

# Earthquake Response of Inelastic Systems

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Motivation

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Cyclic Behaviour of Structural Members

Cyclic Behaviour

Elasto-plastic Idealisation

E-P Idealisation

Earthquake Response of E-P Systems

Normalised Equation of Motion

Earthquake  
Response of E-P  
Systems

Effects of Yielding

Inelastic Response, different values of  $\bar{f}_y$

Effects of Yielding

In Earthquake Engineering it is common practice to design against a large earthquake, that has a given mean period of return (say 500 years), quite larger than the expected life of the construction.

A period of return of 500 years means that in a much larger interval, say 50000 years, you expect say 100 earthquakes that are no smaller (in the sense of some metrics, e.g., the peak ground acceleration) than the design earthquake.

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The reasoning behind such design procedure is that, for the unlikely occurrence of a large earthquake, a large damage in the construction is acceptable as far as no human lives are taken in a complete structural collapse and that, in the mean, the costs for repairing a damaged building are not disproportionate to its value.



To ascertain the amount of acceptable reduction of earthquake loads it is necessary to study

- ▶ the behaviour of structural members and systems subjected to cyclic loading outside the elastic range, to understand the amount of plastic deformation and cumulated plastic deformation that can be sustained before collapse and
- ▶ the global structural behaviour for inelastic response, so that we can relate the reduction in design ordinates to the increase in members' plastic deformation.

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The first part of this agenda pertains to Earthquake Engineering proper, the second part is across EE and Dynamics of Structures, and today's subject.

Investigation of the cyclic behaviour of structural members, sub-assemblages and scaled or real sized building model, either in labs or via numerical simulations, constitutes the bulk of EE.

What is important, at the moment, is the understanding of how different these behaviours can be, due to different materials or structural configurations, with instability playing an important role.

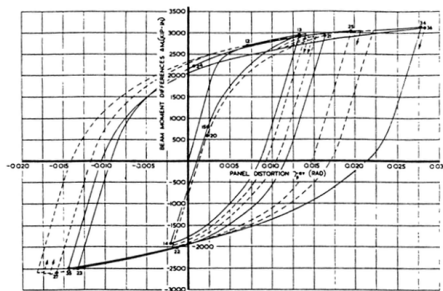
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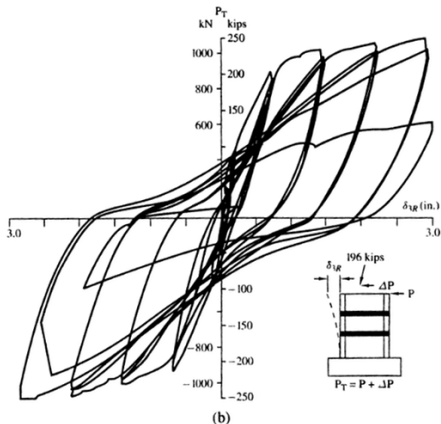
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(a)

# Cyclic behaviour

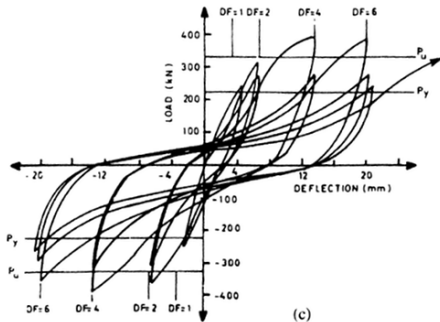
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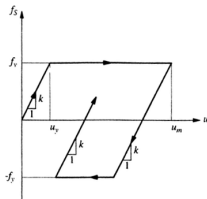
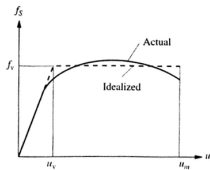
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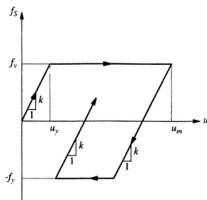
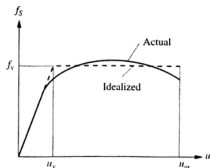


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# E-P model



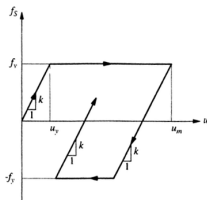
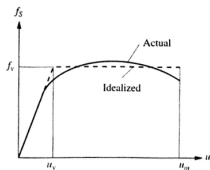
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A more complex behaviour may be represented with an elasto-perfectly plastic (e-p) bilinear idealisation, see figure, where two important requirements are obeyed

1. the initial stiffness of the idealised e-p system is the same of the real system, so that the natural frequencies of vibration for small deformation are equal,
2. the yielding strength is chosen so that the sum of stored and dissipated energy in the e-p system is the same as the energy stored and dissipated in the real system.





In perfect plasticity, when yielding (a) the force is constant,  $f_s = f_y$  and (b) the stiffness is null,  $k_y = 0$ . The force  $f_y$  is the yielding force, the displacement  $x_y = f_y/k$  is the yield deformation.

In the right part of the figure, you can see that at unloading ( $dx = 0$ ) the stiffness is equal to the initial stiffness, and we have  $f_s = k(x - x_{p_{tot}})$  where  $x_{p_{tot}}$  is the total plastic deformation.

# Definitions

For a given seismic excitation, we give the following definitions  
**equivalent system** a linear system with the same characteristics  
 $(\omega_n, \zeta)$  of the non-linear system

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**normalised yield strength**,  $\bar{f}_y$  is the ratio of the yield strength to the peak force of the equivalent system,

$$\bar{f}_y = \min \left\{ \frac{f_y}{f_0} = \frac{x_y}{x_0}, 1 \right\}.$$

It is  $\bar{f}_y \leq 1$  because for  $f_y \geq f_0$  there is no yielding, and in such case we define  $\bar{f}_y = 1$ .

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**yield strength reduction factor**,  $R_y$  it comes handy to define  $R_y$ , as the reciprocal of  $\bar{f}_y$ ,

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**normalised spring force**,  $\bar{f}_S$  the ratio of the e-p spring force to the yield strength,

$$\bar{f}_S = \frac{f_S}{f_y}.$$

equivalent acceleration,  $a_y$  the (pseudo-)acceleration required to yield the system,

$$a_y = \omega_n^2 x_y = f_y/m.$$

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Note that the ratio between the e-p and elastic peak responses is given by

$$\frac{x_m}{x_0} = \frac{x_m}{x_y} \frac{x_y}{x_0} = \mu \bar{f}_y = \frac{\mu}{R_y} \rightarrow \mu = R_y \frac{x_m}{x_0}.$$

# Normalising the force

For an e-p system, the equation of motion (*EOM*) is

$$m\ddot{x} + c\dot{x} + f_S(x, \dot{x}) = -m\ddot{u}_g(t)$$

with  $f_S$  as shown in a previous slide. The *EOM* must be integrated numerically to determine the time history of the e-p response,  $x(t)$ .

For a given excitation  $\ddot{x}_g(t)$ , the response depends on 3 parameters,  $\omega_n = \sqrt{k/m}$ ,  $\zeta = c/(2\omega_n m)$  and  $x_y$ .

If we divide the *EOM* by  $m$ , recalling our definition of the normalised spring force, the last term is

$$\frac{f_S}{m} = \frac{1}{m} \frac{f_y}{f_y} f_S = \frac{1}{m} k x_y \frac{f_S}{f_y} = \omega_n^2 x_y \bar{f}_S$$

and we can write

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x_y \bar{f}_S(x, \dot{x}) = -\ddot{u}_g(t)$$

# Normalising the displacements

With the position  $x(t) = \mu(t) x_y$ , substituting in the *EOM* and dividing all terms by  $x_y$ , it is

$$\ddot{\mu} + 2\omega_n \zeta \dot{\mu} + \omega_n^2 \bar{f}_S(\mu, \dot{\mu}) = -\frac{\omega_n^2 \ddot{x}_g}{\omega_n^2 x_y} = -\omega_n^2 \frac{\ddot{x}_g}{a_y}$$

It is now apparent that the input function for the ductility response is the acceleration ratio: doubling the ground acceleration or halving the yield strength leads to exactly the same response  $\mu(t)$  and the same peak value  $\mu$ .

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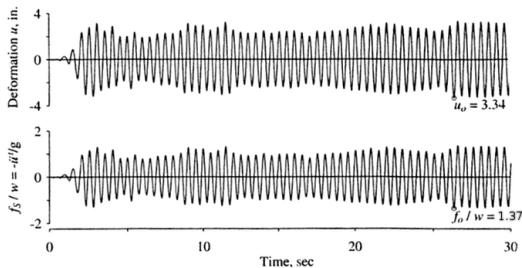
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The equivalent acceleration can be expressed in terms of the normalised yield strength  $\bar{f}_y$ ,

$$a_y = \frac{f_y}{m} = \frac{\bar{f}_y f_0}{m} = \frac{\bar{f}_y k x_0}{m} = \bar{f}_y \omega_n^2 x_0$$

and recognising that  $x_0$  depends only on  $\zeta$  and  $\omega_n$  we conclude that, for given  $\ddot{x}_g(t)$  and  $\bar{f}_S(\mu, \dot{\mu})$  the ductility response depends only on  $\zeta$ ,  $\omega_n$ ,  $\bar{f}_y$ .

# Elastic response, required parameters

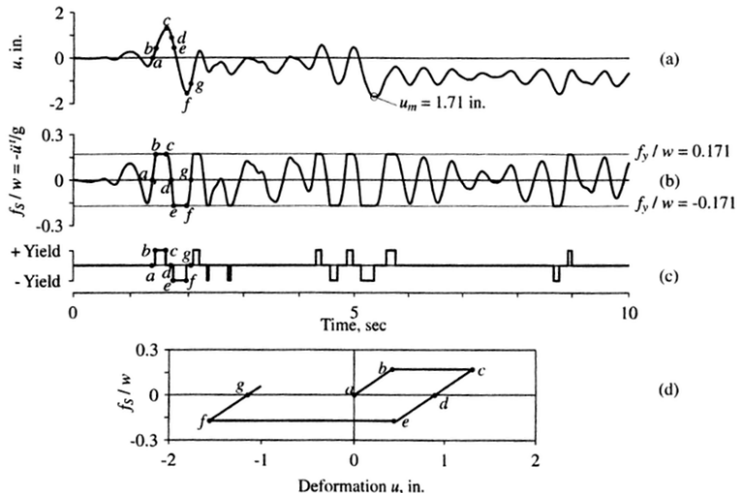


Response of linear system with  $T_n = 0.5$  sec and  $\zeta = 0$  to El Centro

In the figure above, the elastic response of an undamped,  $T_n = 0.5$  s system to the NS component of the El Centro 1940 ground motion (all our examples will be based on this input motion).

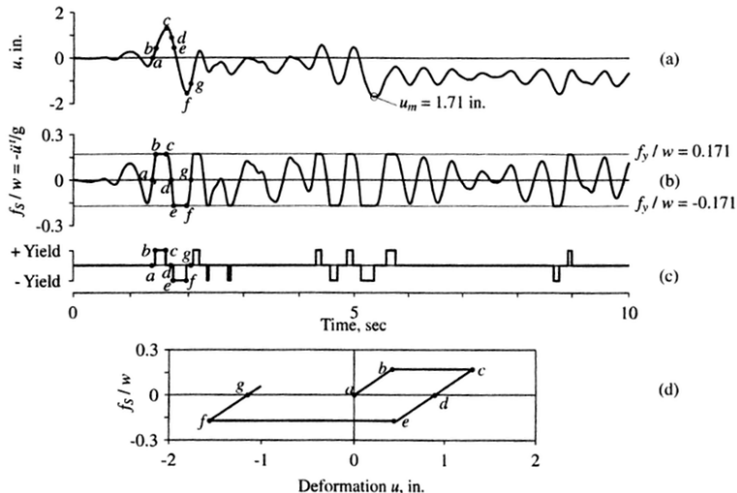
Top, the deformations, bottom the elastic force normalised with respect to weight, from the latter peak value we know that all e-p systems with  $f_y < 1.37w$  will experience plastic deformations during the EC1940NS ground motion.

# Inelastic response, $\bar{f}_y = 1/8$



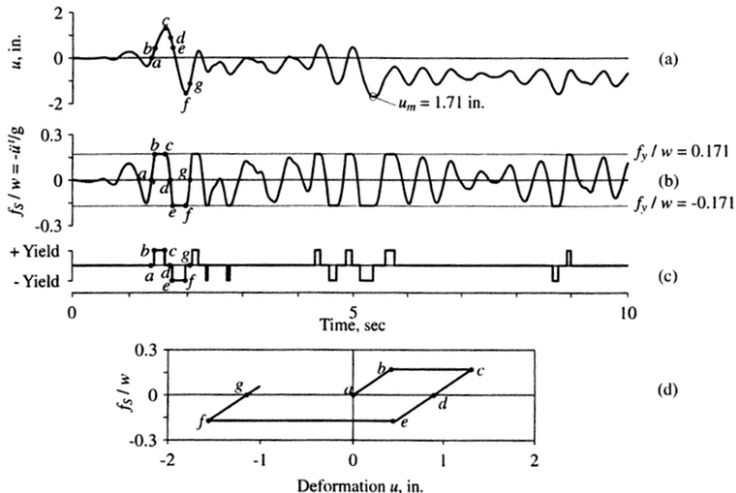
The various response graphs above were computed for  $\bar{f}_y = 0.125$  (i.e.,  $R_y = 8$  and  $f_y = \frac{1.37}{8}w = 0.171w$ ) and  $\zeta = 0$ ,  $T_n = 0.5$  s.

# Inelastic response, $\bar{f}_y = 1/8$



Top, the deformation response, note that the peak response is  $x_m = 1.71$  in, different from  $x_0 = 3.34$  in; it is  $\mu = R_y \frac{x_m}{x_0} = 4.09$

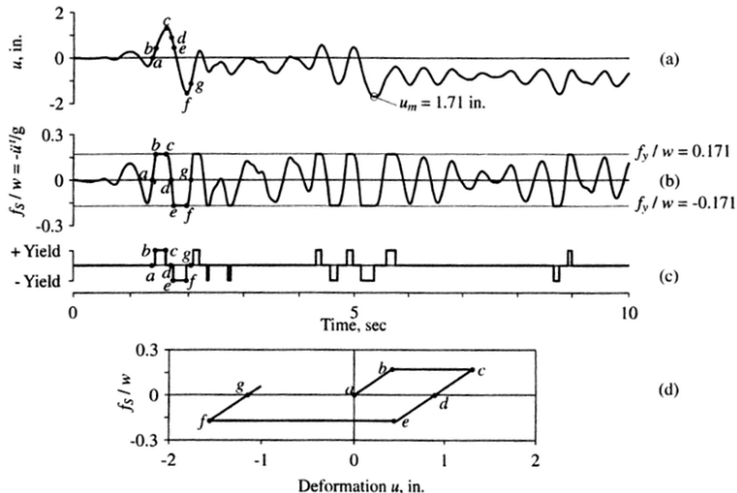
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Second row, normalised force  $f_S/w$ , note that the response is *clipped* at  $f_S = f_y = 1.171w$



# Inelastic response, $\bar{f}_y = 1/8$



Third row, response in terms of yielding state, positive or negative depending on the sign of velocity

# Inelastic response, $\bar{f}_y = 1/8$

The force-deformation diagram for the first two excursions in plastic domain, the time points  $a, b, c, d, e, f$  and  $g$  are the same in all 4 graphs:

- ▶ until  $t = b$  we have an elastic behaviour,
- ▶ until  $t = c$  the velocity is positive and the system accumulates positive plastic deformations,
- ▶ until  $t = e$  we have an elastic unloading (note that for  $t = d$  the force is zero, the deformation is equal to the total plastic deformation),
- ▶ until  $t = f$  we have another plastic excursion, cumulating negative plastic deformations
- ▶ until at  $t = g$  we have an inversion of the velocity and an elastic reloading.

# Response for different $\bar{f}_y$ 's

$\bar{f}_y$	$x_m$	$x_{\text{perm}}$	$\mu$
1.000	2.25	0.00	1.00
0.500	1.62	0.17	1.44
0.250	1.75	1.10	3.11
0.125	2.07	1.13	7.36

This table was computed for  $T_n = 0.5$  s and  $\zeta = 5\%$  for the EC1940NS excitation.

Elastic response was computed first, with peak response  $x_0 = 2.25$  in and peak force  $f_0 = 0.919w$ , later the computation was repeated for  $\bar{f}_y = 0.5, 0.25, 0.125$ .

In our example, all the peak values of the e-p responses are smaller than the elastic one, but this is not always true, and shouldn't be generalised.

The permanent displacements increase for decreasing yield strengths, and also this fact shouldn't be generalised.

Last, the ductility ratios increase for decreasing yield strengths, for our example it is  $\mu \approx R_y$ .

# Ductility demand and capacity

We can say that, for a given value of the normalised yield strength  $\bar{f}_y$  or of the yield strength reduction factor  $R_y$ , there is a *ductility demand*, a measure of the extension of the plastic behaviour that is required when we reduce the strength of the construction.

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Corresponding to this ductility demand our structure must be designed so that there is a sufficient *ductility capacity*.

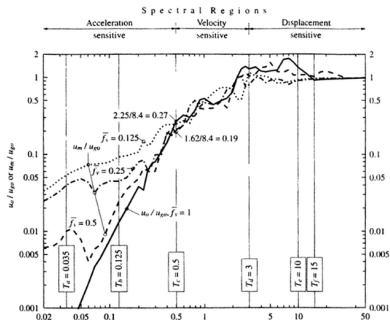
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Corresponding to this ductility demand our structure must be designed so that there is a sufficient *ductility capacity*.

Ductility capacity is, in the first instance, the ability of individual members to sustain the plastic deformation demand without collapsing, the designer must verify that the capacity is greater than the demand for all structural members that go non linear during the seismic excitation.

# Effects of $T_n$



$$x_m / x_{g,0}$$

There are two distinct zones: left there is a strong dependency on  $\bar{f}_y$ , the peak responses grow with  $R_y$ ; right the 4 curves intersect with each other and there is no clear dependency on  $\bar{f}_y$ .

For EC1940NS, for  $\zeta = .05$ , for different values of  $T_n$  and for  $\bar{f}_y = 1.0, 0.5, 0.25, 0.125$  the peak response  $x_0$  of the equivalent system (in black) and the peak responses of the 3 inelastic systems has been computed.

Motivation

Cyclic Behaviour

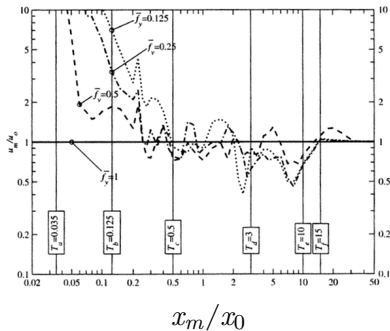
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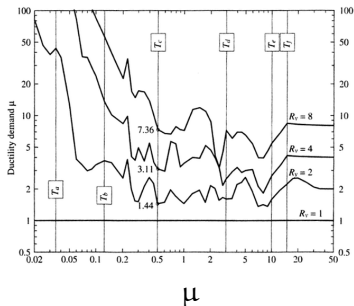


With the same setup as before, here it is the ratio of the  $x_m$ 's to  $x_0$ , what is evident is the fact that, for large  $T_n$ , this ratio is equal to 1... this is justified because, for large  $T_n$ 's, the mass is essentially at rest, and the deformation, either elastic or elasto-plastic, are equal and opposite to the ground displacement.

Also in the central part, where elastic spectrum ordinates are dominated by the ground velocity, there is a definite tendency for the  $x_m/x_0$  ratio, that is  $x_m/x_0 \approx 1$



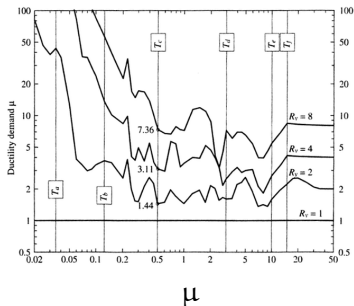
# Effects of $T_n$



With the same setup as before, in this graph are reported the values of the ductility factor  $\mu$ . The values of  $\mu$  are almost equal to  $R_y$  for large values of  $T_n$ , and in the limit, for  $T_n \rightarrow \infty$ , there is a strict equality. An even more interesting observation regard the interval  $T_c \leq T_n \leq T_f$ , where the values of  $\mu$  oscillate near the value of  $R_y$ .

On the other hand, the behaviour is completely different in the acceleration controlled zone, where  $\mu$  grows very fast, and the ductility demand is very high even for low values (0.5) of the yield strength reduction factor.

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The results we have discussed are relative to one particular excitation, nevertheless research and experience confirmed that these propositions are true also for different earthquake records, taking into account the differences in the definition of spectral regions.

The first step in an anti seismic design is to set an available ductility (based on materials, conception, details).

In consequence, we desire to know the yield displacement  $u_y$  or the yield force  $f_y$

$$f_y = ku_y = m\omega_n^2 u_y$$

for which the ductility demand imposed by the ground motion is not greater than the available ductility.

# Response Spectrum for Yield States

For each  $T_n$ ,  $\zeta$  and  $\mu$ , the *Yield-Deformation Response Spectrum* ( $D_y$ ) ordinate is the corresponding value of  $u_y$ :  $D_y = u_y$ . Following the ideas used for Response and Design Spectra, we define  $V_y = \omega_n u_y$  and  $A_y = \omega_n^2 u_y$ , that we will simply call pseudo-velocity and pseudo-acceleration spectra. Using our definitions of pseudo acceleration, we can find a more significant expression for the design force:

$$f_y = ku_y = m\omega_n^2 u_y = mA_y = w \frac{A_y}{g},$$

where  $w$  is the weight of the structure.

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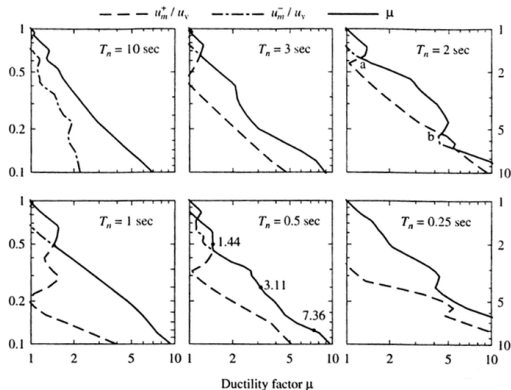
For each  $T_n$ ,  $\zeta$  and  $\mu$ , the *Yield-Deformation Response Spectrum* ( $D_y$ ) ordinate is the corresponding value of  $u_y$ :  $D_y = u_y$ . Following the ideas used for Response and Design Spectra, we define  $V_y = \omega_n u_y$  and  $A_y = \omega_n^2 u_y$ , that we will simply call pseudo-velocity and pseudo-acceleration spectra. Using our definitions of pseudo acceleration, we can find a more significant expression for the design force:

$$f_y = k u_y = m \omega_n^2 u_y = m A_y = w \frac{A_y}{g},$$

where  $w$  is the weight of the structure.

Our definition of inelastic spectra is compatible with the definition of elastic spectra, because for  $\mu = 1$  it is  $u_y = u_0$ . Finally, the  $D_y$  spectrum and its derived pseudo spectra can be plotted on the tripartite log-log graph.

# Computing $D_y$



On the left, for different  $T_n$ 's and  $\mu = 5\%$ , the independent variable is in the ordinates, either  $\bar{f}_y$  (left) or  $R_y$  (right) the strength reduction factor. Dash-dot lines is  $u_m^+ / u_y$ , dash-dot is  $u_m^- / u_y$ .

$u_m^+$  and  $u_m^-$  are the peaks of positive and negative displacements of the inelastic system, the maximum of their ratios to  $u_y$  is the ductility  $\mu$ .

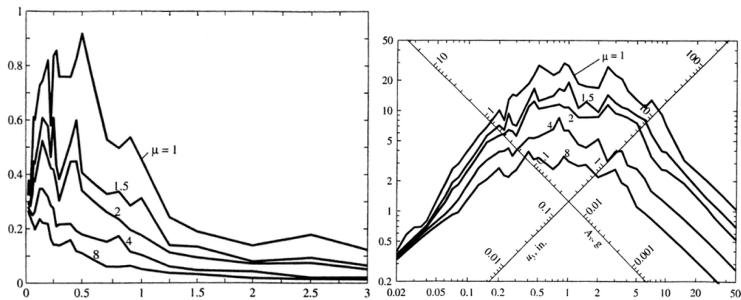
If we look at these graphs using  $\mu$  as the independent variable, it is possible that for a single value of  $\mu$  there are different  $\bar{f}_y$  values on the tick line: in this case, for security reasons, the designer must design for the higher value of  $\bar{f}_y$ .

# Example

For EC1940NS,  $z = 5\%$ , the yield-strength response spectra for  $\mu = 1.0, 1.5, 2.0, 4.0, 8.0$ .

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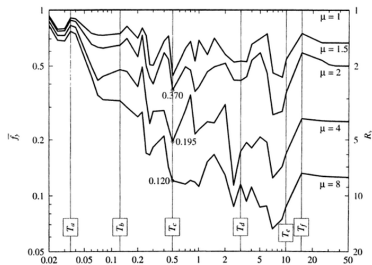


On the left, a lin-lin plot of the pseudo-acceleration normalized (and adimensionalised) with respect to  $g$ , the acceleration of gravity.

On the right, a log-log tripartite plot of the same spectrum. Even a small value of  $\mu$  produces a significant reduction in the required strength.



We have seen that  $\bar{f}_y = \bar{f}_y(\mu, T_n, \zeta)$  is a monotonically increasing function of  $\mu$  for fixed  $T_n$  and  $\zeta$ .

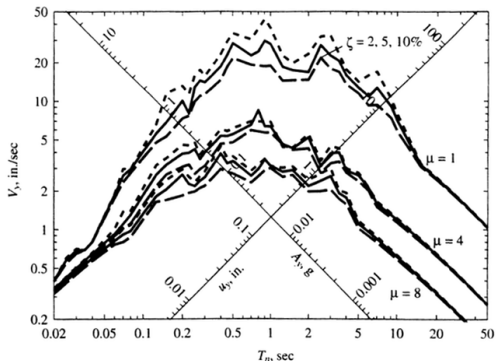


Left the same spectra of the previous slides, plotted in a different format,  $\bar{f}_y$  vs  $T_n$  for different values of  $\mu$ .

The implication of this figure is that an anti seismic design can be based on strength, ductility or a combination of the two.

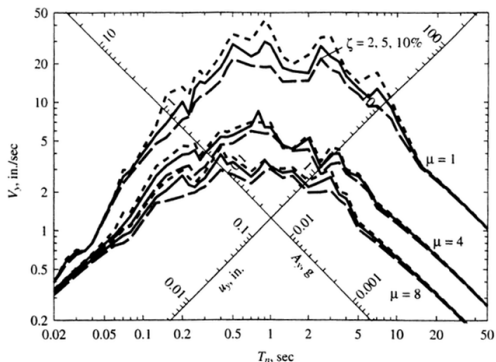
For  $T_n = 1.0$ , the peak force for EC1940NS in an elastic system is  $f_0 = 0.919 w$ , so it is possible to design for  $\mu = 1.0$ , hence  $f_y = 0.919 w$  or for an high value of ductility,  $\mu = 8.0$ , hence  $f_y = 0.120 \cdot 0.919 w$  or, if such an high value of ductility cannot be easily reached, design for  $\mu = 4.0$  and a yielding force of  $0.195$  times  $f_0$ .

# Yielding and Damping



El Centro 1940 NS, elastic response spectra and inelastic spectra for  $\mu = 4$  and  $\mu = 8$ , for different values of  $\zeta$  (2%, 5% and 10%). Effects of damping are relatively important and only in the velocity controlled area of the spectra, while effects of ductility are always important except in the high frequency range.

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Overall, the ordinates reduction due to modest increases in ductility are much stronger than those due to increases in damping.

$$\int^{x(t)} m \ddot{x} dx + \int^{x(t)} c \dot{x} dx + \int^{x(t)} f_S(x, \dot{x}) dx = - \int^{x(t)} m \ddot{x}_g dx$$

This is an energy balance, between the input energy  $\int m \ddot{x}_g$  and the sum of the kinetic, damped, elastic and dissipated by yielding energy.

In every moment, the elastic energy  $E_S(t) = \frac{f_S^2(t)}{2k}$  so the yielded energy is

$$E_y = \int f_S(x, \dot{x}) dx - \frac{f_S^2(t)}{2k}.$$

The damped energy can be written as a function of  $t$ , as  $dx = \dot{x} dt$ :

$$E_D = \int c \dot{x}^2(t) dt$$

# Energy Dissipation

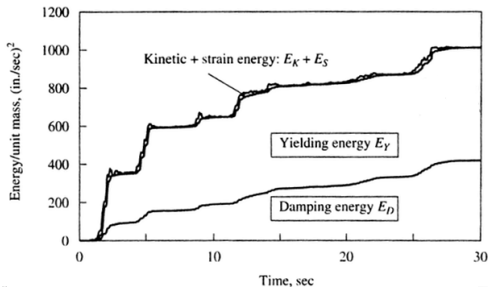
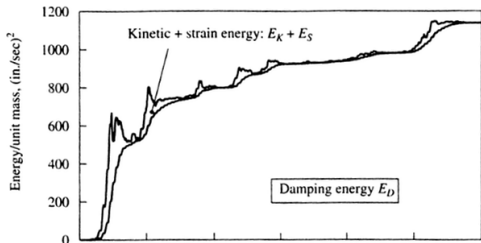
For a system with

$m = 1$  and

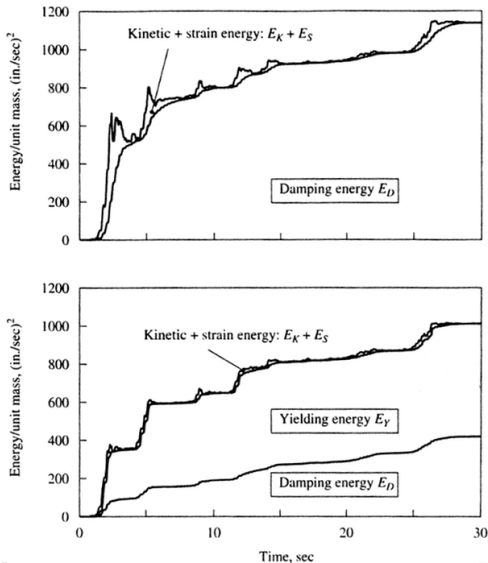
a)  $\bar{f}_y = 1$

b)  $\bar{f}_y = 0.25$

the energy  
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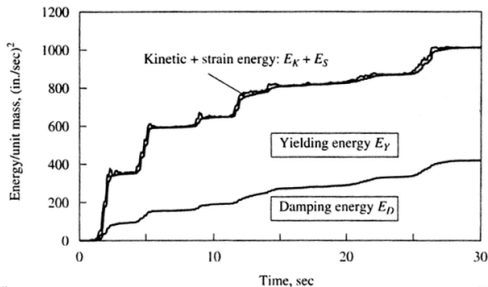
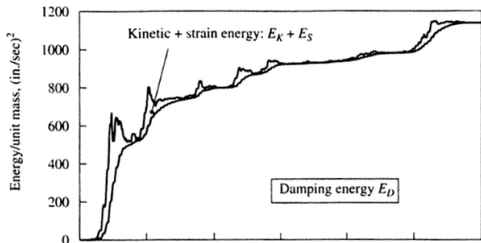
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# Energy Dissipation



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the energy contributions during the EC1940NS,  $T_n = 0.5$  s and  $\zeta = 5\%$ .

In a), input energy is stored in kinetic+elastic energy during strong motion phases and is subsequently dissipated by damping.

In b), yielding energy is dissipated by means of some structural damage.

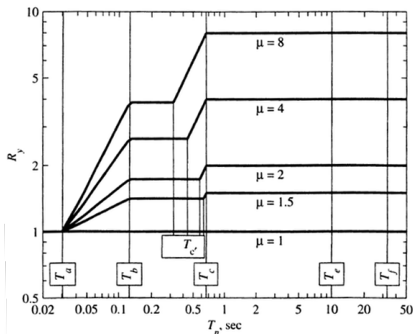
Two possible approaches:

1. compute response spectra for constant ductility demand for many consistent records, compute response parameters statistics and derive inelastic design spectra from these statistics, as in the elastic design spectra procedures;
2. directly modify the elastic design spectra to account for the ductility demand values.

The first procedure is similar to what we have previously seen, so we will concentrate on the second procedure, that it is much more used in practice.



# $R_y - \mu - T_n$ equations

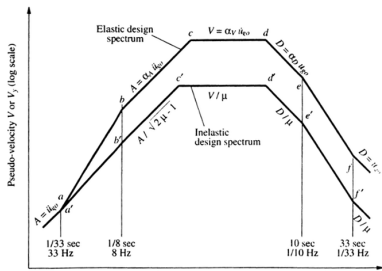


Based on observations and energetic considerations, the plots of  $R_y$  vs  $T_n$  for different  $\mu$  values can be approximated with straight lines in a log-log diagram, where the constant pieces are defined in terms of the key periods in  $D - V - A$  graphs.

$$R_y = \begin{cases} 1 & T_n < T_a \\ \sqrt{2\mu - 1} & T_b < T_n < T_c' \\ \mu & T_c < T_n \end{cases}$$

The key period  $T_{c'}$  is different from  $T_c$ , as we will see in the next slide; the constant pieces are joined with straight lines in the log-log diagram.

# Construction of Design Spectrum



Start from a given elastic design spectrum, defined by the points a-b-c-d-e-f.

Choose a value  $\mu$  for the ductility demand.

Reduce all ordinates right of  $T_c$  by the factor  $\mu$ , reduce the ordinates in the interval  $T_b < T_n < T_c$  by  $\sqrt{2\mu/1}$ .

Draw the two lines  $A = \frac{\alpha_A \ddot{x}_{g0}}{\sqrt{2\mu-1}}$  and  $A = \frac{\alpha_V \dot{x}_{g0}}{\mu}$ , their intersection define the key point  $T_{c'}$ .

Connect the point  $(T_a, A = \ddot{x}_{g0})$  and the point  $(T_b, A = \frac{\alpha_V \dot{x}_{g0}}{\mu})$  with a straight line.

As we already know (at least in principles) the procedure to compute the elastic design spectra for a given site from the peak values of the ground motion, using this simple procedure it is possible to derive the inelastic design spectra for any ductility demand level.

# Important Relationships

For different zones on the  $T_n$  axis, the simple relationships we have previously defined can be made explicit using the equations that define  $R_y$ , in particular we want relate  $u_m$  to  $u_0$  and  $f_y$  to  $f_0$  for the elastoplastic system and the equivalent system.

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Similar equations can be established also for the inclined connection segments in the  $R_y$  vs  $T_n$  diagram.

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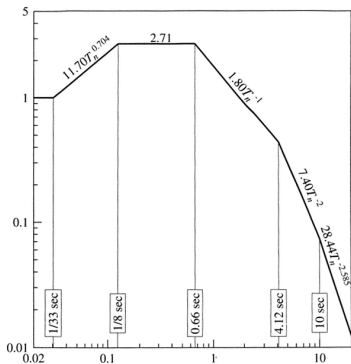
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- ▶ The design peak deformation,  $u_m = \mu D_y / R_y$ , is

$$u_m = \frac{\mu}{R_y(\mu, T_n)} \frac{A_y}{\omega_n^2}.$$

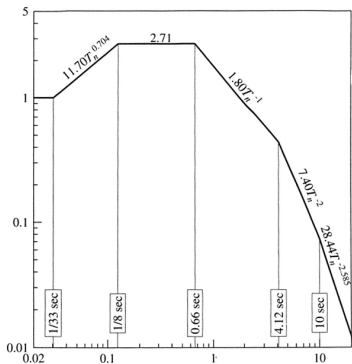
# Example



One storey frame, weight  $w$ , period is  $T_n = 0.25$  s, damping ratio is  $\zeta = 5\%$ , peak ground acceleration is  $\ddot{x}_{g0} = 0.5$  g. Find design forces for

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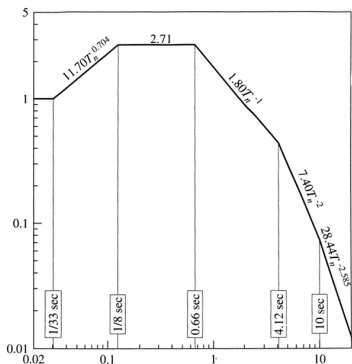
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spectrum for  $\ddot{x}_{g0} = 1$  g,

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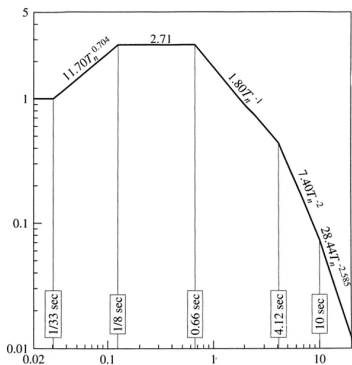
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$$\mu = 1 \quad f_y = 1.355w, \quad u_m = 2.104 \text{ cm},$$

$$\mu = 4 \quad f_y = 0.512w, \quad u_m = 3.182 \text{ cm},$$

$$\mu = 8 \quad f_y = 0.350w, \quad u_m = 4.347 \text{ cm}.$$