Multi Degrees of Freedom Systems
MDOF's

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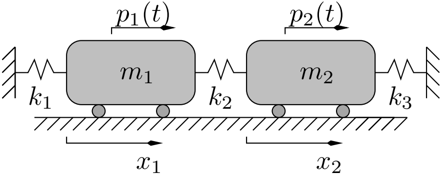
Outline

Introductory Remarks

Consider an undamped system with two masses and two degrees of freedom.



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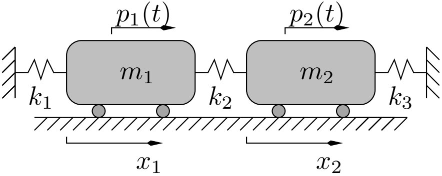

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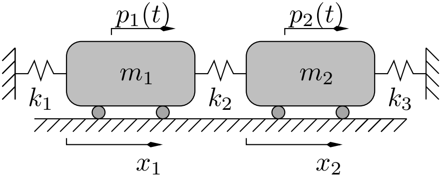

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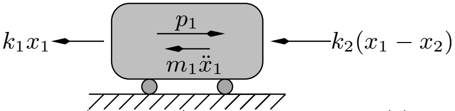

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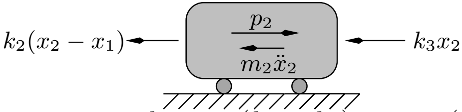

$$m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t)$$

The equation of motion of a 2DOF system

With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

The equation of motion of a 2DOF system

Introducing the loading vector $\boldsymbol{p}$, the vector of inertial forces $\boldsymbol{f}_I$ and the vector of elastic forces $\boldsymbol{f}_S$,

$$\boldsymbol{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \boldsymbol{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \boldsymbol{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\boldsymbol{f}_I + \boldsymbol{f}_S = \boldsymbol{p}(t).$$

$\boldsymbol{f}_S = \boldsymbol{K}\,\boldsymbol{x}$

It is possible to write the linear relationship between $\boldsymbol{f}_S$ and the vector of displacements $\boldsymbol{x} = \{x_1 x_2\}^T$ in terms of a matrix product.
In our example it is

$$\boldsymbol{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{K}\,\boldsymbol{x}$$

introducing the stiffness matrix $\boldsymbol{K}$.
The stiffness matrix $\boldsymbol{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\boldsymbol{K}$ is hence a *square matrix* $\underset{\text{ndof}\times\text{ndof}}{\boldsymbol{K}}$

$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}$

Analogously, introducing the mass matrix $\boldsymbol{M}$ that, for our example, is

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}.$$

Also the mass matrix $\boldsymbol{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

Matrix Equation

Finally it is possible to write the equation of motion in matrix format:

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t).$$

Of course, we can consider the damping forces too, taking into account the velocity vector $\dot{\boldsymbol{x}}$, introducing a *damping matrix* $\boldsymbol{C}$ and writing

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t),$$

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Properties of $\boldsymbol{K}$

- $\boldsymbol{K}$ is symmetrical, because the elastic force that acts on mass i due to an unit displacement of mass j, $f_{S,i} = k_{ij}$ is equal to the force on mass j due to unit diplacement of mass i, $f_{S,j} = k_{ji}$ in virtue of *Betti's theorem*.
- The strain energy V for a discrete system can be written

$$V = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{f}_S = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\,\boldsymbol{x},$$

because the strain energy is positive it follows that $\boldsymbol{K}$ is a positive definite matrix.

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Properties of $\boldsymbol{M}$

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive, as well as the stiffness matrix is symmetrical and definite positive.

En passant, take note that the kinetic energy for a discrete system is

$$T = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{M}\, \dot{\boldsymbol{x}}.$$

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Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with one exception.

For a general structural system, M could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy could be zero.

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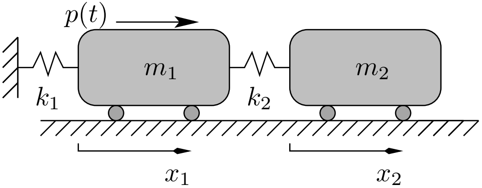
The problem

Graphical statement of the problem



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$$p(t) = p_0 \sin \omega t.$$

The equations of motion

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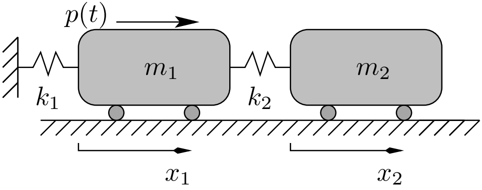

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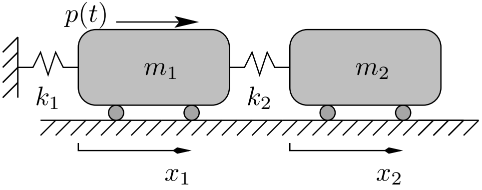

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The steady state solution

because using the matrix notation we can follow the same steps we used to find the steady-state response of a *SDOF* system. First, the equation of motion

$$m\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\ddot{\boldsymbol{x}}+k\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\boldsymbol{x}=p_0\begin{Bmatrix}1\\0\end{Bmatrix}\sin\omega t$$

substituting $x(t)=\boldsymbol{\xi}\sin\omega t$ and simplifying $\sin\omega t$, dividing by k, with $\omega_0^2=k/m$, $\beta^2=\omega^2/\omega_0^2$ and $\Delta_{\text{st}}=p_0/k$ the above equation can be written

$$\left(\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}-\beta^2\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\right)\boldsymbol{\xi}=\begin{bmatrix}3-2\beta^2 & -1\\-1 & 1-\beta^2\end{bmatrix}\boldsymbol{\xi}=\Delta_{\text{st}}\begin{Bmatrix}1\\0\end{Bmatrix}$$

solving for $\boldsymbol{\xi}/\Delta_{\text{st}}$ gives

$$\frac{\boldsymbol{\xi}}{\Delta_{\text{st}}}=\frac{\begin{bmatrix}1-\beta^2 & 1\\1 & 3-2\beta^2\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix}}{(\beta^2-\frac{1}{2})(\beta^2-2)}=\frac{\begin{Bmatrix}1-\beta^2\\1\end{Bmatrix}}{(\beta^2-\frac{1}{2})(\beta^2-2)}.$$

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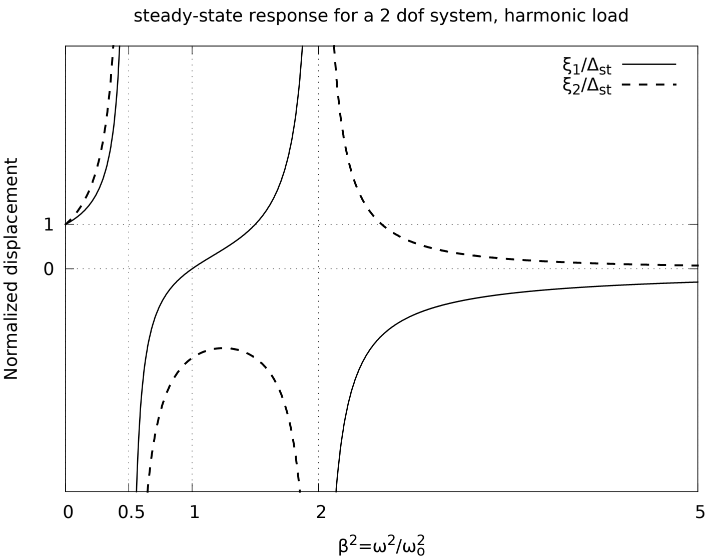

Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we start writing the homogeneous equation of motion,

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{0}.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* $\boldsymbol{\psi}$:

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(A\sin\omega t + B\cos\omega t).$$

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Eigenvalues

The previous equation must hold for every value of t, so it can be simplified removing the time dependency:

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) \boldsymbol{\psi} = \boldsymbol{0}.$$

This is a homogeneous linear equation, with unknowns ψ_i and the coefficients that depends on the parameter ω^2.

Speaking of homogeneous systems, we know that there is always a trivial solution, $\boldsymbol{\psi} = \boldsymbol{0}$, and that different non-zero solutions are available when the determinant of the matrix of coefficients is equal to zero,

$$\det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation (the *equation of frequencies*) is verified.

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Eigenvectors

For a system with N degrees of freedom the expansion of $\det\left(\boldsymbol{K}-\omega^2\boldsymbol{M}\right)$ is an algebraic polynomial of degree N in ω^2, whose roots, ω_i^2, $i=1,\ldots,N$ are all real and greater than zero if both $\boldsymbol{K}$ and $\boldsymbol{M}$ are positive definite matrices, condition that is always satisfied by stable structural systems.

Substituting one of the N roots ω_i^2 in the characteristic equation,

$$\left(\boldsymbol{K}-\omega_i^2\boldsymbol{M}\right)\boldsymbol{\psi}_i=\boldsymbol{0}$$

the resulting system of $N-1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue ω_i^2.

A common choice for the normalisation of the eigenvectors is *normalisation with respect to the mass matrix*, $\boldsymbol{\psi}_i^T\boldsymbol{M}\,\boldsymbol{\psi}_i=1$

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Initial Conditions

The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\boldsymbol{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\boldsymbol{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \boldsymbol{x}_0, \qquad \dot{\boldsymbol{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\boldsymbol{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji} \omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \ldots, N.$$

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and evaluating the displacement and velocity for $t = 0$ it is

$$\boldsymbol{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \boldsymbol{x}_0, \qquad \dot{\boldsymbol{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\boldsymbol{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji} \omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \ldots, N.$$

Orthogonality - 1

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\boldsymbol{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{M}\,\boldsymbol{\psi}_r$$
$$\boldsymbol{K}\,\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{M}\,\boldsymbol{\psi}_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M}\,\boldsymbol{\psi}_r$$
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Orthogonality - 2

The term $\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r\right)^T = \boldsymbol{\psi}_r^T \boldsymbol{K}^T\, \boldsymbol{\psi}_s$$

but $\boldsymbol{K}$ is symmetrical, $\boldsymbol{K}^T = \boldsymbol{K}$ and we have

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\, \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^T \boldsymbol{M}\, \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s.$$

Orthogonality - 3

Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^T \boldsymbol{K}\, \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2)\, \boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\boldsymbol{\psi}_r^T \boldsymbol{M}\, \boldsymbol{\psi}_s = 0, \qquad \text{for } r \neq s.$$

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Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \boldsymbol{K} \, \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M} \, \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \boldsymbol{M} \, \boldsymbol{\psi}_i$$

and

$$\boldsymbol{\psi}_i^T \boldsymbol{K} \, \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

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Eigenvectors are a base

The eigenvectors are linearly independent, so for every vector $\boldsymbol{x}$ we can write

$$\boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j.$$

The coefficients are readily given by premultiplication of $\boldsymbol{x}$ by $\boldsymbol{\psi}_i^T \boldsymbol{M}$, because

$$\boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_j q_j = \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\boldsymbol{\psi}_j^T \boldsymbol{M}\, \boldsymbol{x}}{M_j}.$$

Eigenvectors are a base

Generalising our results for the displacement vector to the acceleration vector, we can write

$$\boldsymbol{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\boldsymbol{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \psi_{ij} \ddot{q}_j(t).$$

Introducing $\boldsymbol{q}(t)$, the vector of *modal coordinates* and $\boldsymbol{\Psi}$, the *eigenvector matrix*, whose columns are the eigenvectors,

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}\, \boldsymbol{q}(t), \qquad \ddot{\boldsymbol{x}}(t) = \boldsymbol{\Psi}\, \ddot{\boldsymbol{q}}(t).$$

EoM in Modal Coordinates...

Substituting the last two equations in the equation of motion,

$$\boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{\Psi}^T\boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{\Psi}^T\boldsymbol{p}(t)$$

introducing the so called *starred* matrices we can finally write

$$\boldsymbol{M}^\star\,\ddot{\boldsymbol{q}} + \boldsymbol{K}^\star\,\boldsymbol{q} = \boldsymbol{p}^\star(t)$$

where $\boldsymbol{p}^\star(t) = \boldsymbol{\Psi}^T\boldsymbol{p}(t)$, and the scalar equation are

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j.$$

... are N independent equations!

We must examine the structure of the starred symbols.
The generic element, with indexes i and j, of the *starred* matrices can be expressed in terms of single eigenvectors,

$$m^\star_{ij} = \boldsymbol{\psi}_i^T \boldsymbol{M}\, \boldsymbol{\psi}_j \qquad = \quad \delta_{ij} M_i,$$
$$k^\star_{ij} = \boldsymbol{\psi}_i^T \boldsymbol{K}\, \boldsymbol{\psi}_j \qquad = \omega_i^2 \delta_{ij} M_i.$$

where δ_{ij} is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p^\star_i = \boldsymbol{\psi}_i^T \boldsymbol{p}(t)$ we have a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p^\star_i(t), \qquad i = 1, \ldots, N$$

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Initial Conditions Revisited

The initial displacements can be written in modal coordinates,

$$\boldsymbol{x}_0 = \boldsymbol{\Psi}\,\boldsymbol{q}_0$$

and premultiplying both members by $\boldsymbol{\Psi}^T\boldsymbol{M}$ we have the following relationship:

$$\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x}_0 = \boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{\Psi}\,\boldsymbol{q}_0 = \boldsymbol{M}^\star\boldsymbol{q}_0.$$

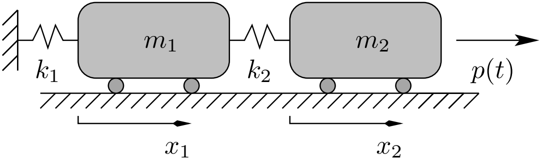
Premultiplying by the inverse of $\boldsymbol{M}^\star$ and taking into account that $\boldsymbol{M}^\star$ is diagonal,

$$\boldsymbol{q}_0 = (\boldsymbol{M}^\star)^{-1}\,\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x}_0 \quad \Rightarrow \quad q_{i0} = \frac{\boldsymbol{\psi}_i^T\boldsymbol{M}\,\boldsymbol{x}_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\boldsymbol{\psi}_i{}^T\boldsymbol{M}\,\dot{\boldsymbol{x}}_0}{M_i}$$

2 DOF System



$$k_1 = k, \quad k_2 = 2k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

$$\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \; \boldsymbol{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\boldsymbol{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; \boldsymbol{K} = k \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

Equation of frequencies

The equation of frequencies is

$$\left\| \boldsymbol{K} - \omega^2 \boldsymbol{M} \right\| = \left\| \begin{matrix} 3k - 2\omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{matrix} \right\| = 0.$$

Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m}\frac{7-\sqrt{33}}{4} \qquad \omega_2^2 = \frac{k}{m}\frac{7+\sqrt{33}}{4}$$

$$\omega_1^2 = 0.31386\frac{k}{m} \qquad \omega_2^2 = 3.18614\frac{k}{m}$$

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$$\omega_1^2 = 0.31386\frac{k}{m} \qquad \omega_2^2 = 3.18614\frac{k}{m}$$

Eigenvectors

Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\,(3 - 2 \times 0.31386)\psi_{11} - 2k\psi_{21} = 0$$

while substituting ω_2^2 gives

$$k\,(3 - 2 \times 3.18614)\psi_{12} - 2k\psi_{22} = 0.$$

Solving with the arbitrary assignment $\psi_{21} = \psi_{22} = 1$ gives the *unnormalized* eigenvectors,

$$\psi_1 = \begin{Bmatrix} +0.84307 \\ +1.00000 \end{Bmatrix}, \quad \psi_2 = \begin{Bmatrix} -0.59307 \\ +1.00000 \end{Bmatrix}.$$

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Normalization

We compute first M_1 and M_2,

$$\begin{aligned} M_1 &= \boldsymbol{\psi}_1^T \boldsymbol{M}\, \boldsymbol{\psi}_1 \\ &= \begin{Bmatrix} 0.84307, & 1 \end{Bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} \\ &= \begin{Bmatrix} 1.68614m, & m \end{Bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} = 2.42153m \end{aligned}$$

$$M_2 = 1.70346m$$

the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{2.42153}, \qquad \alpha_2 = \sqrt{1.70346}.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.54177 & -0.45440 \\ +0.64262 & +0.76618 \end{bmatrix}$$

Modal Loadings

The modal loading is

$$\begin{aligned}\boldsymbol{p}^{\star}(t) &= \boldsymbol{\Psi}^T\,\boldsymbol{p}(t)\\ &= p_0\begin{bmatrix}+0.54177 & +0.64262\\ -0.45440 & +0.76618\end{bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix}\sin\omega t\\ &= p_0\begin{Bmatrix}+0.64262\\+0.76618\end{Bmatrix}\sin\omega t\end{aligned}$$

Modal EoM

Substituting its modal expansion for $\boldsymbol{x}$ into the equation of motion and premultiplying by $\boldsymbol{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} m\ddot{q}_1 + 0.31386k\, q_1 = +0.64262\, p_0 \sin \omega t \\ m\ddot{q}_2 + 3.18614k\, q_2 = +0.76618\, p_0 \sin \omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by m both equations, we have

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = +0.64262\, \dfrac{p_0}{m} \sin \omega t \\ \ddot{q}_2 + \omega_2^2 q_2 = +0.76618\, \dfrac{p_0}{m} \sin \omega t \end{cases}$$

Particular Integral

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 \left(\omega_1^2 - \omega^2\right) \sin \omega t = \frac{p_1^\star}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{p_1^\star}{m} \frac{1}{\omega_1^2 - \omega^2}$$

with $\omega_1^2 = K_1/m \;\Rightarrow\; m = K_1/\omega_1^2$:

$$C_1 = \frac{p_1^\star}{K_1} \frac{\omega_1^2}{\omega_1^2 - \omega^2} = \Delta_{\mathsf{st}}^{(1)} \frac{1}{1 - \beta_1^2} \quad \text{with } \Delta_{\mathsf{st}}^{(1)} = \frac{p_1^\star}{K_1} = 2.047 \frac{p_0}{k} \text{ and } \beta_1 = \frac{\omega}{\omega_1}$$

of course

$$C_2 = \Delta_{\mathsf{st}}^{(2)} \frac{1}{1 - \beta_2^2} \quad \text{with } \Delta_{\mathsf{st}}^{(2)} = \frac{p_2^\star}{K_2} = 0.2404 \frac{p_0}{k} \text{ and } \beta_2 = \frac{\omega}{\omega_2}$$

Integrals

The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + \Delta_{\text{st}}^{(1)} \dfrac{\sin \omega t}{1-\beta_1^2} \\ q_2(t) = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t + \Delta_{\text{st}}^{(2)} \dfrac{\sin \omega t}{1-\beta_2^2} \end{cases}$$

for a system initially at rest

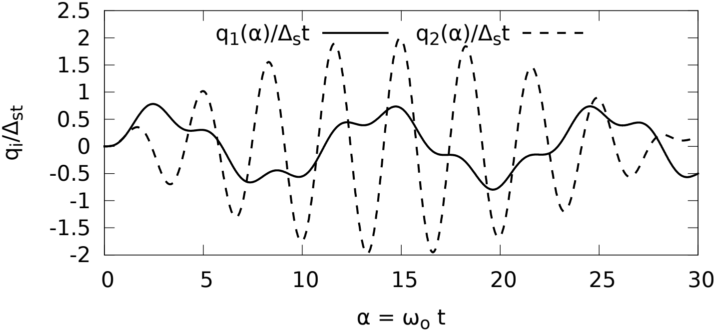
$$\begin{cases} q_1(t) = \Delta_{\text{st}}^{(1)} \dfrac{1}{1-\beta_1^2} (\sin \omega t - \beta_1 \sin \omega_1 t) \\ q_2(t) = \Delta_{\text{st}}^{(2)} \dfrac{1}{1-\beta_2^2} (\sin \omega t - \beta_2 \sin \omega_2 t) \end{cases}$$

we are interested in structural degrees of freedom, too... disregarding transient

$$\begin{cases} x_1(t) = \left(\psi_{11} \dfrac{\Delta_{\text{st}}^{(1)}}{1-\beta_1^2} + \psi_{12} \dfrac{\Delta_{\text{st}}^{(2)}}{1-\beta_2^2} \right) \sin \omega t = \left(\dfrac{1.10926}{1-\beta_1^2} - \dfrac{0.109271}{1-\beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t \\ x_2(t) = \left(\psi_{21} \dfrac{\Delta_{\text{st}}^{(1)}}{1-\beta_1^2} + \psi_{22} \dfrac{\Delta_{\text{st}}^{(2)}}{1-\beta_2^2} \right) \sin \omega t = \left(\dfrac{1.31575}{1-\beta_1^2} + \dfrac{0.184245}{1-\beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t \end{cases}$$

The response in modal coordinates

To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$.
This implies that $\beta_1 = \frac{\omega}{\omega_1} = \frac{2.0}{0.31386} = 6.37226$ and $\beta_2 = \frac{\omega}{\omega_2} = \frac{2.0}{3.18614} = 0.62771$.



In the graph above, the responses are plotted against an adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{\text{st}} = \frac{p_0}{k}$

The response in structural coordinates

Using the same normalisation factors, here are the response functions in terms of $x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$: