An Introduction to Dynamics of Structures

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Outline

Part I

Introduction

Our aim is to develop some analytical and numerical methods for the analysis of the stresses and deflections that the application of a time varying set of loads induces in a generic structure that moves in a neighborhood of a point of equilibrium.
We will see that these methods are extensions of the methods of standard static analysis, or to say it better, that static analysis is a special case of *dynamic analysis*.
If we restrict ourselves to analysis of *linear systems*, however, it is so convenient to use the principle of superposition to study the combined effects of static and dynamic loadings that different methods, of different character, are applied to these different loadings.

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Dynamic Loading a Loading that varies over time

Dynamic Response the Response of a structural system to a dynamic loading, expressed in terms of stresses and/or deflections; a Dynamic Response varies over time.

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Types of Dynamic Analysis

Taking into account linear systems only, we must consider two different definitions of the loading to define two types of dynamic analysis

Deterministic Analysis applies when the time variation of the loading is fully known and we can determine the complete time variation of all the response quantities that are required in our analysis

Non-deterministic Analysis applies when the time variation of the loading is essentially random and is known *only* in terms of some *statistics*, also the structural response can be known only in terms of some statistics of the response quantities.

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Types of Dynamic Loadings

Dealing with deterministic loadings, we will study, in order of complexity,

Harmonic Loading it is modulated by a harmonic function, characterized by a frequency and a phase, $p(t) = p_0 \sin(\omega t - \varphi)$.

Periodic Loading it repeats itself with a fixed period T, $p(t) = p(t + T)$ holds for every t.

Non Periodic Loading here we see two sub-cases,

- the loading can be described in terms of analytical functions, e.g., $p(t) = p_o \exp(\alpha t)$,
- the loading is (a) experimentally measured and (b) is known only in a discrete set of instants; in this case, we say that we have a *time-history*.

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Characteristics of a Dynamical Problem

As both load and response vary over time, our methods of analysis have to provide the dynamical problem solution for every instant in the response.

A dynamical problem is essentially characterized by the relevance of *inertial forces*, arising from the accelerated motion of structural or serviced masses.

A dynamic analysis is *required* only when the inertial forces represent a significant portion of the total load.

On the other hand, if the loads and the deflections are varying *slowly*, a static analysis will often provide an acceptable approximation.

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Formulation of a Dynamical Problem

The inertial forces depend on deflections, the deflections depend on inertial forces, the natural statement of the problem is hence conveniently written in terms of *differential equations*.

If the mass is distributed along the structure, also the inertial forces are distributed and the formulation of our problem must be in terms of *partial differential equations*, to take into account the spatial variations of both loading and response.

In many situations it is however possible to simplify the formulation of the problem, using *ordinary* differential equations.

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Formulation of a Dynamical Problem

In many structural problems, we can say that the mass is concentrated in a discrete set of *lumped masses*.

Under this assumption, the analytical problem is greatly simplified:

1. the inertial forces are applied only at the lumped masses,
2. the only deflections that influence the inertial forces are the deflections of the lumped masses,

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Dynamic Degrees of Freedom

The *dynamic degrees of freedom* (DDOF) in a discretized system are the displacements components of the lumped masses associated with the components of the inertial forces.
If a lumped mass can be regarded as a *point* mass then 3 translational DDOFs will suffice to represent the associated inertial force.
On the contrary, if a lumped mass has a discrete volume its inertial force depends also on its rotations (inertial couples) and we need 6 DDOFs to represent the mass deflections and the inertial force.
Of course, a continuous system has an infinite number of degrees of freedom.

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Generalized Displacements

The lumped mass procedure that we have outlined is effective if a large proportion of the total mass is concentrated in a few points.
A primary example is a multi-storey building, where one can consider a lumped mass for each storey.
When the mass is distributed, we can simplify our problem using *generalized coordinates*. The deflections are expressed in terms of a linear combination of assigned functions of position, with the coefficients of the linear combination being the generalized coordinates. E.g., the deflections of a rectilinear beam can be expressed with a trigonometric series.

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Generalized Displacements, cont.

To fully describe a displacement field, we need to combine an infinity of linearly independent *base functions*, but in practice a good approximation can be achieved using just a small number of functions and degrees of freedom.

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Finite Element Method

The *finite elements method* (FEM) combines aspects of lumped mass and generalized coordinates methods, providing a simple and reliable method of analysis, that can be easily programmed on a digital computer.
In the FEM, the structure is subdivided in a number of non-overlapping pieces, called the *finite elements*, delimited by common *nodes*.
The FEM uses *piecewise approximations* to the field of displacements: in each *element* the displacement field is derived from the displacements of the *nodes* that surround each particular element, using *interpolating functions*, so that the displacement, deformation and stress fields in each element can be expressed in terms of the unknown *nodal displacements*.
The *nodal displacements* are the dynamical DOFs.

Finite Element Method

Some of the most prominent advantages of the FEM method are

1. The desired level of approximation can be achieved by further subdividing the structure.
2. The resulting equations are only loosely coupled, leading to an easier computer solution.
3. For a particular type of finite element (e.g., beam, solid, etc) the procedure to derive the displacement field and the element characteristics does not depend on the particular geometry of the elements, and can easily be implemented in a computer program.

Writing the equation of motion

In a deterministic dynamic analysis, given a prescribed load, we want to evaluate the displacements in each instant of time.

In many cases a limited number of DDOFs gives a sufficient accuracy; further, the dynamic problem can be reduced to the determination of the time-histories of some selected component of the response.

The mathematical expressions, ordinary or partial differential equations, that we are going to write express the *dynamic equilibrium* of the structural system and are known as the *Equations of Motion* (EOM).

The solution of the EOM gives the requested displacements.

The formulation of the EOM is the most important, often the most difficult part of a dynamic analysis.

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Writing the EOM, cont.

We have a choice of techniques to help us in writing the EOM, namely:

- the D'Alembert Principle,
- the Principle of Virtual Displacements,
- the Variational Approach.

D'Alembert principle

By Newton's II law of motion, for any particle the rate of change of momentum is equal to the external force,

$$\vec{p}(t) = \frac{d}{dt}\left(m\frac{d\vec{u}}{dt}\right),$$

where $\vec{u}(t)$ is the particle displacement.
In structural dynamics, we may regard the mass as a constant, and thus write

$$\vec{p}(t) = m\ddot{\vec{u}},$$

where each operation of differentiation with respect to time is denoted with a dot.
If we write

$$\vec{p}(t) - m\ddot{\vec{u}} = 0$$

and interpret the term $-m\ddot{\vec{u}}$ as an *Inertial Force* that contrasts the acceleration of the particle, we have an equation of equilibrium for the particle.

D'Alembert principle, cont.

The concept that a mass develops an inertial force opposing its acceleration is known as the D'Alembert principle, and using this principle we can write the *EOM* as a simple equation of equilibrium.
The term $\vec{p}(t)$ must comprise each different force acting on the particle, including the reactions of kinematic or elastic constraints, opposing displacement, viscous forces and external, autonomous forces.
In many simple problems, D'Alembert principle is the most direct and convenient method for the formulation of the *EOM*.

Principle of virtual displacements

In a reasonably complex dynamic system, with articulated rigid bodies and external/internal constraints, the direct formulation of the *EOM* using D'Alembert principle may result difficult.
In these cases, application of the *Principle of Virtual Displacements* is very convenient, because the reactive forces do not enter the equations of motion, that are directly written in terms of the motions compatible with the restraints/constraints of the system.

Principle of Virtual Displacements, cont.

For example, considering an assemblage of rigid bodies, the *pvd* states that necessary and sufficient condition for equilibrium is that, for every *virtual displacement* (any infinitesimal displacement compatible with the restraints) the total work done by all the external forces is zero.
For an assemblage of rigid bodies, writing the *EOM* requires

1. to identify all the external forces, comprising the inertial forces, and to express their values in terms of the *ddof*;
2. to compute the work done by these forces for different virtual displacements, one for each *ddof*;
3. to equate to zero all these work expressions.

The *pvd* is particularly convenient also because we have only scalar equations, even if the forces and displacements are of vectorial nature.

Variational approach

Variational approaches do not consider directly the forces acting on the dynamic system, but are concerned with the variations of kinetic and potential energy and lead, as well as the *pvd*, to a set of scalar equations.

The method to be used in a particular problem is mainly a matter of convenience and also of personal taste.

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Part II

Single Degree of Freedom System

1 DOF System

Structural dynamics is all about a motion in the neighborhood of a point of equilibrium.
We'll start by studying a generic single degree of freedom system, with *constant* mass m, subjected to a non-linear generic force $F = F(y, \dot{y})$, where y is the displacement and $\dot{y}$ the velocity of the particle. The equation of motion is

$$\ddot{y} = \frac{1}{m} F(y, \dot{y}) = f(y, \dot{y})$$

.
It is difficult to integrate the above equation in the general case, but it's easy when the motion occurs in a small neighborhood of the equilibrium position.

1 DOF System, cont.

In a position of equilibrium, $y_{\text{eq.}}$, the velocity and the acceleration are zero, and hence $f(y_{\text{eq.}}, 0) = 0$.
The force can be linearized in a neighborhood of $y_{\text{eq.}}$, 0:

$$f(y, \dot{y}) = f(y_{\text{eq.}}, 0) + \frac{\partial f}{\partial y}(y - y_{\text{eq.}}) + \frac{\partial f}{\partial \dot{y}}(\dot{y} - 0) + O(y, \dot{y}).$$

Assuming that $O(y, \dot{y})$ is small in a neighborhood of $y_{\text{eq.}}$, we can write the equation of motion

$$\ddot{x} + a\dot{x} + bx = 0$$

where $x = y - y_{\text{eq.}}$, $a = -\frac{\partial f}{\partial \dot{y}}$ and $b = -\frac{\partial f}{\partial y}$.
In an infinitesimal neighborhood of $y_{\text{eq.}}$, the equation of motion can be studied in terms of a linear differential equation of second order.

1 DOF System, cont.

A linear constant coefficient differential equation has the integral $x = A\exp(st)$, that substituted in the equation of motion gives

$$s^2 + as + b = 0$$

whose solutions are

$$s_{1,2} = -\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b}.$$

The general integral is

$$x(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t).$$

Given that for a free vibration problem $A_1,\ A_2$ are given by the initial conditions, the nature of the solution depends on the sign of the real part of s_1, s_2.

1 DOF System, cont.

If we write $s_i = r_i + \imath q_i$, then we have

$$\exp(s_i t) = \exp(\imath q_i t)\exp(r_i t).$$

If one of the $r_i > 0$, the response grows infinitely over time, even for an infinitesimal perturbation of the equilibrium, so that in this case we have an *unstable equilibrium*.
If both $r_i < 0$, the response decrease over time, so we have a *stable equilibrium*.
Finally, if both $r_i = 0$ the s's are imaginary, the response is harmonic with constant amplitude.

1 DOF System, cont.

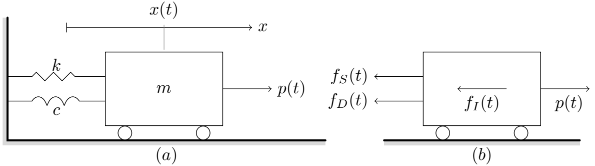
If $a > 0$ and $b > 0$, both roots are negative or complex conjugate with negative real part, the system is asymptotically stable.
If $a = 0$ and $b > 0$, the roots are purely imaginary, the equilibrium is indifferent, the oscillations are harmonic.
If $a < 0$ or $b < 0$ at least one of the roots has a positive real part, and the system is unstable.

The basic dynamic system

A linear system is characterized by its mass distribution,, its elastic properties and its energy-loss mechanism.

In a single degree of freedom (*sdof*) system each property can be conveniently represented in a single physical element

- The entire mass, m, is concentrated in a rigid block, its position completely described by the coordinate $x(t)$.
- The elastic resistance to displacement is provided by a massless spring of stiffness k
- The energy loss is represented by a massless damper, its damping constant being c.
- Finally, the external loading is the time-varying force $p(t)$.



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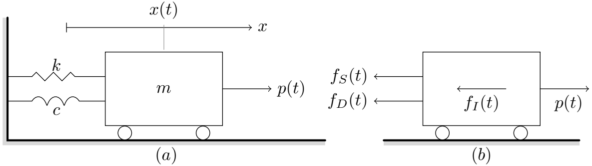

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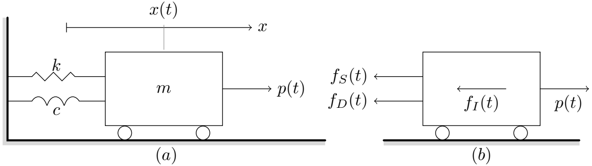

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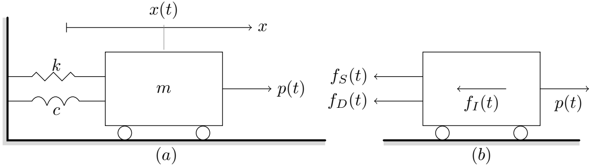

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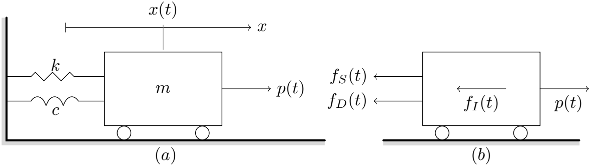

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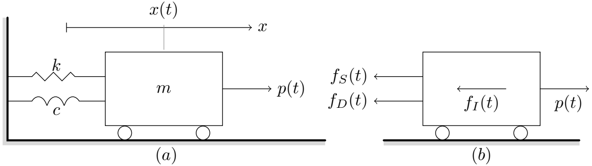
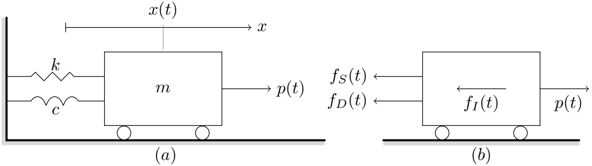


Equation of motion of the basic dynamic system



The equation of motion can be written using the D'Alembert Principle, expressing the equilibrium of all the forces acting on the mass including the inertial force.

The forces, positive if acting in the direction of the motion, are the external force, $p(t)$, and the resisting forces due to motion, i.e., the inertial force $f_{\mathrm{I}}(t)$, the damping force $f_{\mathrm{D}}(t)$ and the elastic force, $f_{\mathrm{S}}(t)$.

The equation of motion, merely expressing the equilibrium of these forces, is

$$f_{\mathrm{I}}(t) + f_{\mathrm{D}}(t) + f_{\mathrm{S}}(t) = p(t)$$

EOM of the basic dynamic system, cont.

The resisting forces in

$$f_{\mathrm{I}}(t) + f_{\mathrm{D}}(t) + f_{\mathrm{S}}(t) = p(t)$$

are functions of the displacement $x(t)$ or of one of its derivatives.
Note that the positive sense of these forces is opposite to the direction of motion.

EOM of the basic dynamic system, cont.

In accordance to D'Alembert principle, the inertial force is the product of the mass and acceleration

$$f_\mathrm{I}(t) = m\,\ddot{x}(t).$$

Assuming a viscous damping mechanism, the damping force is the product of the damping constant c and the velocity,

$$f_\mathrm{D}(t) = c\,\dot{x}(t).$$

Finally, the elastic force is the product of the elastic stiffness k and the displacement,

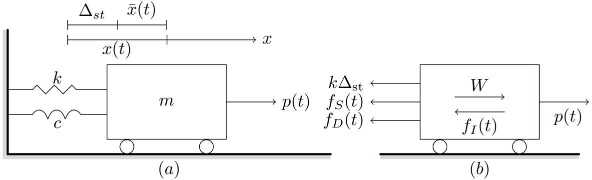
$$f_\mathrm{S}(t) = k\,x(t).$$

The differential equation of dynamic equilibrium

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t).$$

Note that this differential equation is a linear differential equation of the second order, with constant coefficients.

Influence of static forces



Considering the presence of a constant force, let's say W, the equation of motion is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t) + W,$$

but expressing the total displacement as the sum of a constant, static displacement and a dynamic displacement,

$$x(t) = \Delta_{\rm st} + \bar{x}(t),$$

substituting in we have

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,\Delta_{\rm st} + k\,\bar{x}(t) = p(t) + W.$$

Influence of static forces, cont.

Recognizing that $k\,\Delta_{\text{st}} = W$ (so that the two terms, on opposite sides of the equal sign, cancel each other), that $\dot{x} = \dot{\bar{x}}$ and that $\ddot{x} = \ddot{\bar{x}}$ the *EOM* is now

$$m\,\ddot{\bar{x}}(t) + c\,\dot{\bar{x}}(t) + k\,\bar{x}(t) = p(t)$$

The equation of motion expressed with reference to the static equilibrium position is not affected by static forces.
For this reasons, all displacements in further discussions will be referenced from the equilibrium position and denoted, for simplicity, with $x(t)$.
Note that the *total* displacements, stresses. etc. *are influenced* by the static forces, and must be computed using the superposition of effects.

Influence of static forces, cont.

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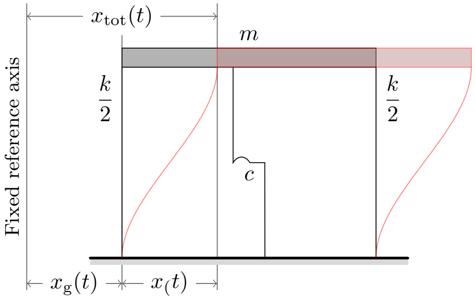
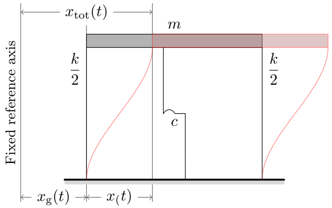
Influence of support motion



Displacements, deformations and stresses in a structure are induced also by a motion of its support.
Important examples of support motion are the motion of a building foundation due to earthquake and the motion of the base of a piece of equipment due to vibrations of the building in which it is housed.

Influence of support motion, cont.



Considering a support motion $x_g(t)$, defined with respect to a inertial frame of reference, the total displacement is

$$x_{\text{tot}}(t) = x_g(t) + x(t)$$

and the total acceleration is

$$\ddot{x}_{\text{tot}}(t) = \ddot{x}_g(t) + \ddot{x}(t).$$

While the elastic and damping forces are still proportional to *relative* displacements and velocities, the inertial force is proportional to the total acceleration,

$$f_I(t) = -m\ddot{x}_{\text{tot}}(t) = m\ddot{x}_g(t) + m\ddot{x}(t).$$

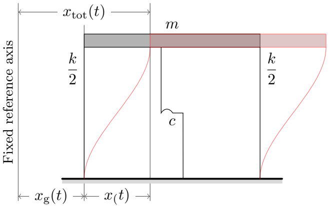
Writing the *EOM* for a null external load, $p(t) = 0$, is hence

$$m\,\ddot{x}_{\text{tot}}(t) + c\,\dot{x}(t) + k\,x(t) = 0, \qquad \text{or,}$$

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = -m\,\ddot{x}_g(t) \equiv p_{\text{eff}}(t).$$

Support motion is sufficient to excite a dynamic system: $p_{\text{eff}}(t) = -m\,\ddot{x}_g(t)$.

Influence of support motion, cont.



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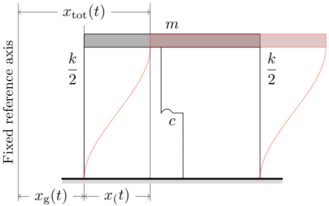

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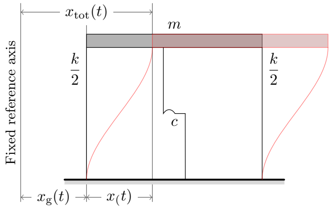

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Free Vibrations

The equation of motion,

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = p(t)$$

is a linear differential equation of the second order, with constant coefficients.
Its solution can be expressed in terms of a superposition of a *particular solution*, depending on $p(t)$, and a *free vibration* solution, that is the solution of the so called *homogeneous problem*, where $p(t) = 0$.
In the following, we will study the solution of the homogeneous problem, the so-called *homogeneous* or *complementary* solution, that is the *free vibrations* of the SDOF after a perturbation of the position of equilibrium.

Free vibrations of an undamped system

An undamped system, where $c = 0$ and no energy dissipation takes place, is just an ideal notion, as it would be a realization of *motus perpetuum*. Nevertheless, it is an useful idealization. In this case, the homogeneous equation of motion is

$$m\,\ddot{x}(t) + k\,x(t) = 0$$

which solution is of the form $\exp st$; substituting this solution in the above equation we have

$$(k + s^2 m)\,\exp st = 0$$

noting that $\exp st \neq 0$, we finally have

$$(k + s^2 m) = 0 \Rightarrow s = \pm\sqrt{-\frac{k}{m}}$$

As m and k are positive quantities, s must be purely imaginary.

Undamped Free Vibrations

Introducing the *natural* circular frequency ω_n

$$\omega_n^2 = \frac{k}{m},$$

the solution of the algebraic equation in s is

$$s = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n$$

where $i = \sqrt{-1}$ and the general integral of the homogeneous equation is

$$x(t) = G_1 \exp(i\omega_n t) + G_2 \exp(-i\omega_n t).$$

The solution has an imaginary part?

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Undamped Free Vibrations

The solution is derived from the general integral imposing the (real) initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0$$

Evaluating $x(t)$ for $t = 0$ and substituting in (40), we have

$$\begin{cases} G_1 + G_2 & = x_0 \\ i\omega_n G_1 - i\omega_n G_2 & = \dot{x}_0 \end{cases}$$

Solving the linear system we have

$$G_1 = \frac{ix_0 + \dot{x}_0/\omega_n}{2i}, \qquad G_2 = \frac{ix_0 - \dot{x}_0/\omega_n}{2i},$$

substituting these values in the general solution and collecting x_0 and $\dot{x}_0$, we finally find

$$x(t) = \frac{\exp(i\omega_n t) + \exp(-i\omega_n t)}{2}\, x_0 + \frac{\exp(i\omega_n t) - \exp(-i\omega_n t)}{2i}\, \frac{\dot{x}_0}{\omega_n}$$

Undamped Free Vibrations

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Using the Euler formulas relating the imaginary argument exponentials and the trigonometric functions, can be rewritten in terms of the elementary trigonometric functions

$$x(t) = x_0 \cos(\omega_n t) + (\dot{x}_o/\omega_n) \sin(\omega_n t).$$

Considering that for every conceivable initial conditions we can use the above representation, it is indifferent, and perfectly equivalent, to represent the general integral either in the form of exponentials of imaginary argument or as a linear combination of sine and cosine of circular frequency ω_n

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Undamped Free Vibrations

Our preferred representation of the general integral of undamped free vibrations is

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

For the usual initial conditions, we have already seen that

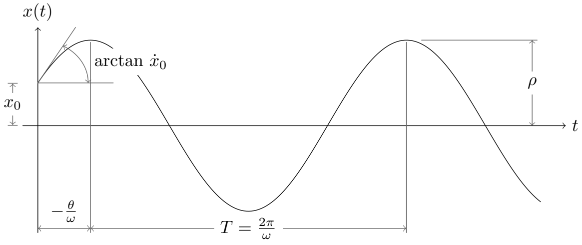
$$A = x_0, \qquad B = \frac{\dot{x}_0}{\omega_n}.$$

Sometimes we prefer to write, using the formula for summing a sine and a cosine, introducing a *phase difference* ϕ

$$x(t) = C\cos(\omega_n t - \varphi), \quad \text{with} \begin{cases} C = \sqrt{A^2 + B^2} \\ \varphi = \arctan(B/A) \end{cases}$$

so that the amplitude of the motion, C, is put in evidence.

Undamped Free Vibrations



It is worth noting that the coefficients A, B and C have the dimension of a length, the coefficient ω_n has the dimension of the reciprocal of time and that the coefficient φ is an angle, or in other terms is adimensional.

Behavior of Damped Systems

The viscous damping modifies the response of a *sdof* system introducing a decay in the amplitude of the response. Depending on the amount of damping, the response can be oscillatory or not. The amount of damping that separates the two behaviors is denoted as *critical damping*.

The solution of the EOM

The equation of motion for a free vibrating damped system is

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = 0,$$

substituting the solution $\exp st$ in the preceding equation and simplifying, we have that the parameter s must satisfy the equation

$$m\,s^2 + c\,s + k = 0$$

or, after dividing both members by m,

$$s^2 + \frac{c}{m}\,s + \omega_\text{n}^2 = 0$$

whose solutions are

$$s = \frac{c}{2m} \mp \sqrt{\frac{c^2}{m^2} - 4\omega_\text{n}^2}.$$

Critical Damping

The behavior of the solution of the free vibration problem depends of course on the sign of the radicand $\Delta = \frac{c^2}{m^2} - 4\omega_\mathrm{n}^2$:

- $\Delta < 0$ the roots s are complex conjugate,
- $\Delta = 0$ the roots are identical, double root,
- $\Delta > 0$ the roots are real.

The value of c that make the radicand equal to zero is known as the *critical damping*,

$$c_\mathrm{cr} = 2m\omega_\mathrm{n} = 2\sqrt{mk}.$$

Critical Damping

A single degree of freedom system is denoted as *critically damped*, *under-critically damped* or *over-critically damped* depending on the value of the damping coefficient with respect to the critical damping.

In the field of dynamics of structures we may assume that every system is undercritically damped, that is a system where the damping coefficient is smaller than the critical damping.

Damping Ratio

If we introduce the ratio of the damping to the critical damping, or *critical damping ratio* ζ,

$$\zeta = \frac{c}{c_{\mathrm{cr}}}, \quad c = \zeta c_{\mathrm{cr}} = 2\zeta\omega_{\mathrm{n}} m$$

the roots $s_{1,2}$ can be rewritten as

$$s = -\zeta\omega_{\mathrm{n}} \mp \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}$$

and the equation of motion itself can be rewritten as

$$\ddot{x}(t) + 2\zeta\omega_{\mathrm{n}}\dot{x}(t) + \omega_{\mathrm{n}}^2 x(t) = 0$$

Free Vibrations of Under-critically Damped Systems

We start studying the free vibration response of under-critically damped SDOF, as this is the most important case in structural dynamics.
Defining the *damped frequency*

$$\omega_{\mathrm{D}} = \omega_{\mathrm{n}}\sqrt{1-\zeta^2}.$$

the $s_{1,2}$ can now be written

$$s = -\zeta\omega_{\mathrm{n}} \mp \omega_{\mathrm{n}}\sqrt{-1}\sqrt{1-\zeta^2} = -\zeta\omega_{\mathrm{n}} \mp i\omega_{\mathrm{D}}$$

and the general integral of the equation of motion is, collecting the terms in $\exp(-\zeta\omega_{\mathrm{n}}t)$

$$x(t) = \exp(-\zeta\omega_{\mathrm{n}}t)\left[G_1\exp(-i\omega_{\mathrm{D}}t) + G_2\exp(+i\omega_{\mathrm{D}}t)\right]$$

Initial Conditions

By imposing the initial conditions, $u(0) = u_0$, $\dot{u}(0) = v_0$, after a bit of algebra we can write the equation of motion for the given initial conditions, namely

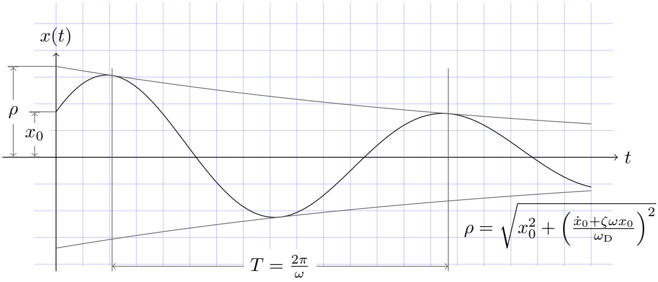
$$x(t) = \exp(-\zeta\omega_n t)\left[\frac{\exp(i\omega_D t) + \exp(-i\omega_D t)}{2} u_0 + \frac{\exp(i\omega_D t) - \exp(-i\omega_D t)}{2i} \frac{v_0 + \zeta\omega_n u_0}{\omega_D}\right].$$

Using the Euler formulas, we finally have the preferred format of the general integral:

$$x(t) = \exp(-\zeta\omega_n t)\left[A\cos(\omega_D t) + B\sin(\omega_D t)\right]$$

with

$$A = u_0, \qquad B = \frac{v_0 + \zeta\omega_n u_0}{\omega_D}.$$



$x(t)$
t
ρ
x_0
$\rho = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega x_0}{\omega_D}\right)^2}$
$T = \frac{2\pi}{\omega}$

Critically damped SDOF

In this case, $\zeta = 1$ and $s_{1,2} = -\omega_n$, so that the general integral must be written in the form

$$x(t) = \exp(-\omega_n t)(A + Bt).$$

The solution for given initial condition is

$$x(t) = \exp(-\omega_n t)(u_0 + (v_0 + \omega_n u_0)t),$$

note that, if $v_0 = 0$, the solution asymptotically approaches zero without crossing the zero axis.

Over-critically damped SDOF

In this case, $\zeta > 1$ and

$$s = -\zeta\omega_n \mp \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \mp \hat{\omega}$$

where

$$\hat{\omega} = \omega_n\sqrt{\zeta^2 - 1}$$

and, after some rearrangement, the general integral for the over-damped SDOF can be written

$$x(t) = \exp(-\zeta\omega_n t)\,(A\cosh(\hat{\omega}t) + B\sinh(\hat{\omega}t))$$

Note that:

- as $\zeta\omega_n > \hat{\omega}$, for increasing t the general integral goes to zero, and that
- as for increasing ζ we have that $\hat{\omega} \to \zeta\omega_n$, the velocity with which the response approaches zero slows down for increasing ζ.

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Measuring damping

The real dissipative behavior of a structural system is complex and very difficult to assess.
For convenience, it is customary to express the real dissipative behavior in terms of an *equivalent viscous damping*.
In practice, we *measure* the response of a SDOF structural system under controlled testing conditions and find the value of the viscous damping (or damping ratio) for which our simplified model best matches the measurements.
For example, we could require that, under free vibrations, the real structure and the simplified model exhibit the same decay of the vibration amplitude.

Logarithmic Decrement

Consider a SDOF system in free vibration and two positive peaks, u_n and u_{n+m}, occurring at times $t_n = n\,(2\pi/\omega_\mathrm{D})$ and $t_{n+m} = (n+m)\,(2\pi/\omega_\mathrm{D})$.
The ratio of these peaks is

$$\frac{u_n}{u_{n+m}} = \frac{\exp(-\zeta\omega_\mathrm{n} n 2\pi/\omega_\mathrm{D})}{\exp(-\zeta\omega_\mathrm{n}(n+m)2\pi/\omega_\mathrm{D})} = \exp(2m\pi\zeta\omega_\mathrm{n}/\omega_\mathrm{D})$$

Substituting $\omega_\mathrm{D} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$ and taking the logarithm of both members we obtain

$$\ln\left(\frac{u_n}{u_{n+m}}\right) = 2m\pi\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Solving for ζ, we finally get

$$\zeta = \frac{\ln\left(\frac{u_n}{u_{n+m}}\right)}{\sqrt{(2m\pi)^2 + \left(\ln\left(\frac{u_n}{u_{n+m}}\right)\right)^2}}.$$

Simplified solution

Starting from

$$\frac{u_n}{u_{n+m}} = \exp\frac{2m\pi\zeta}{\sqrt{1-\zeta^2}},$$

we note that, for the small values of ζ that are typical of structural systems, $\sqrt{1-\zeta^2} \approx 1$.
Using this approximation and the Taylor series representation of $\exp at$,

$$\frac{u_n}{u_{n+m}} \approx \exp\frac{2m\pi\zeta}{1} \approx 1 + 2m\pi\zeta + \cdots$$

Solving for ζ,

$$\zeta \approx \frac{1}{2m\pi}\frac{u_n - u_{n+m}}{u_{n+m}}.$$

Due to the uncertainties in the measurements and the modelization, for small values of ζ we can use the above approximation.

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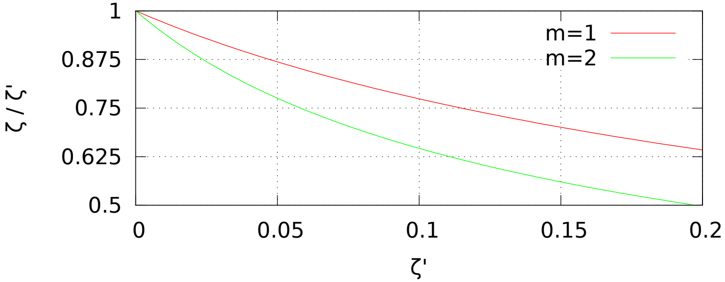
Errors

In abscissae the *approximate* value of ζ, in ordinates the ratio of true to approximate values



As you an see, the error grows with m, and the approximation is good only for really small values of ζ and $m = 1$.
A good alternative is to seek an iterative solution,

$$\delta = \frac{1}{2m\pi} \log \frac{u_n}{u_{n+m}}, \quad \zeta_0 = 0.0, \quad \zeta_{n+1} = \delta\sqrt{1-\zeta_n^2}.$$