Continuous Systems

an example

A constant load P is moving on a simply supported beam of length L with constant velocity, $v(t) = v = \text{const}$. The load enters the beam at $t = 0$ and exits at $t = L/v = t_0$; the beam is uniform, i.e., $m(x) = m = \text{const}$ and $EJ(x) = EJ = \text{const}$.

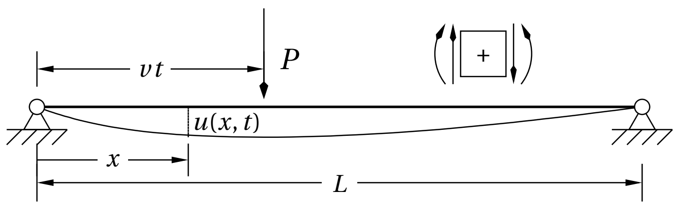

Figure 1: the beam, the load and the sign conventions.

Assume that the beam is horizontal and the load P is vertical, indicate the transverse, vertical displacements of the beam with $u(x,t)$, positive if directed as the vertical load, the bending moment with $M_b(x,t)$, positive if it extends the bottom fibers of the beam and the shear force with $V(x,t)$, positive if clockwise.

Plot the response in the interval $0 \le t \le t_0 = L/v$ in terms of $u(L/2, t)$ and $M_b(L/2, t)$.

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Solution

The equation of dynamic equilibrium (specialized for an uniform beam) is

$$m\frac{\partial^2 u(x,t)}{\partial t^2} + EJ\frac{\partial^4 u(x,t)}{\partial x^4} = p(x,t)$$

where

$$p(x,t) = P\delta(x - vt)$$

with the Dirac's delta defined by $\delta(x - x_0) \equiv 0$ and $\int f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0)$.

The solution will be computed by separation of variables

$$u(x,t) = q(t)\phi(x)$$

and modal analysis,

$$u(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\beta_n = \frac{n\pi}{L}, \qquad \phi_n(x) = \sin\beta_n x = \sin\frac{n\pi x}{L},$$
$$m_n = \frac{mL}{2}, \qquad \omega_n^2 = \beta_n^4\frac{EJ}{m} = n^4\pi^4\frac{EJ}{mL^4}.$$

The orthogonality relationships are

$$\int_0^L \phi_n(x)m(x)\phi_m(x)\,\mathrm{d}x = m_n\delta_{nm},$$
$$\int_0^L \phi_n(x)[EJ(x)\phi_m''(x)]''\,\mathrm{d}x = k_n\delta_{nm} = m_n\omega_n^2\delta_{nm}$$

(the Kroneker's δ is a completely different thing from Dirac's δ, OK?).

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates

$$m_n\ddot{q}(t)+k_nq(t) = \int_0^L \phi_n(x)p(x,t)\,\mathrm{d}x = P\phi_n(vt) = P\sin\frac{n\pi\,vt}{L}, \qquad n = 1,2,\ldots,\infty,$$

considering that the initial conditions are nil for all the modal equations, with $\overline{\omega}_n = n\pi v/L$ and $\beta_n = \overline{\omega}_n/\omega_n$ the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n\sin\omega_n t\right), \qquad 0 \le t \le \frac{L}{v}.$$

With the position

$$v = \kappa\omega_1 L/\pi$$

it is $\beta_n = n\kappa\omega_1/n^2\omega_1 = \kappa/n$ and we can rewrite the solution as

$$q_n(t) = \frac{2PL^3}{\pi^4EJ}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(\frac{\kappa}{n}\omega_n t) - \frac{\kappa}{n}\sin\omega_n t\right), \qquad 0 \le t \le \frac{L}{v},$$

and it is apparent that exists a critical velocity $v_\mathrm{c} = \omega_1 L/\pi$ that gives a resonance condition for the first mode response, while for $v = 2\,v_\mathrm{c}$ the second mode is in resonance, etc.

Introducing an adimensional time coordinate ξ with $t = t_0\xi$, noting that $\omega_n = n^2\omega_1$ we can write

$$q_n(\xi) = \frac{2PL^3}{\pi^4 EJ}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right), \qquad 0 \le \xi \le 1.$$

If we denote with $x_\mathrm{P}(t)$ the position of the load at time t, it is $x_\mathrm{P}(t) = vt = \xi L$, or $\xi = x_\mathrm{P}/L$ and the expression $u(x,\xi) = \sum q_n(\xi)\phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in ξL.

The displacement and the bending moment are given by

$$u(x,\xi) = \frac{2PL^3}{\pi^4 EJ}\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\pi\frac{x}{L}),$$

$$M_\mathrm{b}(x,\xi) = -EJ\frac{\partial^2 u(x,\xi)}{\partial x^2}$$

$$= \frac{2PL}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2-\kappa^2}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\pi\frac{x}{L}).$$

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_\mathrm{stat}(L/2,1/2) = \frac{PL^3}{48EJ} \qquad \text{and} \qquad M_{\mathrm{b\,stat}}(L/2,1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\frac{u(L/2,\xi)}{u_\mathrm{stat}(L/2,1/2)} = \Delta(\xi) = \frac{96}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}),$$

the partial sum of the first N terms will be denoted by

$$\Delta_N(\xi) = \frac{96}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\frac{\pi}{2}).$$

Analogously, normalizing with respect to the maximum static bending moment, it is

$$\frac{M_\mathrm{b}(x,\xi)}{M_{\mathrm{b\,stat}}(L/2,1/2)} = \mu(\xi) == \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2-\kappa^2}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\pi\frac{x}{L}),$$

the partial sum being denoted by

$$\mu_N(\xi) = \frac{8}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2-\kappa^2}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)\sin(n\pi\frac{x}{L}).$$

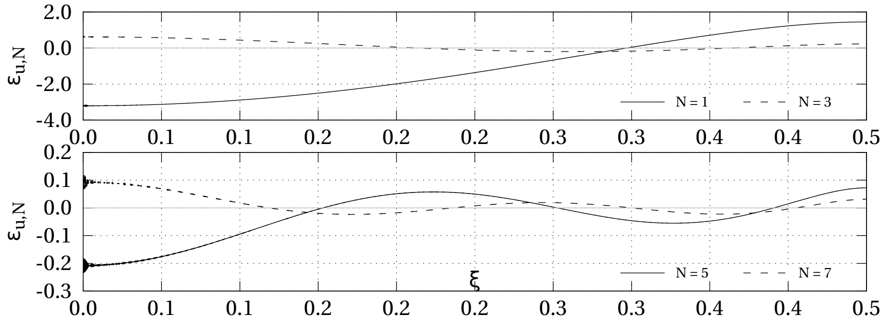

Figure 2: displacement error function.

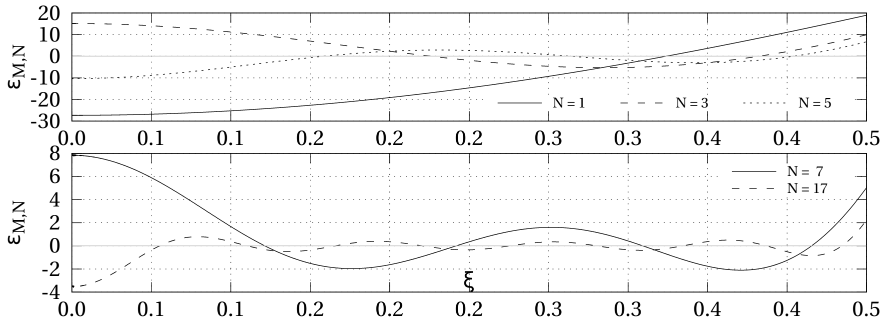

Figure 3: bending moment error function.

The $PL^3/48EJ$ normalized midspan statical displacement, that you can compute using Betti's theorem, is $\Delta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ for $0 \leq \xi \leq 1/2$ and we can define a percent error function (using $\kappa = 10^{-6}$ to obtain a good approximation to the static response)

$$\epsilon_{u,N}(\xi) = 100 \left(1 - \frac{\Delta_N(\xi)|_{\kappa=10^{-6}}}{\Delta_{\text{stat}}(\xi)}\right) \qquad \text{for } 0 \leq \xi \leq 1/2,$$

that you can see plotted in figure 2. With 5 terms in the series, you have an approximation of about $1/1000$.

Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\text{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100 \left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)}\right)$$

that you can see plotted in figure 3. With 17 terms in the series, you have an approximation of about 4%.

Finally, we plot the normalized displacement (figure 4) and the normalized bending moment (figure 5) for different values of κ; note that for the displacement I used $N = 11$ while for the bending moment I used $N = 25$.

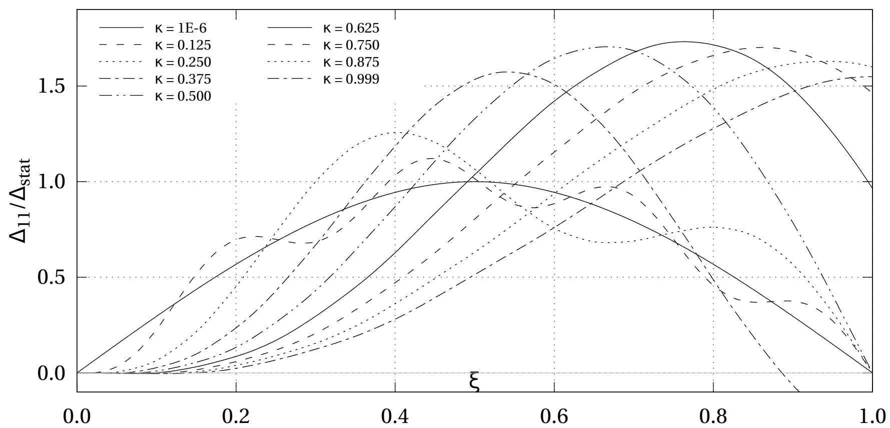

Figure 4: normalized midspan displacement.
(for different velocities $v = \kappa\, v_c$)

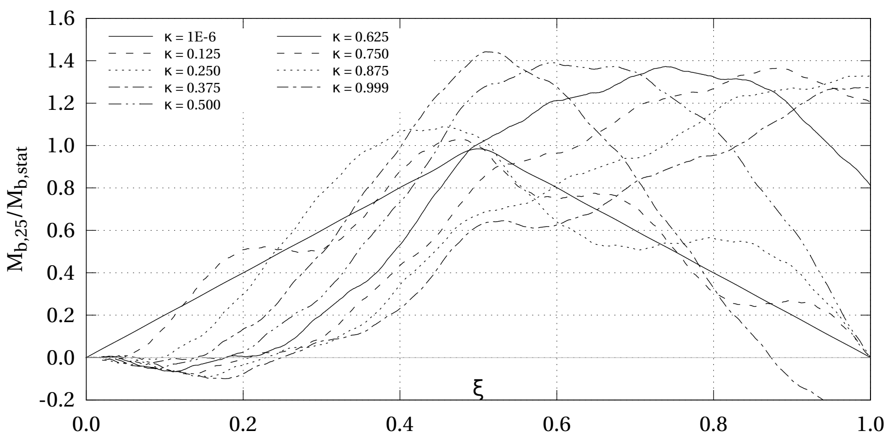

Figure 5: normalized midspan bending moment.
(for different velocities $v = \kappa\, v_c$)