SDOF linear oscillator

Response to Periodic and Non-periodic Loadings

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March 25, 2011

Outline

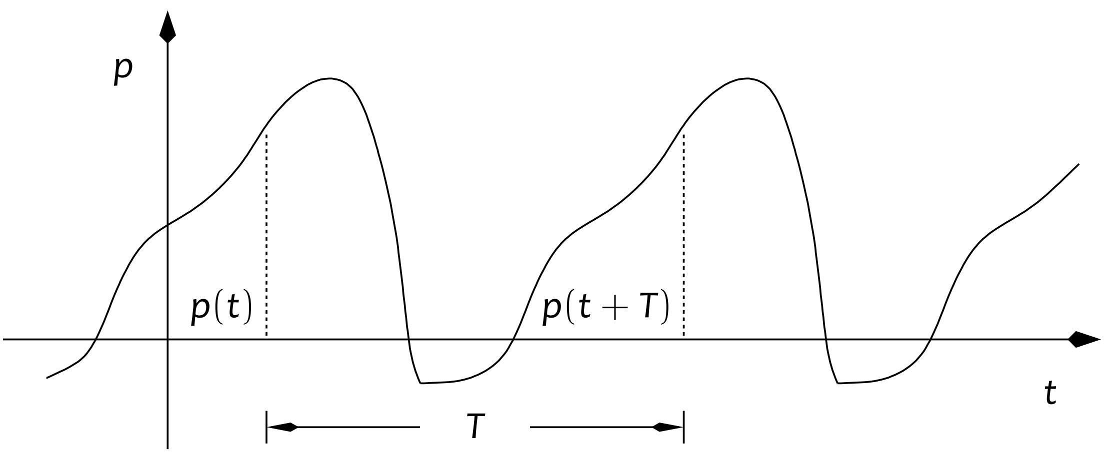
Response to Periodic Loading

Introduction

A periodic loading is characterized by the identity

$$p(t) = p(t + T)$$

where T is the *period* of the loading, and $\omega_1 = \frac{2\pi}{T}$ is its *principal frequency*.



Introduction

Periodic loadings can be expressed as an infinite series of harmonic functions using Fourier theorem, e.g., an antisymmetric loading is

$$p(t) = p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t.$$

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_n$ it is:

$$x_{j,\text{s-s}} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)).$$

In general, it is possible to sum all steady-state responses, the infinite series giving the *SDOF* response to $p(t)$.
Due to the asymptotic behaviour of $D(\beta; \zeta)$ (D goes to zero for large, increasing β) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Fourier Series

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.
Consider a loading of period T_p, its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j\,\omega_1 = j\frac{2\pi}{T_p},$$

where the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_p}\int_0^{T_p} p(t)\ dt, \quad a_j = \frac{2}{T_p}\int_0^{T_p} p(t)\ \cos \omega_j t\ dt,$$

$$b_j = \frac{2}{T_p}\int_0^{T_p} p(t)\ \sin \omega_j t\ dt,$$

as, by orthogonality, $\int_o^{T_p} p(t) cos \omega_j\, dt = \int_o^{T_p} a_j \cos^2 \omega_j t\, dt = \frac{T_p}{2} a_j$, etc etc.

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

(a) to divide the period in N equal parts $\Delta t = T_p/N$,

(b) measure or compute $p(t)$ at a discrete set of instants $t_1, t_2, \ldots, t_N$, with $t_m = m\Delta t$,

obtaining a discrete set of values $p_m,\ m = 1, \ldots, N$ (note that $p_0 = p_N$ by periodicity).
Using the trapezoidal rule of integration, with $p_0 = p_N$ we can write, for example, the cosine-wave amplitude coefficients,

$$a_j \approxeq \frac{2\Delta t}{T_p} \sum_{m=1}^{N} p_m \cos \omega_j t_m$$

$$= \frac{2}{N} \sum_{m=1}^{N} p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^{N} p_m \cos \frac{jm\,2\pi}{N}.$$

It's worth to note that the discrete function $\cos \frac{jm\,2\pi}{N}$ is periodic with period N.

Exponential Form

The Fourier series can be written in terms of the exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp i\omega_j t\ dt, \qquad j = -\infty, \ldots, +\infty.$$

For a sampled p_m we can write, using the trapezoidal integration rule and substituting $t_m = m\Delta t = m\,T_p/N$, $\omega_j = j\,2\pi/T_p$:

$$P_j \approxeq \frac{1}{N} \sum_{m=1}^{N} p_m \exp(-i\frac{2\pi\, j\, m}{N}),$$

Undamped Response

We have seen that the steady-state response to the jth sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\sin\omega_j t, \qquad \beta_j = \omega_j/\omega_n,$$

analogously, for the jth cosine-wave harmonic,

$$x_j = \frac{a_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\cos\omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k}\left\{a_0 + \sum_{j=1}^{\infty}\left[\frac{1}{1-\beta_j^2}\right]\left(a_j\cos\omega_j t + b_j\sin\omega_j t\right)\right\}.$$

Damped Response

In the case of a damped oscillator, we must substitute the steady state response for both the jth sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k}\sum_{j=1}^{\infty}\frac{+(1-\beta_j^2)\,a_j - 2\zeta\beta_j\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\cos\omega_j t+$$
$$+\frac{1}{k}\sum_{j=1}^{\infty}\frac{+2\zeta\beta_j\,a_j + (1-\beta_j^2)\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\sin\omega_j t.$$

As usual, the exponential notation is neater,

$$x(t) = \sum_{j=-\infty}^{\infty}\frac{P_j}{k}\frac{\exp i\omega_j t}{(1-\beta_j^2)+i\,(2\zeta\beta_j)}.$$

Example

As an example, consider the loading
$p(t) = \max\{p_0 \sin \frac{2\pi t}{T_p},\ 0\}$

$$a_0 = \frac{1}{T_p}\int_0^{T_p/2} p_o \sin\frac{2\pi t}{T_p}\,dt = \frac{p_0}{\pi},$$

$$a_j = \frac{2}{T_p}\int_0^{T_p/2} p_o \sin\frac{2\pi t}{T_p}\cos\frac{2\pi j t}{T_p}\,dt = \begin{cases} 0 & \text{for } j \text{ odd} \\ \frac{p_0}{\pi}\left[\frac{2}{1-j^2}\right] & \text{for } j \text{ even,} \end{cases}$$

$$b_j = \frac{2}{T_p}\int_0^{T_p/2} p_o \sin\frac{2\pi t}{T_p}\sin\frac{2\pi j t}{T_p}\,dt = \begin{cases} \frac{p_0}{2} & \text{for } j = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

Example cont.

Assuming $\beta_1 = 3/4$, from
$p = \frac{p_0}{\pi}\left(1 + \frac{\pi}{2}\sin\omega_1 t - \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_2 t - \dots\right)$ with the dynamic amplification factors

$$D_1 = \frac{1}{1-(1\frac{3}{4})^2} = \frac{16}{7},$$

$$D_2 = \frac{1}{1-(2\frac{3}{4})^2} = -\frac{4}{5},$$

$$D_4 = \frac{1}{1-(4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \dots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi}\left(1 + \frac{8\pi}{7}\sin\omega_1 t + \frac{8}{15}\cos 2\omega_1 t + \frac{1}{60}\cos 4\omega_1 t + \dots\right)$$

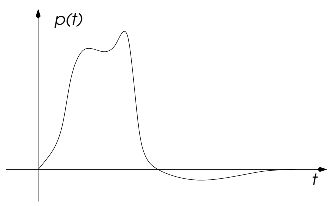
Take note, these solutions are particular solutions! If your solution has to respect given initial conditions, you must consider also the homogeneous solution.

Response to Impulsive Loadings

Nature of Impulsive Loadings

An impulsive load is characterized

- by a single principal impulse, and
- by a relatively short duration.



- Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
- Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
- For this reason, in the following we'll consider only the undamped response to impulsive loads.

Definition of Maximum Response

In general, when dealing with impulse response characterized by its duration t_0 we are interested either in

- *a* the maximum of the absolute values of maxima (named also the *peak value*) of the response ratio $R(t)$ in $0 < t < t_0$ or,
- *b* if we have no maxima during the excitation phase (i.e., $\dot{x} \neq 0$ in $0 < t < t_0$) we want to know the amplitude of the free vibrations that are excited by the impulse.

Half-sine Wave Impulse

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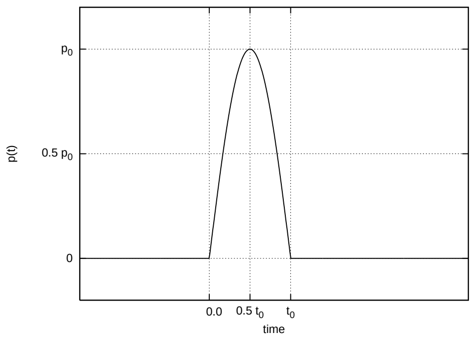
The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load. Note that $\omega\, t_0 = \pi$.

Response to sine-wave impulse

Consider an undamped *SDOF* initially at rest, with natural circular frequency ω_n and stiffness k. With reference to a half-sine impulse with duration t_0, the frequency ratio β is $\omega/\omega_n = T_n/2t_0$.
Its response ratio in the interval $0 < t < t_0$ is

$$R(t) = \frac{1}{1-\beta^2}\left(\sin\omega t - \beta \sin\frac{\omega t}{\beta}\right) \qquad [\text{NB: } \frac{\omega}{\beta} = \omega_n]$$

while for $t > t_0$ the response ratio is

$$R(t) = \frac{-\beta}{1-\beta^2}\left((1+\cos\frac{\pi}{\beta})\sin\omega_n(t-t_0) + \sin\frac{\pi}{\beta}\cos\omega_n(t-t_0)\right)$$

Maximum response to sine impulse

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(a) Since we are interested in the maximum response ratio during the excitation, we need to know when velocity is zero in the time interval $0 \le t \le t_0$; from

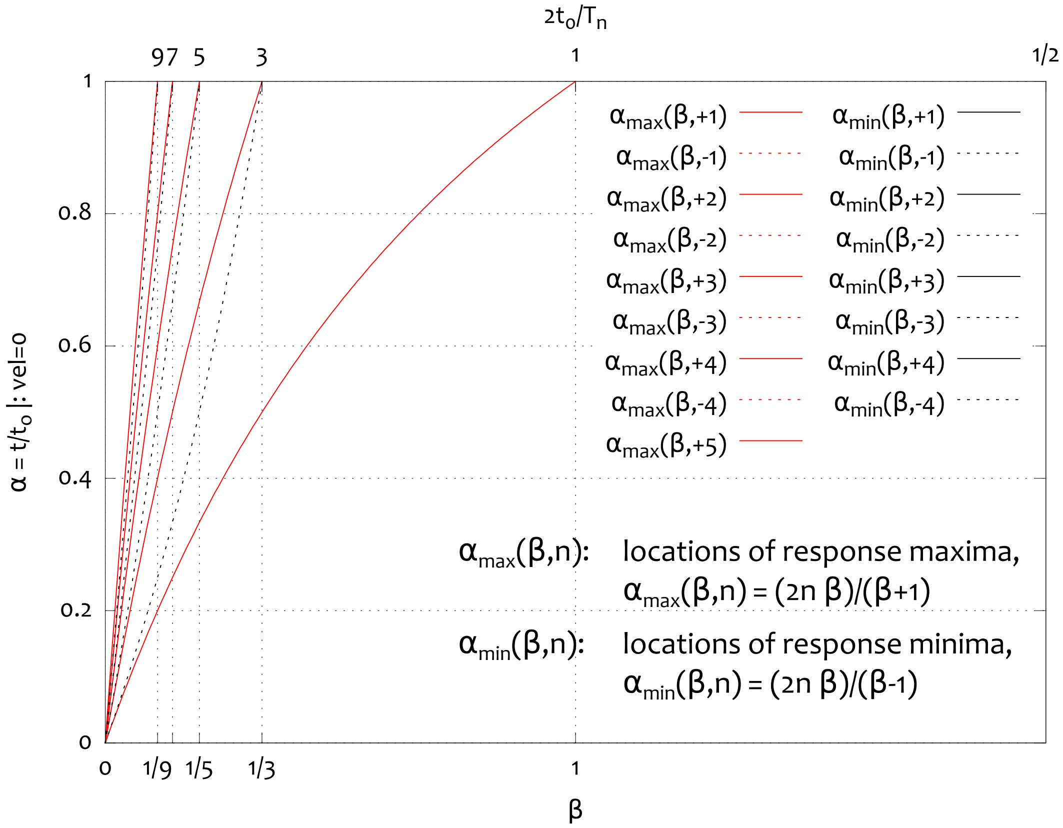
$$\dot{R}(t) = \frac{\omega}{1-\beta^2}\left(\cos\omega t - \cos\frac{\omega t}{\beta}\right) = 0.$$

we can see that the roots are
$\omega t = \mp\omega t/\beta + 2n\pi,\;\; n = 0, \mp 1, \mp 2, \mp 3, \ldots$; it is convenient to substitute $\omega t = \pi\alpha$, where $\alpha = t/t_0$; substituting and solving for α one has

$$\alpha = \frac{2n\beta}{\beta \mp 1}, \qquad \text{with } n = 0, \mp 1, \mp 2, \ldots, \text{ for } 0 < \alpha < 1.$$

The next slide regards the characteristics of these roots.

$\alpha(\beta, n)$



$\alpha(\beta, n)$

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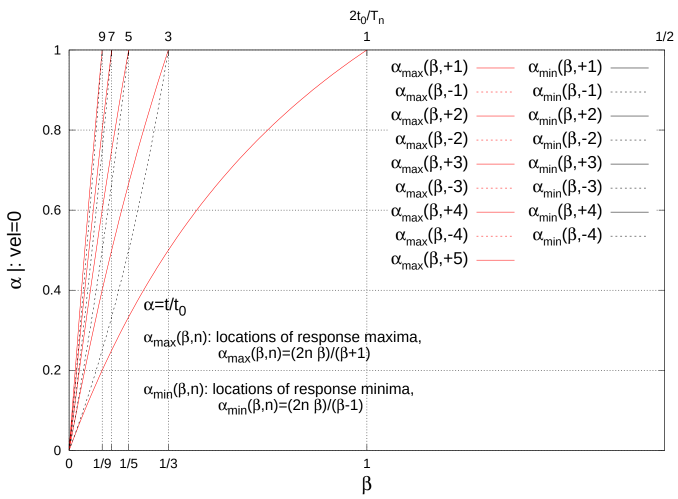


- No roots of type α_{min} for $n > 0$;
- no roots of type α_{max} for $n < 0$;
- no roots for $\beta > 1$, i.e., no roots for $t_0 < \frac{T_n}{2}$;
- only one root of type α_{max} for $\frac{1}{3} < \beta < 1$, i.e., $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$;
- three roots, two maxima and one minimum, for $\frac{1}{5} < \beta < \frac{1}{3}$;
- five roots, three maxima and two minima, for $\frac{1}{7} < \beta < \frac{1}{5}$;
- etc etc.

In summary, to find the maximum of the response for an assigned $\beta < 1$, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k t_0$, (c) choose the maximum of the maxima.

Maximum response for $\beta > 1$

For $\beta > 1$, the maximum response takes place for $t > t_0$, and its absolute value (see slide *Response to sine-wave impulse*) is

$$R_{max} = \frac{\beta}{1-\beta^2}\sqrt{(1+\cos\frac{\pi}{\beta})^2 + \sin^2\frac{\pi}{\beta}},$$

using a simple trigonometric identity we can write

$$R_{max} = \frac{\beta}{1-\beta^2}\sqrt{2+2\cos\frac{\pi}{\beta}}$$

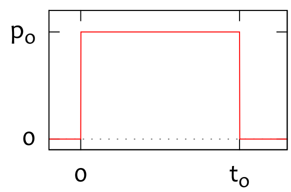
but
$1+\cos 2\phi = (\cos^2\phi + \sin^2\phi) + (\cos^2\phi - \sin^2\phi) = 2\cos^2\phi$,
so that

$$R_{max} = \frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}.$$

Rectangular Impulse

Cosider a rectangular impulse of duration t_0,

$$p(t) = p_0 \begin{cases} 1 & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



The response ratio and its time derivative are

$$R(t) = 1 - \cos\omega_n t, \qquad \dot{R}(t) = \omega_n \sin\omega_n t,$$

and we recognize that we have maxima $R_{max} = 2$ for $\omega_n t = n\pi$, with the condition $t \leq t_0$. Hence we have no maximum during the loading phase for $t_0 < T_n/2$, and at least one maximum, of value $2\Delta_{st}$, if $t_0 \geq T_n/2$.

Rectangular Impulse (2)

For shorter impulses, the maximum response ratio is not attained during loading, so we have to compute the amplitude of the free vibrations after the end of loading (remember, as $t_0 \le T_n/2$ the velocity is positive at $t = t_0$!).

$$R(t) = (1-\cos\omega_n t_0)\cos\omega_n(t-t_0) + (\sin\omega_n t_0)\sin\omega_n(t-t_0).$$

The amplitude of the response ratio is then

$$A = \sqrt{(1-\cos\omega_n t_0)^2 + \sin^2\omega_n t_0} = \\ = \sqrt{2(1-\cos\omega_n t_0)} = 2\sin\frac{\omega_n t_0}{2}.$$

Triangular Impulse

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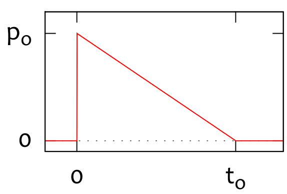
Let's consider the response of a *SDOF* to a triangular impulse,

$$p(t) = p_0\,(1 - t/t_0) \text{ for } 0 < t < t_0$$



As usual, we must start finding the minimum duration that gives place to a maximum of the response in the loading phase, that is

$$R(t) = \frac{1}{\omega_n t_0}\sin\omega_n\frac{t}{t_0} - \cos\omega_n\frac{t}{t_0} + 1 - \frac{t}{t_0}, \qquad 0 < t < t_0.$$

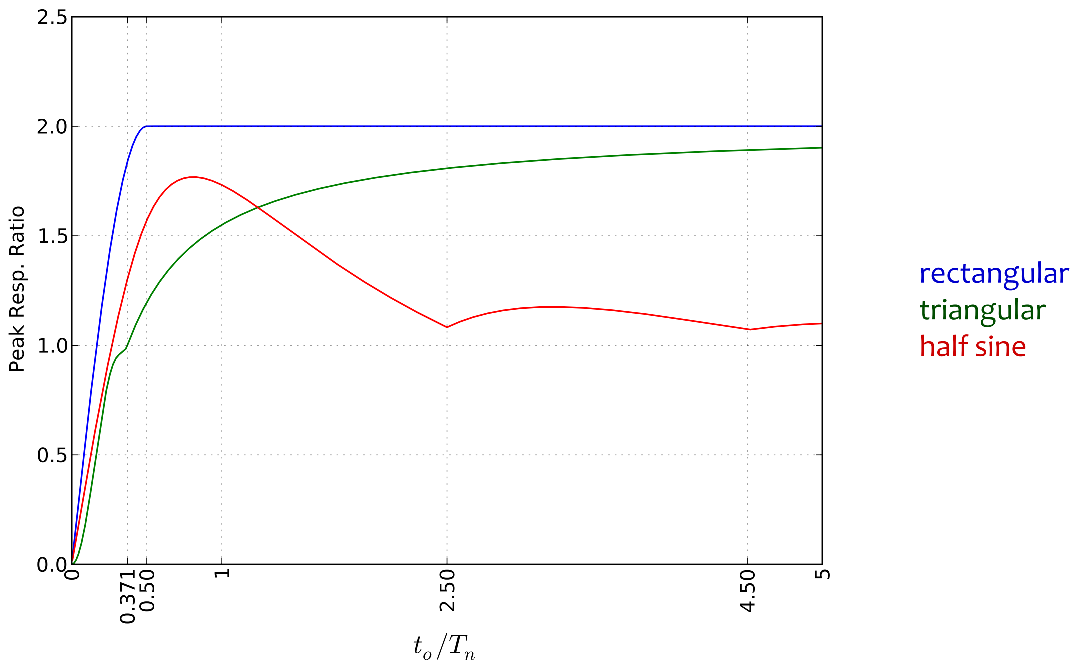
Taking the first derivative and setting it to zero, one can see that the first maximum occurs for $t = t_0$ for $t_0 = 0.37101T_n$, and substituting one can see that $R_{max} = 1$.

Triangular Impulse (2)

For load durations shorter than $0.37101T_n$, the maximum occurs after loading and it's necessary to compute the displacement and velocity at the end of the load phase. For longer loads, the maxima are in the load phase, so that one has to find the all the roots of $\dot{R}(t)$, compute all the extreme values and finally sort out the absolute value maximum.

Shock or response spectra

We have seen that the response ratio is determined by the ratio of the impulse duration to the natural period of the oscillator. One can plot the maximum displacement ratio R_{max} as a function of t_o/T_n for various forms of impulsive loads.



Such plots are commonly known as displacement-response spectra, or simply as response spectra.

Approximate Analysis

For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor $\approxeq 1$

On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the load.
The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Approximate Analysis (2)

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)]\ dt.$$

When one notes that, for small t_0, the displacement is of the order of t_0^2 while the velocity is in the order of t_0, it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approxeq \int_0^{t_0} p(t)\ dt.$$

Approximate Analysis (3)

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m}\int_0^{t_0} p(t)\ dt,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{1}{m\omega_n}\int_0^{t_0} p(t)\ dt\ \ \sin\omega_n(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading (and will be discovered again in a few minutes).

Response to General Dynamic Loading

Undamped SDOF

For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau)\,d\tau$ and the response is

$$dx(t-\tau) = \frac{p(\tau)\,d\tau}{m\omega_n} \sin \omega_n (t-\tau), \qquad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \, \sin \omega_n (t-\tau)\, d\tau, \qquad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Damped SDOF

The derivation of the equation of motion for a generic load is analogous to what we have seen for undamped SDOF, the infinitesimal contribution to the response at time t of the load at time τ is

$$dx(t) = \frac{p(\tau)}{m\omega_D} d\tau \, \sin \omega_D (t-\tau) \exp(-\zeta\omega_n(t-\tau)) \qquad t \geq \tau$$

and integrating all infinitesimal contributions one has

$$x(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \, \sin \omega_D (t-\tau) \exp(-\zeta\omega_n(t-\tau))\, d\tau, \qquad t \geq 0.$$

Evaluation of Duhamel integral, undamped

Using the trig identity

$$\sin(\omega_n t - \omega_n \tau) = \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau$$

the Duhamel integral is rewritten as

$$\begin{aligned} x(t) &= \frac{\int_0^t p(\tau) \cos \omega_n \tau \, d\tau}{m\omega_n} \sin \omega_n t - \frac{\int_0^t p(\tau) \sin \omega_n \tau \, d\tau}{m\omega_n} \cos \omega_n t \\ &= \mathcal{A}(t) \sin \omega_n t - \mathcal{B}(t) \cos \omega_n t \end{aligned}$$

where

$$\begin{cases} \mathcal{A}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \cos \omega_n \tau \, d\tau \\ \mathcal{B}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n \tau \, d\tau \end{cases}$$

Numerical evaluation of Duhamel integral, undamped

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Usual numerical procedures can be applied to the evaluation of $\mathcal{A}$ and $\mathcal{B}$, e.g., using the trapezoidal rule, one can have, with $\mathcal{A}_N = \mathcal{A}(N\Delta\tau)$ and $y_N = p(N\Delta\tau)\cos(N\Delta\tau)$

$$\mathcal{A}_{N+1} = \mathcal{A}_N + \frac{\Delta\tau}{2m\omega_n} \left(y_N + y_{N+1}\right).$$

Evaluation of Duhamel integral, damped

For a damped system, it can be shown that

$$x(t) = \mathcal{A}(t)\sin\omega_D t - \mathcal{B}(t)\cos\omega_D t$$

with

$$\mathcal{A}(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\frac{\exp\zeta\omega_n\tau}{\exp\zeta\omega_n t}\cos\omega_D\tau\,d\tau,$$
$$\mathcal{B}(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\frac{\exp\zeta\omega_n\tau}{\exp\zeta\omega_n t}\sin\omega_D\tau\,d\tau.$$

Numerical evaluation of Duhamel integral, damped

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Numerically, using e.g. Simpson integration rule and $y_N = p(N\Delta\tau)\cos\omega_D\tau$,

$$\mathcal{A}_{N+2} = \mathcal{A}_N\exp(-2\zeta\omega_n\Delta\tau) + \frac{\Delta\tau}{3m\omega_D}\left[y_N\exp(-2\zeta\omega_n\Delta\tau) + 4y_{N+1}\exp(-\zeta\omega_n\Delta\tau) + y_{N+2}\right]$$

$$N = 0, 2, 4, \cdots$$