

SDOF linear oscillator

Response to Periodic and Non-periodic Loadings

Giacomo Boffi

Dipartimento di Ingegneria Strutturale, Politecnico di Milano

March 25, 2011

Response to Periodic Loading

Response to Impulsive Loading

Response to General Dynamic Loadings

Response to
Periodic Loading

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

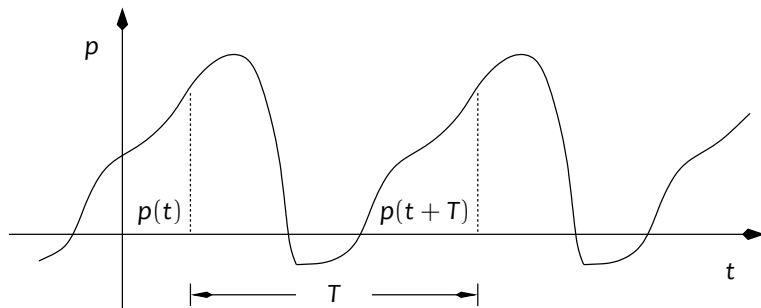
Response to Impulsive Loading

Response to General Dynamic Loadings

A periodic loading is characterized by the identity

$$p(t) = p(t + T)$$

where T is the *period* of the loading, and $\omega_1 = \frac{2\pi}{T}$ is its *principal frequency*.



Periodic loadings can be expressed as an infinite series of harmonic functions using Fourier theorem, e.g., an antisymmetric loading is

$$p(t) = p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t.$$

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_n$ it is:

$$x_{j,s-s} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)).$$

In general, it is possible to sum all steady-state responses, the infinite series giving the SDOF response to $p(t)$.

Due to the asymptotic behaviour of $D(\beta; \zeta)$ (D goes to zero for large, increasing β) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

Periodic loadings can be expressed as an infinite series of harmonic functions using Fourier theorem, e.g., an antisymmetric loading is

$$p(t) = p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t.$$

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_n$ it is:

$$x_{j,s-s} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)).$$

In general, it is possible to sum all steady-state responses, the infinite series giving the SDOF response to $p(t)$.

Due to the asymptotic behaviour of $D(\beta; \zeta)$ (D goes to zero for large, increasing β) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

Periodic loadings can be expressed as an infinite series of harmonic functions using Fourier theorem, e.g., an antisymmetric loading is

$$p(t) = p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t.$$

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j / \omega_n$ it is:

$$x_{j,s-s} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)).$$

In general, it is possible to sum all steady-state responses, the infinite series giving the SDOF response to $p(t)$.

Due to the asymptotic behaviour of $D(\beta; \zeta)$ (D goes to zero for large, increasing β) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.

Consider a loading of period T_p , its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j \omega_1 = j \frac{2\pi}{T_p},$$

where the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t) dt, \quad a_j = \frac{2}{T_p} \int_0^{T_p} p(t) \cos \omega_j t dt,$$
$$b_j = \frac{2}{T_p} \int_0^{T_p} p(t) \sin \omega_j t dt,$$

as, by orthogonality, $\int_0^{T_p} p(t) \cos \omega_j t dt = \int_0^{T_p} a_j \cos^2 \omega_j t dt = \frac{T_p}{2} a_j$, etc etc.

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.

Consider a loading of period T_p , its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j \omega_1 = j \frac{2\pi}{T_p},$$

where the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t) dt, \quad a_j = \frac{2}{T_p} \int_0^{T_p} p(t) \cos \omega_j t dt,$$

$$b_j = \frac{2}{T_p} \int_0^{T_p} p(t) \sin \omega_j t dt,$$

as, by orthogonality, $\int_0^{T_p} p(t) \cos \omega_j t dt = \int_0^{T_p} a_j \cos^2 \omega_j t dt = \frac{T_p}{2} a_j$, etc etc.

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.

Consider a loading of period T_p , its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j \omega_1 = j \frac{2\pi}{T_p},$$

where the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t) dt, \quad a_j = \frac{2}{T_p} \int_0^{T_p} p(t) \cos \omega_j t dt,$$

$$b_j = \frac{2}{T_p} \int_0^{T_p} p(t) \sin \omega_j t dt,$$

as, by orthogonality, $\int_0^{T_p} p(t) \cos \omega_j t dt = \int_0^{T_p} a_j \cos^2 \omega_j t dt = \frac{T_p}{2} a_j$, etc etc.

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Impulsive Loading

Response to General Dynamic Loadings

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

- (a) to divide the period in N equal parts $\Delta t = T_p/N$,
- (b) measure or compute $p(t)$ at a discrete set of instants t_1, t_2, \dots, t_N , with $t_m = m\Delta t$,

obtaining a discrete set of values p_m , $m = 1, \dots, N$ (note that $p_0 = p_N$ by periodicity).

Using the trapezoidal rule of integration, with $p_0 = p_N$ we can write, for example, the cosine-wave amplitude coefficients,

$$\begin{aligned} a_j &\approx \frac{2\Delta t}{T_p} \sum_{m=1}^N p_m \cos \omega_j t_m \\ &= \frac{2}{N} \sum_{m=1}^N p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^N p_m \cos \frac{jm 2\pi}{N}. \end{aligned}$$

It's worth to note that the discrete function $\cos \frac{jm 2\pi}{N}$ is periodic with period N .

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

- (a) to divide the period in N equal parts $\Delta t = T_p/N$,
- (b) measure or compute $p(t)$ at a discrete set of instants t_1, t_2, \dots, t_N , with $t_m = m\Delta t$,

obtaining a discrete set of values p_m , $m = 1, \dots, N$ (note that $p_0 = p_N$ by periodicity).

Using the trapezoidal rule of integration, with $p_0 = p_N$ we can write, for example, the cosine-wave amplitude coefficients,

$$\begin{aligned} a_j &\approx \frac{2\Delta t}{T_p} \sum_{m=1}^N p_m \cos \omega_j t_m \\ &= \frac{2}{N} \sum_{m=1}^N p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^N p_m \cos \frac{jm 2\pi}{N}. \end{aligned}$$

It's worth to note that the discrete function $\cos \frac{jm 2\pi}{N}$ is periodic with period N .

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

- (a) to divide the period in N equal parts $\Delta t = T_p/N$,
- (b) measure or compute $p(t)$ at a discrete set of instants t_1, t_2, \dots, t_N , with $t_m = m\Delta t$,

obtaining a discrete set of values p_m , $m = 1, \dots, N$ (note that $p_0 = p_N$ by periodicity).

Using the trapezoidal rule of integration, with $p_0 = p_N$ we can write, for example, the cosine-wave amplitude coefficients,

$$\begin{aligned} a_j &\approx \frac{2\Delta t}{T_p} \sum_{m=1}^N p_m \cos \omega_j t_m \\ &= \frac{2}{N} \sum_{m=1}^N p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^N p_m \cos \frac{jm 2\pi}{N}. \end{aligned}$$

It's worth to note that the discrete function $\cos \frac{jm 2\pi}{N}$ is periodic with period N .

Exponential Form

The Fourier series can be written in terms of the exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp i\omega_j t dt, \quad j = -\infty, \dots, +\infty.$$

For a sampled p_m we can write, using the trapezoidal integration rule and substituting $t_m = m\Delta t = m T_p/N$, $\omega_j = j 2\pi/T_p$:

$$P_j \approx \frac{1}{N} \sum_{m=1}^N p_m \exp\left(-i \frac{2\pi j m}{N}\right),$$

Exponential Form

The Fourier series can be written in terms of the exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_p} \int_0^{T_p} p(t) \exp i\omega_j t dt, \quad j = -\infty, \dots, +\infty.$$

For a sampled p_m we can write, using the trapezoidal integration rule and substituting $t_m = m\Delta t = m T_p/N$, $\omega_j = j 2\pi/T_p$:

$$P_j \cong \frac{1}{N} \sum_{m=1}^N p_m \exp\left(-i \frac{2\pi j m}{N}\right),$$

Undamped Response

We have seen that the steady-state response to the j th sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k} \left[\frac{1}{1 - \beta_j^2} \right] \sin \omega_j t, \quad \beta_j = \omega_j / \omega_n,$$

analogously, for the j th cosine-wave harmonic,

$$x_j = \frac{a_j}{k} \left[\frac{1}{1 - \beta_j^2} \right] \cos \omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k} \left\{ a_0 + \sum_{j=1}^{\infty} \left[\frac{1}{1 - \beta_j^2} \right] (a_j \cos \omega_j t + b_j \sin \omega_j t) \right\}.$$

Undamped Response

We have seen that the steady-state response to the j th sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k} \left[\frac{1}{1 - \beta_j^2} \right] \sin \omega_j t, \quad \beta_j = \omega_j / \omega_n,$$

analogously, for the j th cosine-wave harmonic,

$$x_j = \frac{a_j}{k} \left[\frac{1}{1 - \beta_j^2} \right] \cos \omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k} \left\{ a_0 + \sum_{j=1}^{\infty} \left[\frac{1}{1 - \beta_j^2} \right] (a_j \cos \omega_j t + b_j \sin \omega_j t) \right\}.$$

Damped Response

In the case of a damped oscillator, we must substitute the steady state response for both the j th sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{j=1}^{\infty} \frac{+(1 - \beta_j^2) a_j - 2\zeta\beta_j b_j}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2} \cos \omega_j t + \\ + \frac{1}{k} \sum_{j=1}^{\infty} \frac{+2\zeta\beta_j a_j + (1 - \beta_j^2) b_j}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2} \sin \omega_j t.$$

As usual, the exponential notation is neater,

$$x(t) = \sum_{j=-\infty}^{\infty} \frac{P_j}{k} \frac{\exp i\omega_j t}{(1 - \beta_j^2) + i(2\zeta\beta_j)}.$$

Response to
Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

In the case of a damped oscillator, we must substitute the steady state response for both the j th sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{j=1}^{\infty} \frac{+(1 - \beta_j^2) a_j - 2\zeta\beta_j b_j}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2} \cos \omega_j t + \\ + \frac{1}{k} \sum_{j=1}^{\infty} \frac{+2\zeta\beta_j a_j + (1 - \beta_j^2) b_j}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2} \sin \omega_j t.$$

As usual, the exponential notation is neater,

$$x(t) = \sum_{j=-\infty}^{\infty} \frac{P_j}{k} \frac{\exp i\omega_j t}{(1 - \beta_j^2) + i(2\zeta\beta_j)}.$$

Response to
Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

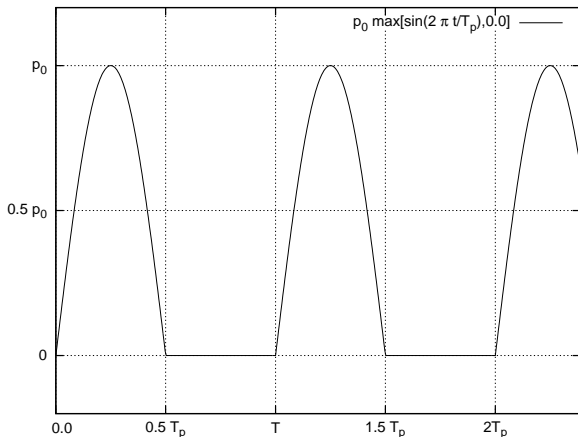
Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Example

As an example, consider the loading

$$p(t) = \max\left\{p_0 \sin \frac{2\pi t}{T_p}, 0\right\}$$



Response to
Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Example

As an example, consider the loading

$$p(t) = \max\left\{p_0 \sin \frac{2\pi t}{T_p}, 0\right\}$$

$$a_0 = \frac{1}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi t}{T_p} dt = \frac{p_0}{\pi},$$

$$a_j = \frac{2}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi t}{T_p} \cos \frac{2\pi j t}{T_p} dt = \begin{cases} 0 & \text{for } j \text{ odd} \\ \frac{p_0}{\pi} \left[\frac{2}{1-j^2} \right] & \text{for } j \text{ even,} \end{cases}$$

$$b_j = \frac{2}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi t}{T_p} \sin \frac{2\pi j t}{T_p} dt = \begin{cases} \frac{p_0}{2} & \text{for } j = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

Response to
Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Example cont.

Assuming $\beta_1 = 3/4$, from

$p = \frac{p_0}{\pi} \left(1 + \frac{\pi}{2} \sin \omega_1 t - \frac{2}{3} \cos 2\omega_1 t - \frac{2}{15} \cos 4\omega_2 t - \dots \right)$ with the dynamic amplification factors

$$D_1 = \frac{1}{1 - (1\frac{3}{4})^2} = \frac{16}{7},$$

$$D_2 = \frac{1}{1 - (2\frac{3}{4})^2} = -\frac{4}{5},$$

$$D_4 = \frac{1}{1 - (4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \dots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi} \left(1 + \frac{8\pi}{7} \sin \omega_1 t + \frac{8}{15} \cos 2\omega_1 t + \frac{1}{60} \cos 4\omega_1 t + \dots \right)$$

Take note, these solutions are particular solutions! If your solution has to respect given initial conditions, you must consider also the homogeneous solution.

Example cont.

Assuming $\beta_1 = 3/4$, from

$p = \frac{p_0}{\pi} \left(1 + \frac{\pi}{2} \sin \omega_1 t - \frac{2}{3} \cos 2\omega_1 t - \frac{2}{15} \cos 4\omega_2 t - \dots \right)$ with the dynamic amplification factors

$$D_1 = \frac{1}{1 - (1\frac{3}{4})^2} = \frac{16}{7},$$

$$D_2 = \frac{1}{1 - (2\frac{3}{4})^2} = -\frac{4}{5},$$

$$D_4 = \frac{1}{1 - (4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \dots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi} \left(1 + \frac{8\pi}{7} \sin \omega_1 t + \frac{8}{15} \cos 2\omega_1 t + \frac{1}{60} \cos 4\omega_1 t + \dots \right)$$

Take note, these solutions are particular solutions! If your solution has to respect given initial conditions, you must consider also the homogeneous solution.

Response to
Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the
Response

An example

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Response to Impulsive Loadings

SDOF linear
oscillator

G. Boffi

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to
Periodic Loading

Response to
Impulsive Loading

Introduction

Response to Half-Sine Wave
Impulse

Response for Rectangular
and Triangular Impulses

Shock or response spectra

Approximate Analysis of
Response Peak

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to
General Dynamic
Loadings

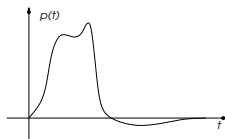
Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

An impulsive load is characterized

- ▶ by a single principal impulse, and
 - ▶ by a relatively short duration.
-
- ▶ Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
 - ▶ Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
 - ▶ For this reason, in the following we'll consider only the undamped response to impulsive loads.



Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

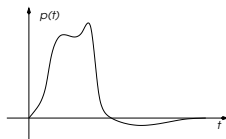
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Nature of Impulsive Loadings

An impulsive load is characterized

- ▶ by a single principal impulse, and
 - ▶ by a relatively short duration.
- ▶ Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
 - ▶ Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
 - ▶ For this reason, in the following we'll consider only the undamped response to impulsive loads.



Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Definition of Maximum Response

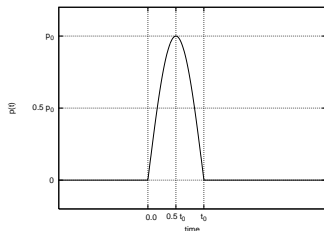
In general, when dealing with impulse response characterized by its duration t_0 we are interested either in

- a** the maximum of the absolute values of maxima (named also the *peak value*) of the response ratio $R(t)$ in $0 < t < t_0$ or,
- b** if we have no maxima during the excitation phase (i.e., $\dot{x} \neq 0$ in $0 < t < t_0$) we want to know the amplitude of the free vibrations that are excited by the impulse.

Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load. Note that $\omega t_0 = \pi$.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

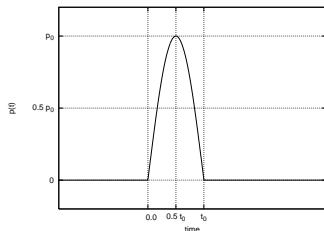
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load. Note that $\omega t_0 = \pi$.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

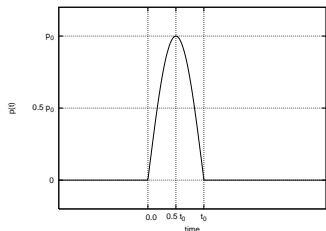
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load. Note that $\omega t_0 = \pi$.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Response to sine-wave impulse

Consider an undamped SDOF initially at rest, with natural circular frequency ω_n and stiffness k . With reference to a half-sine impulse with duration t_0 , the frequency ratio β is $\omega/\omega_n = T_n/2t_0$.

Its response ratio in the interval $0 < t < t_0$ is

$$R(t) = \frac{1}{1 - \beta^2} \left(\sin \omega t - \beta \sin \frac{\omega t}{\beta} \right) \quad [\text{NB: } \frac{\omega}{\beta} = \omega_n]$$

while for $t > t_0$ the response ratio is

$$R(t) = \frac{-\beta}{1 - \beta^2} \left(\left(1 + \cos \frac{\pi}{\beta} \right) \sin \omega_n (t - t_0) + \sin \frac{\pi}{\beta} \cos \omega_n (t - t_0) \right)$$

Maximum response to sine impulse

(a) Since we are interested in the maximum response ratio during the excitation, we need to know when velocity is zero in the time interval $0 \leq t \leq t_0$; from

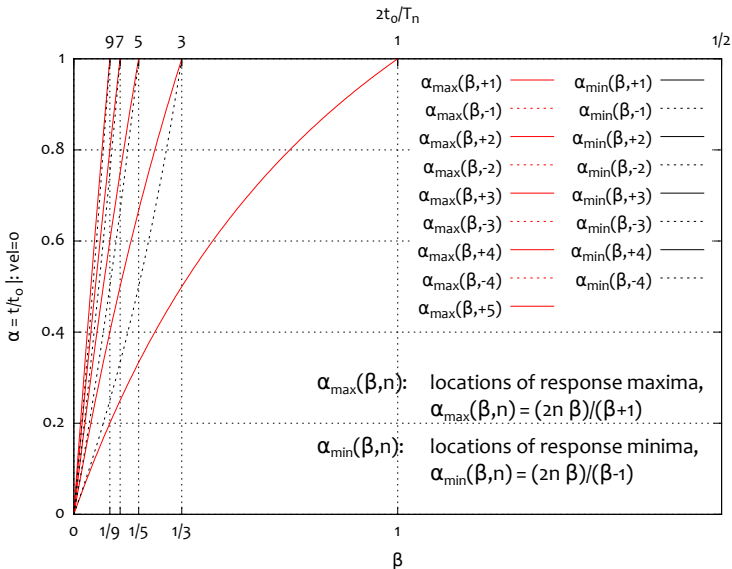
$$\dot{R}(t) = \frac{\omega}{1 - \beta^2} (\cos \omega t - \cos \frac{\omega t}{\beta}) = 0.$$

we can see that the roots are $\omega t = \mp \omega t / \beta + 2n\pi$, $n = 0, \mp 1, \mp 2, \mp 3, \dots$; it is convenient to substitute $\omega t = \pi\alpha$, where $\alpha = t/t_0$; substituting and solving for α one has

$$\alpha = \frac{2n\beta}{\beta \mp 1}, \quad \text{with } n = 0, \mp 1, \mp 2, \dots, \text{ for } 0 < \alpha < 1.$$

The next slide regards the characteristics of these roots.

$\alpha(\beta, n)$



Response to Periodic Loading

Response to Impulsive Loading

Introduction

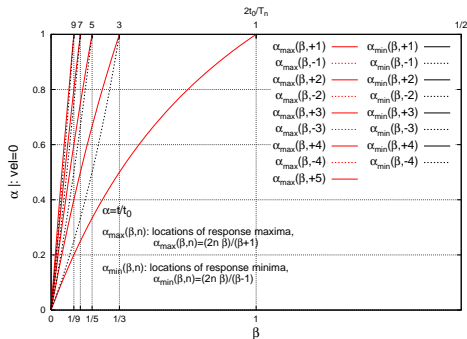
Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings



- No roots of type α_{\min} for $n > 0$;
- no roots of type α_{\max} for $n < 0$;
- no roots for $\beta > 1$, i.e., no roots for $t_0 < \frac{T_n}{2}$;
- only one root of type α_{\max} for $\frac{1}{3} < \beta < 1$, i.e., $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$;
- three roots, two maxima and one minimum, for $\frac{1}{5} < \beta < \frac{1}{3}$;
- five roots, three maxima and two minima, for $\frac{1}{7} < \beta < \frac{1}{5}$;
- etc etc.

In summary, to find the maximum of the response for an assigned $\beta < 1$, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k t_0$, (c) choose the maximum of the maxima.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

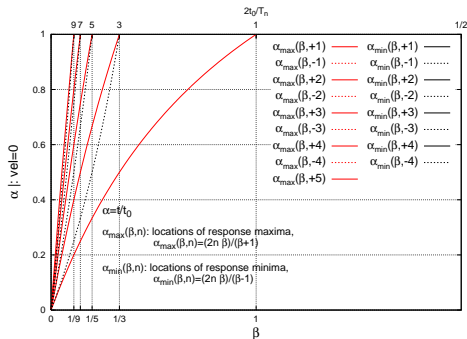
Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings



- No roots of type α_{\min} for $n > 0$;
- no roots of type α_{\max} for $n < 0$;
- no roots for $\beta > 1$, i.e., no roots for $t_0 < \frac{T_n}{2}$;
- only one root of type α_{\max} for $\frac{1}{3} < \beta < 1$, i.e., $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$;
- three roots, two maxima and one minimum, for $\frac{1}{5} < \beta < \frac{1}{3}$;
- five roots, three maxima and two minima, for $\frac{1}{7} < \beta < \frac{1}{5}$;
- etc etc.

In summary, to find the maximum of the response for an assigned $\beta < 1$, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k t_0$, (c) choose the maximum of the maxima.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Maximum response for $\beta > 1$

For $\beta > 1$, the maximum response takes place for $t > t_0$, and its absolute value (see slide *Response to sine-wave impulse*) is

$$R_{\max} = \frac{\beta}{1 - \beta^2} \sqrt{\left(1 + \cos \frac{\pi}{\beta}\right)^2 + \sin^2 \frac{\pi}{\beta}},$$

using a simple trigonometric identity we can write

$$R_{\max} = \frac{\beta}{1 - \beta^2} \sqrt{2 + 2 \cos \frac{\pi}{\beta}}$$

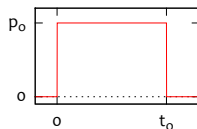
but

$1 + \cos 2\phi = (\cos^2 \phi + \sin^2 \phi) + (\cos^2 \phi - \sin^2 \phi) = 2 \cos^2 \phi$,
so that

$$R_{\max} = \frac{2\beta}{1 - \beta^2} \cos \frac{\pi}{2\beta}.$$

Consider a rectangular impulse of duration t_0 ,

$$p(t) = p_0 \begin{cases} 1 & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



The response ratio and its time derivative are

$$R(t) = 1 - \cos \omega_n t, \quad \dot{R}(t) = \omega_n \sin \omega_n t,$$

and we recognize that we have maxima $R_{\max} = 2$ for $\omega_n t = n\pi$, with the condition $t \leq t_0$. Hence we have no maximum during the loading phase for $t_0 < T_n/2$, and at least one maximum, of value $2\Delta_{st}$, if $t_0 \geq T_n/2$.

Response to
Periodic Loading

Response to
Impulsive Loading

Introduction

Response to Half-Sine Wave
Impulse

Response for Rectangular
and Triangular Impulses

Shock or response spectra

Approximate Analysis of
Response Peak

Response to
General Dynamic
Loadings

Rectangular Impulse (2)

For shorter impulses, the maximum response ratio is not attained during loading, so we have to compute the amplitude of the free vibrations after the end of loading (remember, as $t_0 \leq T_n/2$ the velocity is positive at $t = t_0$!).

$$R(t) = (1 - \cos \omega_n t_0) \cos \omega_n (t - t_0) + (\sin \omega_n t_0) \sin \omega_n (t - t_0).$$

The amplitude of the response ratio is then

$$\begin{aligned} A &= \sqrt{(1 - \cos \omega_n t_0)^2 + \sin^2 \omega_n t_0} = \\ &= \sqrt{2(1 - \cos \omega_n t_0)} = 2 \sin \frac{\omega_n t_0}{2}. \end{aligned}$$

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

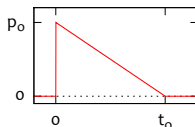
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Triangular Impulse

Let's consider the response of a SDOF to a triangular impulse,

$$p(t) = p_0 \left(1 - t/t_0\right) \text{ for } 0 < t < t_0$$



As usual, we must start finding the minimum duration that gives place to a maximum of the response in the loading phase, that is

$$R(t) = \frac{1}{\omega_n t_0} \sin \omega_n \frac{t}{t_0} - \cos \omega_n \frac{t}{t_0} + 1 - \frac{t}{t_0}, \quad 0 < t < t_0.$$

Taking the first derivative and setting it to zero, one can see that the first maximum occurs for $t = t_0$ for $t_0 = 0.37101T_n$, and substituting one can see that $R_{\max} = 1$.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Triangular Impulse (2)

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

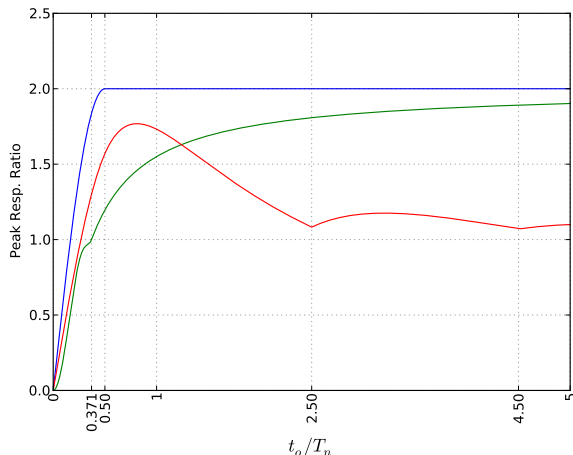
Approximate Analysis of Response Peak

Response to General Dynamic Loadings

For load durations shorter than $0.37101T_n$, the maximum occurs after loading and it's necessary to compute the displacement and velocity at the end of the load phase. For longer loads, the maxima are in the load phase, so that one has to find the all the roots of $\dot{R}(t)$, compute all the extreme values and finally sort out the absolute value maximum.

Shock or response spectra

We have seen that the response ratio is determined by the ratio of the impulse duration to the natural period of the oscillator. One can plot the maximum displacement ratio R_{\max} as a function of t_o/T_n for various forms of impulsive loads.



rectangular
triangular
half sine

Such plots are commonly known as displacement-response spectra, or simply as response spectra.

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor ≈ 1

On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the load. The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Response to
Periodic Loading

Response to
Impulsive Loading

Introduction

Response to Half-Sine Wave
Impulse

Response for Rectangular
and Triangular Impulses

Shock or response spectra

Approximate Analysis of
Response Peak

Response to
General Dynamic
Loadings

For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor ≈ 1

On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the load. The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Response to
Periodic Loading

Response to
Impulsive Loading

Introduction

Response to Half-Sine Wave
Impulse

Response for Rectangular
and Triangular Impulses

Shock or response spectra

Approximate Analysis of
Response Peak

Response to
General Dynamic
Loadings

Approximate Analysis (2)

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)] dt.$$

When one notes that, for small t_0 , the displacement is of the order of t_0^2 while the velocity is in the order of t_0 , it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \cong \int_0^{t_0} p(t) dt.$$

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Approximate Analysis (3)

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t) dt,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approx 0$, one has

$$x(t - t_0) \approx \frac{1}{m\omega_n} \int_0^{t_0} p(t) dt \sin \omega_n(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading (and will be discovered again in a few minutes).

Response to Periodic Loading

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to General Dynamic Loadings

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t) dt,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \cong 0$, one has

$$x(t - t_0) \cong \frac{1}{m\omega_n} \int_0^{t_0} p(t) dt \sin \omega_n(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading (and will be discovered again in a few minutes).

Response to
Periodic Loading

Response to
Impulsive Loading

Introduction

Response to Half-Sine Wave
Impulse

Response for Rectangular
and Triangular Impulses

Shock or response spectra

Approximate Analysis of
Response Peak

Response to
General Dynamic
Loadings

Response to General Dynamic Loading

SDOF linear oscillator

G. Boffi

Response to Periodic Loading

Introduction

Fourier Series Representation

Fourier Series of the Response

An example

Response to Periodic Loading

Response to Impulsive Loading

Response to General Dynamic Loadings

Response to Impulsive Loading

Introduction

Response to Half-Sine Wave Impulse

Response for Rectangular and Triangular Impulses

Shock or response spectra

Approximate Analysis of Response Peak

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau) d\tau$ and the response is

$$dx(t - \tau) = \frac{p(\tau) d\tau}{m\omega_n} \sin \omega_n(t - \tau), \quad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n(t - \tau) d\tau, \quad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Response to
Periodic Loading

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Response to infinitesimal
impulse

Numerical integration of
Duhamel integral

Undamped SDOF systems

Damped SDOF systems

For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau) d\tau$ and the response is

$$dx(t - \tau) = \frac{p(\tau) d\tau}{m\omega_n} \sin \omega_n(t - \tau), \quad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n(t - \tau) d\tau, \quad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Response to
Periodic Loading

Response to
Impulsive Loading

Response to
General Dynamic
Loadings

Response to infinitesimal
impulse

Numerical integration of
Duhamel integral

Undamped SDOF systems

Damped SDOF systems

The derivation of the equation of motion for a generic load is analogous to what we have seen for undamped SDOF, the infinitesimal contribution to the response at time t of the load at time τ is

$$dx(t) = \frac{p(\tau)}{m\omega_D} d\tau \sin \omega_D(t - \tau) \exp(-\zeta\omega_n(t - \tau)) \quad t \geq \tau$$

and integrating all infinitesimal contributions one has

$$x(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \sin \omega_D(t - \tau) \exp(-\zeta\omega_n(t - \tau)) d\tau, \quad t \geq 0.$$

Evaluation of Duhamel integral, undamped

Using the trig identity

$$\sin(\omega_n t - \omega_n \tau) = \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau$$

the Duhamel integral is rewritten as

$$\begin{aligned} x(t) &= \frac{\int_0^t p(\tau) \cos \omega_n \tau d\tau}{m\omega_n} \sin \omega_n t - \frac{\int_0^t p(\tau) \sin \omega_n \tau d\tau}{m\omega_n} \cos \omega_n t \\ &= \mathcal{A}(t) \sin \omega_n t - \mathcal{B}(t) \cos \omega_n t \end{aligned}$$

where

$$\begin{cases} \mathcal{A}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \cos \omega_n \tau d\tau \\ \mathcal{B}(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n \tau d\tau \end{cases}$$

Response to Periodic Loading

Response to Impulsive Loading

Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

Numerical evaluation of Duhamel integral, undamped

SDOF linear oscillator

G. Boffi

Response to Periodic Loading

Response to Impulsive Loading

Response to General Dynamic Loadings

Response to infinitesimal impulse

Numerical integration of Duhamel integral

Undamped SDOF systems

Damped SDOF systems

Usual numerical procedures can be applied to the evaluation of \mathcal{A} and \mathcal{B} , e.g., using the trapezoidal rule, one can have, with $\mathcal{A}_N = \mathcal{A}(N\Delta\tau)$ and $y_N = p(N\Delta\tau) \cos(N\Delta\tau)$

$$\mathcal{A}_{N+1} = \mathcal{A}_N + \frac{\Delta\tau}{2m\omega_n} (y_N + y_{N+1}) .$$

Evaluation of Duhamel integral, damped

For a damped system, it can be shown that

$$x(t) = \mathcal{A}(t) \sin \omega_D t - \mathcal{B}(t) \cos \omega_D t$$

with

$$\mathcal{A}(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \frac{\exp \zeta \omega_n \tau}{\exp \zeta \omega_n t} \cos \omega_D \tau d\tau,$$

$$\mathcal{B}(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) \frac{\exp \zeta \omega_n \tau}{\exp \zeta \omega_n t} \sin \omega_D \tau d\tau.$$

Numerical evaluation of Duhamel integral, damped

Numerically, using e.g. Simpson integration rule and

$$y_N = p(N\Delta\tau) \cos \omega_D \tau,$$

$$\mathcal{A}_{N+2} = \mathcal{A}_N \exp(-2\zeta\omega_n\Delta\tau) +$$

$$\frac{\Delta\tau}{3m\omega_D} \left[y_N \exp(-2\zeta\omega_n\Delta\tau) + 4y_{N+1} \exp(-\zeta\omega_n\Delta\tau) + y_{N+2} \right]$$

$$N = 0, 2, 4, \dots$$