exp i\omega t$, and the mass total displacement is given by

$$x_{tot} = x_{s\text{-}s} + u_g(t) = u_{g_0}\left(\frac{\beta^2}{(1-\beta^2)+2\,i\,\zeta\beta}+1\right)\exp i\omega t$$
$$= u_{g_0}\,(1+2i\zeta\beta)\frac{1}{(1-\beta^2)+2\,i\,\zeta\beta}\exp i\omega t$$
$$= u_{g_0}\sqrt{1+(2\zeta\beta)^2}\,D\,\exp i\,(\omega t-\varphi).$$

If we define the transmissibility ratio TR as the ratio of the maximum total response to the support displacement amplitude, we find that, as in the previous case,

$$\mathrm{TR} = D\sqrt{1+(2\zeta\beta)^2}.$$

Isolation Effectiveness

Define the isolation effectiveness,

$$\mathrm{IE} = 1 - \mathrm{TR},$$

IE=1 means complete isolation, i.e., $\beta = \infty$, while IE=0 is no isolation, and takes place for $\beta = \sqrt{2}$.
As effective isolation requires low damping, we can approximate $\mathrm{TR} \approxeq 1/(\beta^2-1)$, in which case we have $\mathrm{IE} = (\beta^2-2)/(\beta^2-1)$. Solving for β^2, we have $\beta^2 = (2-\mathrm{IE})/(1-\mathrm{IE})$, but

$$\beta^2 = \omega^2/\omega_n^2 = \omega^2\,(m/k) = \omega^2\,(W/gk) = \omega^2\,(\Delta_{st}/g)$$

where W is the weight of the mass and Δ_{st} is the static deflection under self weight. Finally, from $\omega = 2\pi f$ we have

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{st}}\frac{2-\mathrm{IE}}{1-\mathrm{IE}}}$$

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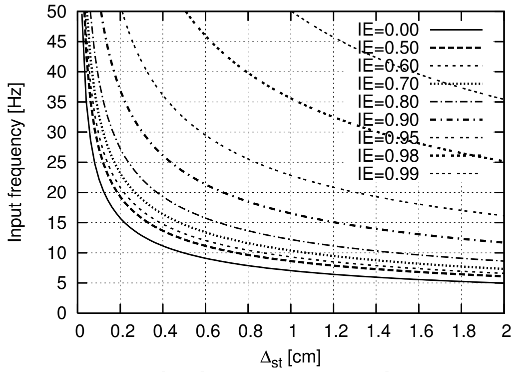
Isolation Effectiveness (2)

The strange looking

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{st}}\frac{2-\mathrm{IE}}{1-\mathrm{IE}}}$$

can be plotted f vs Δ_{st} for different values of IE, obtaining a design chart.



Knowing the frequency of excitation and the required level of vibration isolation efficiency (IE), one can determine the minimum static deflection (proportional to the spring flexibility) required to achieve the required IE. It is apparent that any isolation system must be very flexible to be effective.

Part IV

Evaluation of Viscous Damping Ratio

Evaluation of damping

The mass and stiffness of phisycal systems of interest are usually evaluated easily, but this is not feasible for damping, as the energy is dissipated by different mechanisms, some one not fully understood... it is even possible that dissipation cannot be described in term of viscous-damping, But it generally is possible to measure an equivalent viscous-damping ratio by experimental methods:

- free-vibration decay method,
- resonant amplification method,
- half-power (bandwidth) method,
- resonance cyclic energy loss method.

Free vibration decay

We already have discussed the free-vibration decay method,

$$\zeta = \frac{\delta_m}{2\pi\, m\,(\omega_n/\omega_D)}$$

with $\delta_m = \ln \frac{x_n}{x_{n+m}}$, *logarithmic decrement*. The method is simple and its requirements are minimal, but some care must be taken in the interpretation of free-vibration tests, because the damping ratio decreases with decreasing amplitudes of the response, meaning that for a very small amplitude of the motion the effective values of the damping can be underestimated.

Resonant amplification

This method assumes that it is possible to measure the stiffness of the structure, and that damping is small. The experimenter (a) measures the steady-state response x_{ss} of a SDOF system under a harmonic loading for a number of different excitation frequencies (eventually using a smaller frequency step when close to the resonance), (b) finds the maximum value $D_{\max} = \frac{\max\{x_{ss}\}}{\Delta_{st}}$ of the dynamic magnification factor, (c) uses the approximate expression (good for small ζ) $D_{\max} = \frac{1}{2\zeta}$ to write

$$D_{\max} = \frac{1}{2\zeta} = \frac{\max\{x_{ss}\}}{\Delta_{st}}$$

and finally (d) has

$$\zeta = \frac{\Delta_{st}}{2\max\{x_{ss}\}}.$$

The most problematic aspect here is getting a good estimate of Δ_{st}, if the results of a static test aren't available.

Half Power

The adimensional frequencies where the response is $1/\sqrt{2}$ times the peak value can be computed from the equation

$$\frac{1}{\sqrt{(1-\beta^2)^2+(2\beta\zeta)^2}} = \frac{1}{\sqrt{2}}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

squaring both sides and solving for β^2 gives

$$\beta_{1,2}^2 = 1-2\zeta^2 \mp 2\zeta\sqrt{1-\zeta^2}$$

For small ζ we can use the binomial approximation and write

$$\beta_{1,2} = \left(1-2\zeta^2 \mp 2\zeta\sqrt{1-\zeta^2}\right)^{\frac{1}{2}} \approxeq 1-\zeta^2 \mp \zeta\sqrt{1-\zeta^2}$$

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Half power (2)

From the approximate expressions for the difference of the half power frequency ratios,

$$\beta_2 - \beta_1 = 2\zeta\sqrt{1-\zeta^2} \approx 2\zeta$$

and their sum

$$\beta_2 + \beta_1 = 2(1-\zeta^2) \approx 2$$

we can deduce that

$$\frac{\beta_2-\beta_1}{\beta_2+\beta_1} = \frac{f_2-f_1}{f_2+f_1} \approx \frac{2\zeta\sqrt{1-\zeta^2}}{2(1-\zeta^2)} \approx \zeta, \text{ or } \zeta \approx \frac{f_2-f_1}{f_2+f_1}$$

where f_1, f_2 are the frequencies at which the steady state amplitudes equal $1/\sqrt{2}$ times the peak value, frequencies that can be determined from a dynamic test where detailed test data is available.

Resonance Cyclic Energy Loss

If it is possible to determine the phase of the s-s response, it is possible to measure ζ from the amplitude ρ of the resonant response.
At resonance, the deflections and accelerations are in quadrature with the excitation, so that the external force is equilibrated *only* by the viscous force, as both elastic and inertial forces are also in quadrature with the excitation.
The equation of dynamic equilibrium is hence:

$$p_0 = c\,\dot{x} = 2\zeta\omega_n m\,(\omega_n\rho).$$

Solving for ζ we obtain:

$$\zeta = \frac{p_0}{2m\omega_n^2\rho}.$$