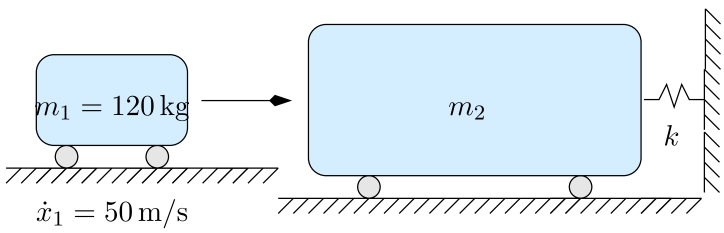
Dynamics of Structures 2010-2011

1st home assignment due on Tuesday 2011-06-17

Solutions

Contents

1 Impact



A body of mass $m_1 = 120\,\text{kg}$ hits an undamped *SDOF* system, of unknown characteristics k and m_2, with velocity $\dot{x}_1 = 50\,{}^{\text{m}}\!/\!_{\text{s}}$.

The collision is anelastic, i.e., the two masses are *glued* together and a measurement of the ensuing free oscillations gives the following results:

$$x_{\max} = 30\,\text{mm}, \qquad \dot{x}_{\max} = 60\,{}^{\text{mm}}\!/\!_{\text{s}}.$$

Compute:

1. the total mass $m = m_1 + m_2$
2. the mass m_2 of the impacted body,
3. the circular frequency of the insuing motion,
4. the spring stiffness k.

1.1 Solution

After the collision the two *glued* bodies have the same velocity, so that by the law of conservation of the momentum we can write

$$m_1 \times \dot{x}_1 + m_2 \times 0 = (m_1 + m_2) \times \dot{x}_0,$$

where we have denoted the initial velocity of the compound body with $\dot{x}_0$.

Observing that the initial conditions for the compound are $x(0) = 0$ and $\dot{x}(0) = \dot{x}_0$ we can write

$$x(t) = \frac{\dot{x}_0}{\omega_n} \sin \omega_n t, \qquad \dot{x}(t) = \dot{x}_0 \cos \omega_n t.$$

$$x_{max} = \frac{\dot{x}_0}{\omega_n} = 30\,\text{mm}, \qquad \dot{x}_{max} = \dot{x}_0 = 60\,\text{mm/s}$$

and hence

$$\dot{x}_0 = 60\,\text{mm/s}, \qquad \omega_n = \frac{\dot{x}_{max}}{x_{max}} = 2\,\text{rad/s}.$$

Substituting $\dot{x}_0 = 60\,\text{mm/s}$ in $m_1 \times \dot{x}_1 + m_2 \times 0 = (m_1 + m_2) \times \dot{x}_0$ we have

$120\,\text{kg} \times 50\,000\,\text{mm/s} = (120\,\text{kg} + m_2) \times 60\,\text{mm/s} \Rightarrow m = 100\,000\,\text{kg} \Rightarrow m_2 = 99\,880\,\text{kg}$.

As for the last question, it is $k = \omega_n^2\, m$ and substituting we find

$$k = (2\,\text{rad/s})^2\, 100\,000\,\text{kg} = 400\,000\,\text{N/m}.$$

2 Vibration Isolation — Numerical Integration

A rotating machine, its mass $M = 35\,000\,\text{kg}$, is rigidly connected to the floor.

Due to unbalances, during steady-state regime the machine is subjected to a harmonic force $p(t) = 1\,\text{kN} \sin(2\pi\, 5\,\text{Hz}\, t)$.

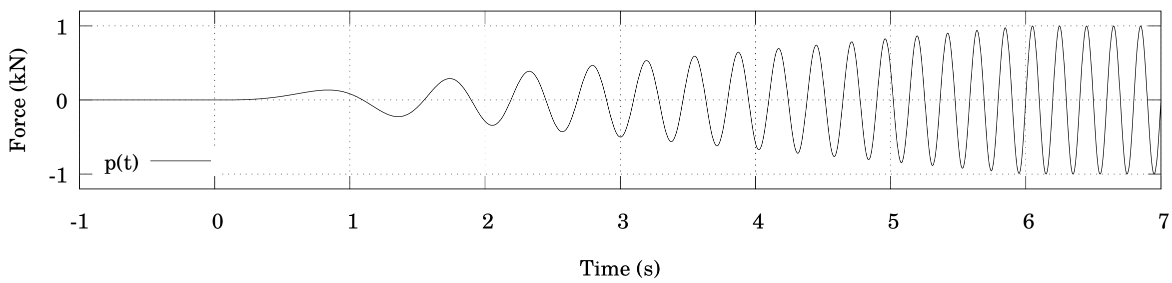
2.1 Vibration Isolation

Considering the floor fixed, design an appropriate suspension system such that the steady-state transmitted force is reduced to $300\,\mathrm{N}$.

2.2 Numerical Integration

When the machine is turned on, its full velocity is reached in $6\,\mathrm{s}$. The angular velocity and the unbalanced load vary linearly, from 0 to their respective maximum values, i.e.,

$$p(t) = \begin{cases} 1\,\mathrm{kN}\,\frac{t}{6\,\mathrm{s}}\ \sin\left(2\pi\,2.5\,\mathrm{Hz}\,\frac{t^2}{6\,\mathrm{s}}\right) & 0\,\mathrm{s} \le t \le 6\,\mathrm{s}, \\ 1\,\mathrm{kN}\ \sin(2\pi\,5\,\mathrm{Hz}\,t) & 6\,\mathrm{s} \le t. \end{cases}$$



Using the stiffness computed in the previous step, find the maximum absolute value of the displacement using either the constant or the linear acceleration method and plot the response in the interval $0\,\mathrm{s} \le t \le 10\,\mathrm{s}$.

2.3 Solution

2.3.1 Vibration Isolation

With ω_n being the natural frequency of the system composed by the machine and the suspension springs, $\beta = \frac{\omega}{\omega_\mathrm{n}}$ the frequency ratio, the condition on the maximum tranmitted load is

$$f_\mathrm{max} = \frac{p_0}{\beta^2 - 1} \le 300\,\mathrm{N}.$$

Substituting $p_0 = 1000\,\mathrm{N}$ in the equation above, we have

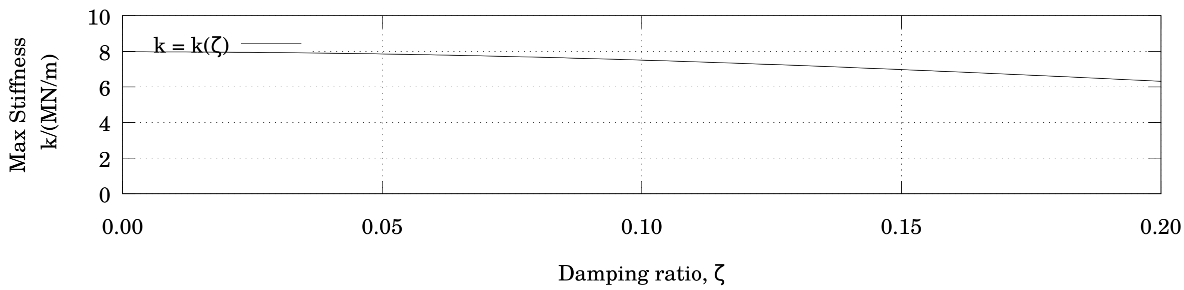
$$\beta^2 = \frac{\omega^2}{k/m} \ge \frac{13}{3} \quad \Rightarrow \quad k \le \frac{3}{13}(\pi\,10\,{}^\mathrm{rad}/_\mathrm{s})^2 35\,000\,\mathrm{kg} = 7\,971\,603\,{}^\mathrm{N}/_\mathrm{m}$$

If we accept a small damping, it must be

$$\mathrm{TR} = \frac{\sqrt{1^2 + (2\beta\zeta)^2}}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}} = \frac{\sqrt{1 + 4\beta^2\zeta^2}}{\sqrt{(1-\beta^2)^2 + 4\beta^2\zeta^2}} \le 0.4$$

with $\omega = 2\pi 5\,^{\mathrm{rad}}\!/\!_{\mathrm{s}}$ and $\omega_{\mathrm{n}}^2 = k/m$, substituting the actual value of m, solving the quadratic equation in k and discarding the negative root, it is finally found

$$k \le \left(\sqrt{(182\zeta^2+9)^2 + 819} - \left(182\zeta^2+9\right)\right)\frac{\pi^2}{26}\,^{\mathrm{MN}}\!/\!_{\mathrm{m}}$$



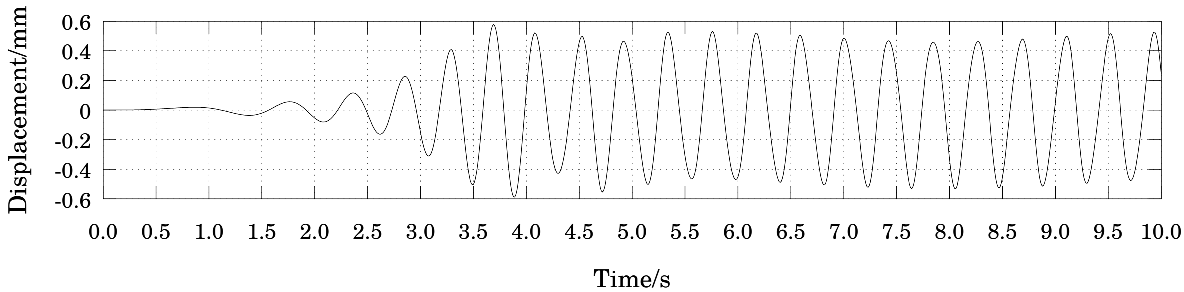
2.3.2 Numerical Integration

The response can be computed and printed with the following program

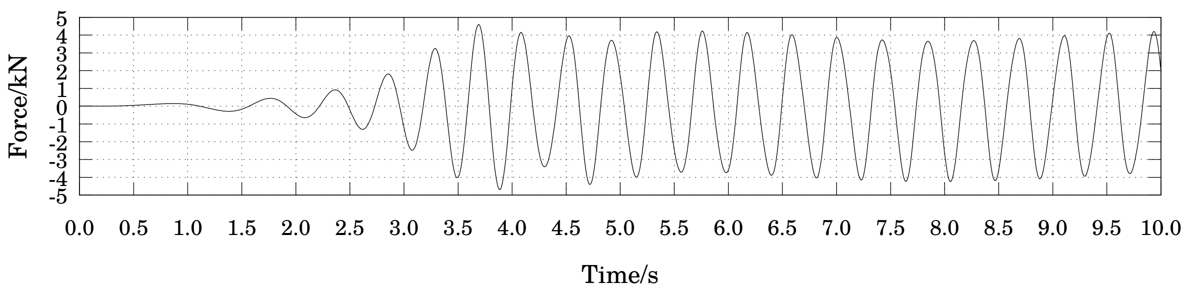
```
from math import pi, sin, sqrt
print "#      t/s          x/m        fs/N",
print "    v/(m/s)  a/(m/s/s)        p/N"
fmt  = "%+10.4f " + 4*"%+10.4g " + "%+10.4f"

m    = 35000.0   # mass of machine, kg
c    = 0.000     # assumed damping, Newton/(meter/second)
p_0  = 1000.0    # applied force, Newton
freq = 2*pi*5.0  # its circular freq., rad/second
t_0  = 6.0       # duration of transient, second
dur  = 12.0      # total duration, second
h    = 0.010     # time step, second
k    = 7971604.0 # design stiffness, Newton/meter
def load(t):
    if t > t_0: return p_0 * sin(freq*t)
    else: return p_0*t/t_0 * sin(freq*t*t/t_0/2)

# linear acceleration coefficients
```

```
a_fac = c*h/2.0 + 3.0*m;    v_fac = 3.0*c + 6.0*m/h
stiff = k + (3.0*c + 6.0*m/h)/h

# initial state
t = 0
x, v, p = 0.0, 0.0, load(t)
a = (p - c*v - k*x) / m
print fmt % (t, x, x*k, v, a, p)
# iterate with linear acceleration algorithm
while (t + h/2) < dur:
    t  = t + h
    dp = load(t) - p
    dx = (dp + a*a_fac + v*v_fac)/stiff
    dv = 3.0*dx/h - 3*v - a*h/2.0
    x, v, p = x + dx, v + dv, p + dp
    a  = (p - c*v - k*x) / m
    print fmt % (t, x, x*k, v, a, p)
```

and the response time history is



but these displacements are a bit meaningless, let's try to plot $f_{\mathrm{S}} = k\,x$



oh my, it's $f_{\mathrm{S}} \approx 5\,\mathrm{kN}$! just a moment, the machine weight is about $350\,\mathrm{kN}$ so this harmonic force is less than 1/70 of the weight, it shouldn't be a structural problem... on the other hand, there is no dissipation and the effects of the transient become permanent, it should be obvious that we have to use a dissipative device.

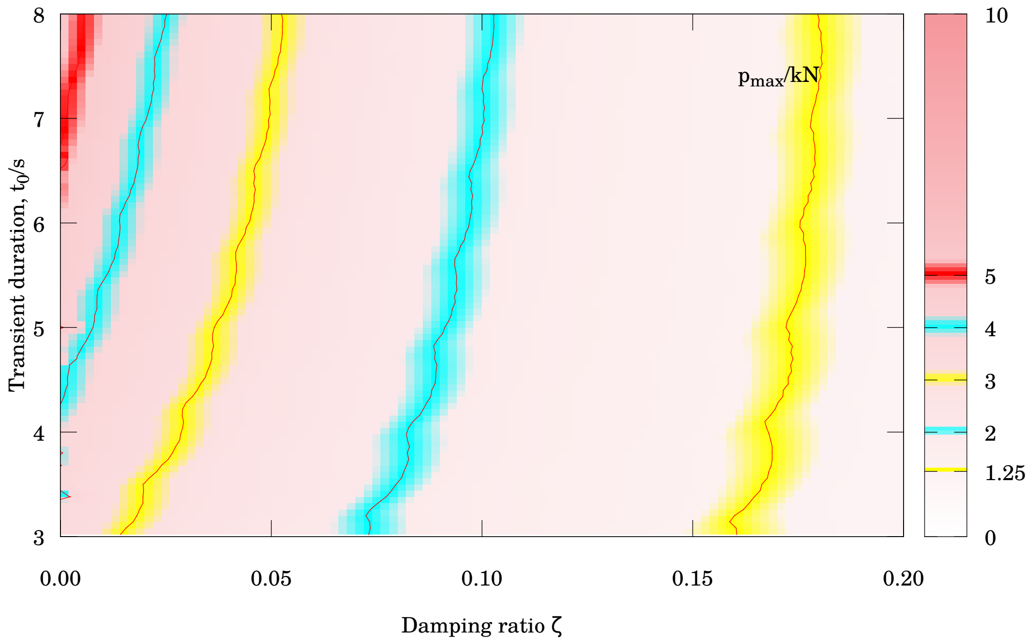
To get an appreciation of the problem, I have modified the previous program so that the integration is run for different values of t_0, the duration of transient, and different values of the damping ratio ζ; for each value of ζ k is given by the formula we have seen before,

$$k \leq \left(\sqrt{(182\zeta^2 + 9)^2 + 819} - \left(182\zeta^2 + 9\right) \right) \frac{\pi^2}{26} \,{}^{\mathrm{MN}}\!/_{\mathrm{m}}$$

and the damping coefficient is computed by

$$\omega_\mathrm{n} = \sqrt{k/m}, \qquad c = 2\zeta\omega_\mathrm{n} m.$$

For each iteration on ζ, t_0 the program prints the peak value of the transmitted force, and finally the results are plotted as a colour map



3 Estimation of damping ratio

You want to determine the mass m, the stiffness k and the damping ratio ζ of a one storey building that can be modeled as a single degree of freedom system.

A series of 4 dynamical test is performed, loading the building with a vibrodyne and measuring the amplitude ρ and the phase difference θ of the steady state motion (note that the measures of ρ and θ are affected by a random measurement error).

In each test the load amplitude is $p_0 = 600\,\mathrm{N}$, while the excitation frequencies ω_n (with $n = 1, \ldots, 4$) are different.

The relevant data is summarized in the following table

n	$\omega_n(\mathrm{rad}/\mathrm{s})$	$\rho_n(\mu\mathrm{m})$	$\theta_n(\mathrm{deg})$
1	40	12.39062	7.58258
2	50	41.09556	33.33505
3	60	18.07490	163.21210
4	70	7.11246	171.69968

Give your best estimate of m, ζ and k.

3.1 Solution

When you have a linear system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ with more equations than unknowns, it is usually solved under the hypotesis that the *best* solution is the solution that minimises the sum of the squares of the residuals $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}$. In a vector notation, the sum of the squares of the residuals is

$$
\begin{aligned}
\boldsymbol{r}^T \cdot \boldsymbol{r} &= (\boldsymbol{b}^T - \boldsymbol{x}^T \boldsymbol{A}^T)(\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}) \\
&= \boldsymbol{b}^T \boldsymbol{b} + \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{b} - \boldsymbol{b}^T \boldsymbol{A}\,\boldsymbol{x}
\end{aligned}
$$

the last term is the transpose of the previous one, both are scalars, so we can write

$$
\boldsymbol{r}^T \cdot \boldsymbol{r} = \boldsymbol{b}^T \boldsymbol{b} + \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{A}\,\boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{b}.
$$

The square of the residual is positive definite, and its minimum is achieved when all the partial derivatives with respect to the x_i are equal to zero,

$$
\frac{\partial(\boldsymbol{r}^T \cdot \boldsymbol{r})}{\partial x_i} = 0, \qquad i = 1, \ldots, N.
$$

These N equations can be conveniently be expressed in matrix format,

$$2(\boldsymbol{A}^T\boldsymbol{A}\,\boldsymbol{x}-\boldsymbol{A}^T\boldsymbol{b})=0 \quad\Rightarrow\quad \boldsymbol{A}^T\boldsymbol{A}\,\boldsymbol{x}=\boldsymbol{A}^T\boldsymbol{b}.$$

In our case, it is

$$1\cdot k-\omega_n^2\,m=p_0\frac{\cos\theta_n}{\rho_n},\qquad n=1,\ldots,4$$

or, matricially

$$\begin{bmatrix}1 & -1600\\ 1 & -2500\\ 1 & -3600\\ 1 & -4900\end{bmatrix}\begin{Bmatrix}k\\ m\end{Bmatrix}=\begin{Bmatrix}+48.000\\ +12.198\\ -31.780\\ -83.475\end{Bmatrix}\,{}^{\mathrm{MN}}\!/_{\mathrm{m}}$$

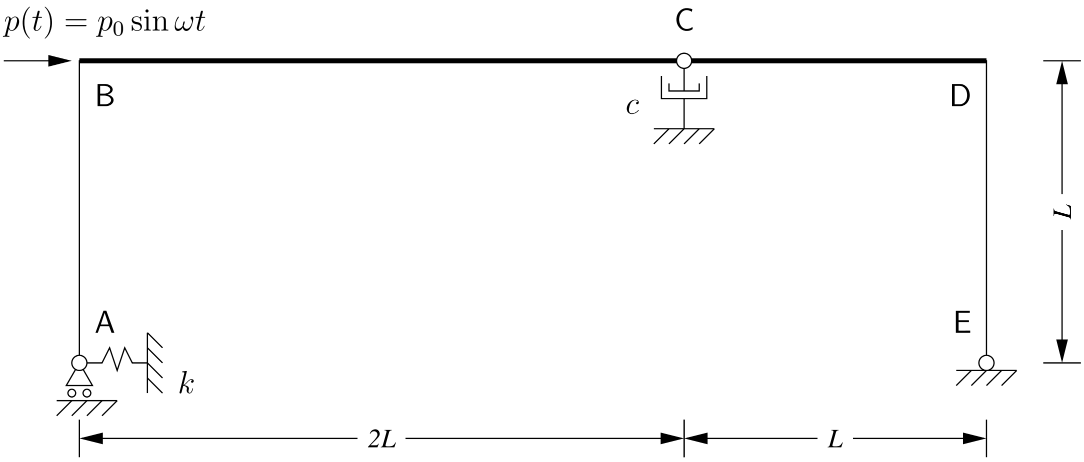
(note that the coefficients of m are, dimensionally, a square frequency) premultiplying both members by the transpose of the coefficient matrix we write

$$\begin{cases} +4\cdot k & -2600\,{}^1\!/_{\mathrm{s}^2}\cdot m & = & -55.058\times10^6\,{}^{\mathrm{N}}\!/_{\mathrm{m}}\\ -2600\cdot k & +45\,780\,000\,{}^1\!/_{\mathrm{s}^2}\cdot m & = & +416.14\times10^{+9}\,{}^{\mathrm{N}}\!/_{\mathrm{m}}\end{cases},$$

where the second equation was multiplied by s^2.

Solving the previos linear system gives the best estimates $k=111.78\times10^6\,{}^{\mathrm{N}}\!/_{\mathrm{m}}$ and $m=39854\mathrm{kg}$, in good agrement with the data entered in the simulation. The damping ratio, computations omitted, is $\zeta\approx3.8\%$.

4 Generalised Coordinates (rigid bodies)



The articulated system in figure, composed by

- two rigid bars, (1) ABC and (2) CDE,
- three fixed constraints, (1) a horizontal roller in A, (2) an internal hinge in C and (3) a hinge in E,
- two deformable constraints, (1) a horizontal spring in A, its stiffness $=$ k and (2) a vertical dashpot in C, its damping coefficient $= c$,

is excited by a horizontal harmonic force applied in B, $p(t) = p_0 \sin \omega t$.

The vertical parts of the two bars, AB and ED, are massless while both the horizontal parts, BC and CD, have a constant unit mass $\overline{m}$, with $\overline{m}\, L = m$.

Using u_A (the horizontal displacement of A) as the generalised coordinate

1. compute the generalised parameters m^*, c^* and k^*,
2. compute the generalised loading $p^*(t)$ and
3. write the equation of dynamic equilibrium.

4.1 Solution

Our sistem of reference will be centred in A, so that the positions of the Center of Instantaneous Rotation (CIR) for the two rigid bodies are $\Omega_1 = (0, 3L)$ and $\Omega_2 = (3L, 0) \equiv$ E. Using $Z = u_\mathsf{A}$ as our free coordinate, the rotation about Ω_1 is $\theta_1 = +1/3\, Z/L$, the rotation about Ω_2 is $\theta_2 = -2/3\, Z/L$ (anticlockwise rotations are positive), and then we can compute the relevant displacements

	u/Z	v/Z
A	1	0
B	2/3	0
C	2/3	2/3
G_1	2/3	1/3
G_2	2/3	1/3

For equilibrium, the external virtual work δW_E (work of the external, spring, damper and inertial forces) equals to the internal virtual work, δW_I but for a rigid system it is $\delta W_\mathrm{I} = 0$, hence our equilibrium equation is

$$\delta W_\mathrm{E} = 0.$$

In detail,

$$
\begin{aligned}
&p(t)\,\frac{2}{3}\delta Z+(-kZ)\,\delta Z+(-c\frac{2}{3}\dot{Z})\,(\frac{2}{3}\delta Z)+\\
&\quad(-(2m)\frac{2}{3}\ddot{Z})\,(\frac{2}{3}\delta Z)+(-(2m)\frac{1}{3}\ddot{Z})\,(\frac{1}{3}\delta Z)+(-\frac{2m(2L)^2}{12}\frac{\ddot{Z}}{3L})\,(\frac{1}{3L}\delta Z)+\\
&\qquad(-m\frac{2}{3}\ddot{Z})\,(\frac{2}{3}\delta Z)+(-m\frac{1}{3}\ddot{Z})\,(\frac{1}{3}\delta Z)+(\frac{mL^2}{12}\frac{2\ddot{Z}}{3L})\,(-\frac{2}{3L}\delta Z)=0
\end{aligned}
$$

simplyfying δZ, collecting Z and its derivatives, moving Z's on the right side of the equation, it is

$$
\begin{aligned}
\frac{2}{3}p(t)&=k\,Z+\frac{4}{9}c\,\dot{Z}+(\frac{8}{9}+\frac{2}{9}+\frac{2}{27}+\frac{4}{9}+\frac{1}{9}+\frac{1}{27})m\,\ddot{Z}\\
&=\frac{16}{9}m\,\ddot{Z}+\frac{4}{9}c\,\dot{Z}+k\,Z
\end{aligned}
$$

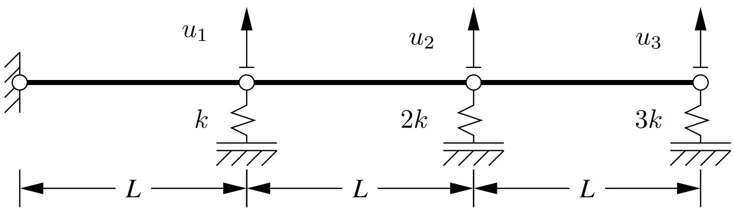
The required answers are

$$m^*=\frac{16}{9}m,\quad c^*=\frac{4}{9}c,\quad k^*=k,\quad p^*=\frac{2}{3}p(t),$$

and

$$\frac{16}{9}m\,\ddot{Z}+\frac{4}{9}c\,\dot{Z}+k\,Z=\frac{2}{3}p(t)$$

5 Rayleigh quotient



The undamped 3 *DOF* system in figure is composed of 3 identical rigid bars, their masses $m_i = m$, and three vertical springs, their stiffnesses as detailed in figure. Starting with a trial shape $\phi = \{1\ \ 1\ \ 1\}^T$ so that $u_1 = u_2 = u_3 = Z_0 \sin\omega t$, give the successive Rayleigh estimates of (squared) free vibration circular frequency R_{00}, R_{01} and R_{11}.

Note (1) that the bars have a not negligible rotatory inertia: $J_i = mL^2/12$, that you must take into account and (2) that the free coordinates are not referred to the centres of mass of the bars (hence a non-diagonal mass matrix).

HINT: the nodal inertial forces are $\boldsymbol{f}_{\mathrm{I}} = \boldsymbol{M}\,\ddot{\boldsymbol{u}}$, the mass matrix's coefficients can be deduced comparing an explicit derivation of the kinetic energy T in terms of the velocities $\dot{u}_i$, the mass m and the inertia J to the matrix expression $T = \frac{1}{2}\dot{\boldsymbol{u}}^T\boldsymbol{M}\,\dot{\boldsymbol{u}} = \frac{1}{2}\left(m_{11}\,\dot{x}_1^2 + \cdots + (m_{12}+m_{21})\,\dot{x}_1\dot{x}_2 + \cdots\right)$, where $m_{ij} = m_{ji}$.

5.1 Solution

The displacements x_i of the 3 centres of mass are

$$x_1 = u_1/2, \qquad x_2 = (u_2+u_1)/2 \qquad x_3 = (u_3+u_2)/2),$$

the rotations θ_i are

$$\theta_1 = u_1/L, \qquad \theta_2 = (x_2-x_1)/L \qquad \theta_3 = (x_3-x_2)/L.$$

The kinetic energy is, summing the contributions from the three identical bars,

$$T = \frac{1}{2}\left(m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) + \frac{mL^2}{12}(\dot{\theta}_1^2+\dot{\theta}_2^2+\dot{\theta}_3^2)\right).$$

Substituting the free coordinates, simplifying and collecting the similar terms, it is

$$T = \frac{1}{2}\,\left(8\,\dot{u}_1^2 + 8\,\dot{u}_2^2 + 4\,\dot{u}_3^2 + 4\,\dot{u}_1\dot{u}_2 + 4\,\dot{u}_2\dot{u}_3 + 0\,\dot{u}_3\dot{u}_1\right)\,\frac{m}{12}.$$

The kinetic energy can be expressed also by a matrix product,

$$T = \frac{1}{2}\,\boldsymbol{u}^T\boldsymbol{M}\,\boldsymbol{u} = \frac{1}{2}\left(m_{11}\dot{u}_1^2 + \cdots + 2\,m_{12}\dot{u}_1\dot{u}_2 + \cdots\right),$$

equating the two right members term by term, we deduce that the mass matrix coefficients are as in

$$\boldsymbol{M} = \frac{m}{6}\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}.$$

The stiffness matrix, simply, is

$$\boldsymbol{K} = k\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

The Rayleigh procedure starts writing

$$\boldsymbol{u}(t) = \boldsymbol{\phi} Z_0 \sin\omega t, \qquad \dot{\boldsymbol{u}}(t) = \omega\boldsymbol{\phi} Z_0 \cos\omega t,$$
$$V = \frac{1}{2}\boldsymbol{\phi}^T \boldsymbol{K}\,\boldsymbol{\phi}\, Z_0^2 \sin^2\omega t, \qquad T = \frac{1}{2}\omega^2\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{\phi}\, Z_0^2 \cos^2\omega t,$$

so that, by equating the maximum values of energies V and T and substituting $\boldsymbol{\phi} = \begin{Bmatrix}1 & 1 & 1\end{Bmatrix}^T$ we have

$$\omega^2 = \frac{\boldsymbol{\phi}^T \boldsymbol{K}\,\boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{\phi}} = \frac{6k}{7/3\; m} = \frac{18}{7}\frac{k}{m} = 2.5714\frac{k}{m}.$$

A better approximation of the strain energy is given by

$$V = \frac{1}{2}\boldsymbol{u}_\mathrm{I}^T \boldsymbol{f}_\mathrm{I},$$

where $\boldsymbol{f}_\mathrm{I} = -\omega^2\boldsymbol{M}\boldsymbol{\phi}Z_0\sin\omega t$ is the vector of the inertial forces and $\boldsymbol{u}_\mathrm{I} = \boldsymbol{K}^{-1}\boldsymbol{f}_\mathrm{I} = -\omega^2\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}Z_0\sin\omega t$ is the vector of displacements produced by $\boldsymbol{f}$I.

Equating the new maximum value of the strain energy to the old kinetic energy maximum, it is

$$\omega^2 = \frac{\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}} = \frac{7/3\; m}{23/18\; m^2/k} = \frac{42}{23}\frac{k}{m} = 1.8261\frac{k}{m}.$$

A better approximation to the kinetic energy is given by

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_\mathrm{I}^T \boldsymbol{M}\,\dot{\boldsymbol{u}}_\mathrm{I},$$

where $\dot{\boldsymbol{u}}_\mathrm{I} = -\omega^3\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}Z_0\cos\omega t$ is the velocity due to application of the inertial forces, equating the new max of T to the new max of V we have

$$\omega^2 = \frac{\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{K}^{-1}\boldsymbol{M}\,\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}} = \frac{23/18\; m^2/k}{29/36\; m^3/k^2} = \frac{46}{29}\frac{k}{m} = 1.5862\frac{k}{m}.$$

6 3 DOF System

With reference to the system of problem 5, using the position $\omega_0^2 = \dfrac{k}{m}$

1. compute the three eigenvalues of the system and the corresponding eigenvectors,
2. normalize the eigenvectors with respect to the mass matrix $\boldsymbol{M}$ (it must be $\boldsymbol{\psi}^T \boldsymbol{M} \boldsymbol{\psi} = m$).

Considering that the system is at rest for $t = 0$ and is then loaded by a load vector $\boldsymbol{p}(t)$,

$$\boldsymbol{p}(t) = \frac{kL}{200} \begin{Bmatrix} 0 \\ -1 \\ +1 \end{Bmatrix} \sin(7\omega_0 t),$$

3. find the analytical expression of $u_3 = u_3(t)$, showing your intermediate results and
4. plot u_3 in the interval $0 \le \omega_0 t \le 6$.

6.1 Solution

Using the previously computed structural matrices, the eigenvalues are the roots of the equation

$$\det\left(k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \omega^2 \frac{m}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) = 0$$

with the position $\omega^2 = \Lambda\omega_0^2$, developing the determinant, simplifying etc it is

$$13\Lambda^3 - 204\Lambda^2 + 720\Lambda - 648 = 0,$$

solving for the Λ_i and substituting it is

$$\omega_1^2 = 1.4185\omega_0^2, \quad \omega_2^2 = 3.1619\omega_0^2, \quad \omega_3^2 = 11.112\omega_0^2.$$

For algebraic manipulations, it is often convenient to collect the eigenvalues in a diagonal matrix

$$\boldsymbol{\Lambda} = \omega_0^2 \begin{bmatrix} 1.4185 & 0 & 0 \\ 0 & 3.1619 & 0 \\ 0 & 0 & 11.112 \end{bmatrix}.$$

The eigenvectors are given by solving the following linear systems

$$\left(k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \omega_i^2 \frac{m}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \boldsymbol{\psi}_i = 0, \quad i = 1, 2, 3.$$

Normalising and collecting the eigenvectors in an eigenvector matrix $\boldsymbol{\Psi}$,

$$\boldsymbol{\Psi} = \begin{bmatrix} +1.1324 & -0.5408 & +0.2015 \\ +0.2594 & +1.1369 & -0.6973 \\ +0.0243 & +0.3079 & +1.8347 \end{bmatrix}.$$

The steady state response, $\boldsymbol{x}_{\text{s-s}}(t) = \boldsymbol{\xi}\,\sin\omega t$, can be computed directly in terms of nodal coordinates

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\,\boldsymbol{\xi}\,\sin\omega t = \boldsymbol{p}\,\sin\omega t \quad \Rightarrow \quad \boldsymbol{\xi} = (\boldsymbol{K} - \omega^2\boldsymbol{M})^{-1}\boldsymbol{p}$$

Substituting $\omega^2 = 49\omega_0^2$ let us write

$$\left(k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \frac{49}{6}\omega_0^2 m \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \boldsymbol{\xi}\sin(7\omega_0 t) = \frac{kL}{200} \begin{Bmatrix} 0 \\ -1 \\ +1 \end{Bmatrix} \sin(7\omega_0 t)$$

dividing all terms by k, simplifying $\sin(7\omega_0 t)$ and $\omega_0^2\frac{m}{k} = 1$ and solving for $\boldsymbol{\xi}$ gives

$$\boldsymbol{\xi} = \frac{L}{200} \begin{bmatrix} -0.01765207 \\ +0.06844680 \\ -0.11692366 \end{bmatrix}, \qquad \boldsymbol{x}_{\text{s-s}}(t) = \frac{L}{200} \begin{bmatrix} -0.01765207 \\ +0.06844680 \\ -0.11692366 \end{bmatrix} \sin\omega t.$$

Now, write the integral of the homogeneous problem as

$$\boldsymbol{q}(t) = \begin{bmatrix} \sin\omega_1 t & 0 & 0 \\ 0 & \sin\omega_2 t & 0 \\ 0 & 0 & \sin\omega_3 t \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} \cos\omega_1 t & 0 & 0 \\ 0 & \cos\omega_2 t & 0 \\ 0 & 0 & \cos\omega_3 t \end{bmatrix} \boldsymbol{b}$$

$$\boldsymbol{\Lambda}^{-\frac{1}{2}}\,\dot{\boldsymbol{q}}(t) = \begin{bmatrix} \cos\omega_1 t & 0 & 0 \\ 0 & \cos\omega_2 t & 0 \\ 0 & 0 & \cos\omega_3 t \end{bmatrix} \boldsymbol{a} - \begin{bmatrix} \sin\omega_1 t & 0 & 0 \\ 0 & \sin\omega_2 t & 0 \\ 0 & 0 & \sin\omega_3 t \end{bmatrix} \boldsymbol{b}$$

and evaluate modal displacements and modal velocities at $t = 0$

$$\boldsymbol{q}_0 = \boldsymbol{b}, \qquad \dot{\boldsymbol{q}}_0 = \boldsymbol{\Lambda}^{\frac{1}{2}}\,\boldsymbol{a}.$$

The initial conditons in terms of nodal displacements, for a system starting from rest conditions, are

$$\boldsymbol{x}_0 = \boldsymbol{\Psi}\,\boldsymbol{q}_0 + \boldsymbol{\xi}\,\sin\omega 0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{x}}_0 = \boldsymbol{\Psi}\,\dot{\boldsymbol{q}}_0 + \omega\boldsymbol{\xi}\,\cos\omega 0 = \boldsymbol{0},$$

substituting the initial values of the modal coordinates

$$\boldsymbol{\Psi}\,\boldsymbol{q}_0 = \boldsymbol{\Psi}\,\boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{\Psi}\,\dot{\boldsymbol{q}}_0 = \boldsymbol{\Psi}\,\boldsymbol{\Lambda}^{\frac{1}{2}}\,\boldsymbol{a} = -\omega\boldsymbol{\xi},$$

solving for $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{a} = -\boldsymbol{\Lambda}^{-\frac{1}{2}}\omega\left(\boldsymbol{\Psi}^T\frac{\boldsymbol{M}}{m}\boldsymbol{\xi}\right).$$

Substituting the numerical values into the last equation we find

$$\boldsymbol{a} = \frac{L}{200}\begin{Bmatrix}-0.02904676\\-0.07119805\\+0.14033766\end{Bmatrix}.$$

Finally, $x_3(t)$ is a linear combination of the modal responses by the third elements of the three eigenvectors, plus the steady state response $\xi_3 \sin 7\omega_0$

$$\begin{aligned}200\,x_3(t)/L &= \psi_{31}a_1\sin\omega_1 t + \psi_{32}a_2\sin\omega_2 t + \psi_{33}a_3\sin\omega_3 t + \xi_3\sin\omega t\\ &= (-0.0243\cdot 0.0290)\ \sin\omega_1 t + (-0.3079\cdot 0.0712)\ \sin\omega_2 t +\\ &\quad (+1.8347\cdot 0.1403)\ \sin\omega_3 t - 0.1169\ \sin\omega t\\ &= -0.000705\ \sin 1.1911\omega_0 - 0.02192\ \sin 1.7782\omega_0 t +\\ &\quad + 0.2574789\ \sin 3.3334\omega_0 - 0.11692366\ \sin 7\omega_0 t.\end{aligned}$$

The following short Python program computes and prints the response using the linear acceleration algorithm, note that it is almost identical to the program used for the *SDOF* system of problem 2, except the use of vectors and matrices and the adimensionalisation of all physical quantities.

```
from scipy import mat, sin, zeros
K = mat('1 0 0;0 2 0;0 0 3')
M = mat('4 1 0;1 4 1;0 1 2')/6.0
r = mat('0;-1;1')
def load(t): return r*sin(7.0*t) #
h = 0.005; duration = 6.0
# linear acceleration coefficients
A = 3.0*M;    V = 6.0*M/h
Flex  = (K + 6.0*M/(h*h)).I
MI = M.I
# initial state
t = 0
x, v, p = mat(zeros((3,1))), mat(zeros((3,1))), r*sin(omega*t)
a = MI*(p - K*x)
print "%12.9f" % t, ' '.join(["%12.9f" % x_i for x_i in x])
# iterate with linear acceleration algorithm
```

```
while (t + h/2) < duration:
    t  = t + h
    dp = r*sin(omega*t) - p
    dx = Flex*(dp + A*a + V*v)
    dv = 3.0*dx/h - 3*v - a*h/2.0
    x, v, p = x + dx, v + dv, p + dp
    a  = MI*(p - K*x)
    print "%12.9f" % t, ' '.join(["%12.9f" % x_i for x_i in x])
```

In the plot below, you can compare the results of the numerical integration with the results of the analytical derivation and gain some confidence in the correctness of both derivations.