Dynamics of Structures 2011-2012

1st home assignment due on Tuesday 2012-05-22

Solutions

Contents

1 Dynamical Testing

A simple structure, which can be modeled as a single degree of freedom system, is subjected to testing to measure its dynamical characteristics:

1. the structure is loaded with a static force $F = 12.0\,\mathrm{kN}$ and the static displacement is measured: $u_0 = 2.0\,\mathrm{mm}$,

2. the force is then suddenly released, the structure oscillates freely and after 12 cycles, corresponding to 3.0 s after the force release, the measured maximum displacement is $u_{12} = 0.6\,\mathrm{mm}$.

What are the values of m, ζ and k?

1.1 Solution

From $\Delta_{\mathrm{stat}} = u_0 = P/k$ it is $k = P/u_0 = \frac{12\,000\,\mathrm{N}}{0.002\,\mathrm{m}} \Rightarrow k = 6\,\mathrm{MN\,m^{-1}}$.

The period of vibration of the damped system being $T_{\mathrm{d}} = \frac{3\,\mathrm{s}}{12} = 0.25\,\mathrm{s}$, the damped frequency is $\omega_{\mathrm{d}} = \frac{2\pi}{T_{\mathrm{d}}} = 25.1327\,\mathrm{rad\,s^{-1}}$.

We derive a recursive formula to compute ζ:

$$\frac{u_n}{u_0} = \exp(-\zeta\omega t_n) \qquad\qquad \log\frac{u_0}{u_n} = \delta_{\log} = \zeta\omega t_n$$

but the natural frequency ω is $\omega_\text{d}/\sqrt{1-\zeta^2}$

$$\delta_\text{log} = \zeta \frac{\omega_\text{d}}{\sqrt{1-\zeta^2}} t_n$$

introducing a sequence of successive approximations to the damping ratio, $\zeta_0, \zeta_1, \ldots, \zeta_n, \zeta_{n+1}, \ldots$, we can write

$$\zeta_{n+1} = \frac{\delta_\text{log}\sqrt{1-\zeta_n^2}}{\omega_\text{d}\, t_n}.$$

Initializing the procedure above with $\zeta_0 = 0$, after a few iterations we converge to the values $\zeta = 1.5966\%$ and $\omega_n = 25.1359\,\text{rad}\,\text{s}^{-1}$.
From $k = \omega_n^2 m$ we find $m = 9496.4395\,\text{kg}$

2 Vibration Isolation

A rotating machine is characterized by its mass $m = 192\,000\,\text{kg}$, its working frequency $f_\text{w} = 100\,\text{Hz}$ and the value of the unbalanced load it exerts on its supports, $f_\text{w} = 4800\,\text{N}$.

Design a suspension system for the machine (i.e., give the values of k and c, the stiffness and the damping constant) knowing that (1) it is necessary to reduce the transmitted force to 400 N, (2) to reduce the vibration amplitude during transients the suspension must have a viscous damping ratio of 6%.

2.1 Solution

The requested transmissibility ratio is $\text{TR}_\text{req} = \frac{400\,\text{N}}{4800\,\text{N}} = \frac{1}{12}$; the effective TR must be less or equal to $1/12$ and, for $\zeta = 0.06$, we have

$$\text{TR} = \frac{\sqrt{1+(0.12\beta)^2}}{\sqrt{(1-\beta^2)^2+(0.12\beta)^2}} \le \frac{1}{12},$$

that is, squaring both members and rearranging,

$$144(1+0.0144\beta^2) \le 1 - 2\beta^2 + \beta^4 + 0.0144\beta^2.$$

Solving for β^2 gives $\beta^2 = \frac{\omega^2}{k/m} \ge 14.1588735215 \Rightarrow 14.1588735215k \le m\omega^2$, with $m = 192\,000\,\text{kg}$ and $\omega = 100 \cdot 2\pi$ it is

$$k = \frac{192\,000 \cdot 10\,000 \cdot 4\pi^2}{14.1588735215} = 5353.431\,\text{kN}\,\text{mm}^{-1},$$

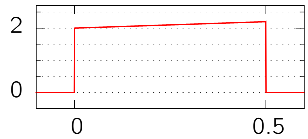
the damping can be computed by $c = 2\zeta\omega_n m$, with $\omega_n = \sqrt{k/m} = \sqrt{\frac{5.353\,431\times10^9\,\text{N}\,\text{m}^{-1}}{192\,000\,\text{kg}}} = 166.98\,\text{rad}\,\text{s}^{-1}$ it is

$$c = 2 \cdot 0.06 \cdot 166.98 \cdot 192000 = 3.847 \times 10^6\,\text{N}\,\text{s}\,\text{m}^{-1}.$$

3 Numerical Integration

A single degree of freedom system, with a mass $m = 1200\,\text{kg}$, a stiffness $k = 50\,\text{kN}\,\text{m}^{-1}$ and a damping ratio $\zeta = 0.05$ is at rest when it is subjected to an external force $p(t)$:

$$p(t) = \begin{cases} 2\,\text{kN}\left(0.95 + \frac{1}{10}\frac{t}{t_0}\right) & \text{for } 0.0 \le t \le t_0 = 0.5\,\text{s}, \\ 0.0 & \text{otherwise.} \end{cases}$$



(1) Write the exact response for $0 \le t \le t_0$ and $t_0 \le t \le 2\,\text{s}$ using superposition of the general integral and appropriate particular integrals. (2) Integrate the equation of motion numerically, using the algorithm of linear acceleration with a time step $h = 0.02\,\text{s}$ for $0 \le t \le 2\,\text{s}$. (3) Plot your results (both the exact response and the numerical solution) in a meaningful manner. (4) [optional] Repeat the numerical integration assuming an elasto-plastic spring with a yield strength $f_y = 3.2\,\text{kN}$ and plot your results.

3.1 Solution

The frequency of free vibration is
$\omega_\text{n} = \sqrt{k/m} = 6.454\,972\,\text{rad}\,\text{s}^{-1}$ and it follows that it is $T_n = 0.973\,386\,919\,\text{s}$. The damped frequency is $\omega_\text{d} = 6.446\,898\,\text{rad}\,\text{s}^{-1}$ and the damping coefficient is $c = 2\zeta\omega_\text{n}m = 774.597\,\text{N}\,\text{s}\,\text{m}^{-1}$

Analytical Elastic Response

For a polynomial loading of degree 1, the particular integral and its time derivatives are

$$\xi(t) = R + St/t_0, \qquad \dot\xi(t) = S/t_0, \qquad \ddot\xi(t) = 0.$$

Substituting ξ, $\dot\xi$, $\ddot\xi$ in the equation of motion we have

$$c\,S/t_0 + k\,R + k\,St/t_0 = 1900 + 200\,t/t_0.$$

Equating the coefficients of the polynomials, substituting the numerical values and solving for R and S,

$$\begin{cases} kS & = 200 \\ cS/t_0 + kR & = 1900 \end{cases} \Rightarrow \begin{cases} S & = 4\,\text{mm} \\ R & = 37.876\,065\,\text{mm} \end{cases}.$$

The general integral is

$$x_1(t) = e^{-\zeta\omega_\text{n}t}\left(A_1\cos(\omega_\text{d}t) + B_1\sin(\omega_\text{d}t)\right) + \xi(t)$$

and, for a system starting from rest, the initial condition are

$$x_1(0) = A_1 + R = 0, \quad \dot x_1(0) = -\zeta\omega_\text{n}A_1 + \omega_\text{d}B_1 + S/t_0 = 0$$

substituting and solving

$$A_1 = -37.876\,065\,\text{mm}, \qquad B_1 = -3.137\,082\,\text{mm},$$

and the forced response is

$$x_1(t) = exp^{-0.323t}(-37.876\cos 6.447t - 3.137\sin 6.447t) + 37.876 + 8t.$$

To find the free response, $x_2(t) = e^{-\zeta\omega_n t}\left(A_2\cos(\omega_d t) + B_2\sin(\omega_d t)\right)$ for $t_0 \le t$, we need

$$x_1(t_0) = 74.217\,946\,\text{mm} = x_2(t_0) \quad \text{and} \quad \dot{x}_1(t_0) = -2.275\,724\,\text{mm/s} = \dot{x}_2(t_0)$$

The initial conditions, with $\mathcal{E} = e^{-\zeta\omega_n t_0}$, $\mathcal{C} = \cos(\omega_d t_0)$ and $\mathcal{S} = \sin(\omega_d t_0)$, are

$$\begin{cases} \mathcal{E}\mathcal{C}A_2 & + & \mathcal{E}\mathcal{S}B_2 & = & 74.217\,946\,\text{mm}, \\ -\mathcal{E}(\omega_d\mathcal{S} + \zeta\omega_n\mathcal{C})A_2 & + & \mathcal{E}(\omega_d\mathcal{C} - \zeta\omega_n\mathcal{S})B_2 & = & -2.275\,724\,\text{mm/s}, \end{cases}$$

after substituting the numerical values and solving, it is

$$A_2 = -86.600\,226\,\text{mm}, \qquad B_2 = -11.069\,365\,\text{mm}.$$

Analytical EP Response

The yielding displacement is $x_y = 3200\,\text{N}/50\,000\,\text{N}\,\text{m}^{-1} = 0.064\,\text{m} = 64\,\text{mm}$ and it is reached at time $t_y = 0.373\,887\,\text{s}$ (t_y is found numerically); the velocity at yelding is $\dot{x}_1(t_y) = 158.394\,93\,\text{mm/s}$.

During the forced phase, if the velocity doesn't become negative, the general integral is

$$x_3 = A_3\exp(-ct/m) + B_3 + R_3\frac{t}{t_0} + S_3\left(\frac{t}{t_0}\right)^2.$$

Substituting the particular integral in the equation of motion

$$\frac{2m}{t_0^2}S_3 + cR_3 + \frac{2c}{t_0}\frac{t}{t_0} + f_y = 1900 + 200\frac{t}{t_0}$$

and equating the polynomial coefficients, it is

$$R_3 = -1239.146\,\text{mm} \qquad \text{and} \qquad S_3 = 64.550\,\text{mm}.$$

Writing the initial conditions for $t = t_y$ gives

$$\begin{cases} A_3\exp(-ct_y/m) & + & B_3 & = & x_1(t_y) - R_3\frac{t_y}{t_0} - S_3\left(\frac{t_y}{t_0}\right)^2 \\ -\frac{c}{m}A_3\exp(-ct_y/m) & & & = & \dot{x}_1(t_y) - \frac{R_3}{t_0} - \frac{S_3}{t_0}\frac{t_y}{t_0} \end{cases}$$

that gives

$$A_3 = -4818.947\,\text{mm}, \quad \text{and} \quad B_3 = 4740.138\,\text{mm}.$$

The resulting motion is then

$$x_3(t) = -4818.947e^{-0.6455t} + 258.20t^2 - 2478.29t + 4740.14.$$

The displacements and velocities (note that the velocity remains positive) at the end of the forced phase are

$$x_3(t_0) = 75.872\,282\,\text{mm}, \qquad \dot{x}_3(t_0) = 32.477\,544\,\text{mm/s},$$

the general integral in the free response phase is

$$x_4 = A_4 \exp(-ct/m) + B_4 + R_4 \frac{t}{t_0}$$

where there is a linear term to take into account the *braking effect* of the constant force in the yielded spring.

R_4 can be computed from the equation of motion, A_4 and B_4 by $x_4(t_0) = x_3(t_0)$ and $\dot{x}_4(t_0) = \dot{x}_3(t_0)$, to have

$$A_4 = -8907.356\,\text{mm}, \quad B_4 = 8591.777\,\text{mm and } R_4 = -2065.591\,\text{mm}.$$

To proceed, we have to find the time $t = t_{\text{max}}$ such that $x_4(t_{\text{max}})$ is maximum and $\dot{x}_4(t_{\text{max}}) = 0$. This must be done numerically, to find $t_{\text{max}} = 0.512\,131\,\text{s}$ and , successively, $x_4(t_{\text{max}}) = x_{\text{max}} = 76.069\,025\,\text{mm}$.

At $t = t_{\text{max}}$ begins a new elastic response phase, with

$$x_5(t) = e^{-\zeta\omega_n t}\left(A_5 \cos(\omega_d t) + B_5 \sin(\omega_d t)\right) + (x_4(t_{\text{max}}) - x_y),$$

where the constant term is required to take into account that $f_S = k \cdot (x - x_p) = k \cdot (x - (x_4(t_{\text{max}}) - x_y)) = kx - k \cdot 12.069025$.

The integration constants A_5 and B_5 can be determined by the initial conditions

$$x_5(t_{\text{max}}) = x_{\text{max}} = 76.069\,025\,\text{mm and } \dot{x}_5(t_{\text{max}}) = 0,$$

and are then

$$A_5 = -73.935\,395\,\text{mm and } B_5 = -15.765\,549\,\text{mm}.$$

The free elastic motion is

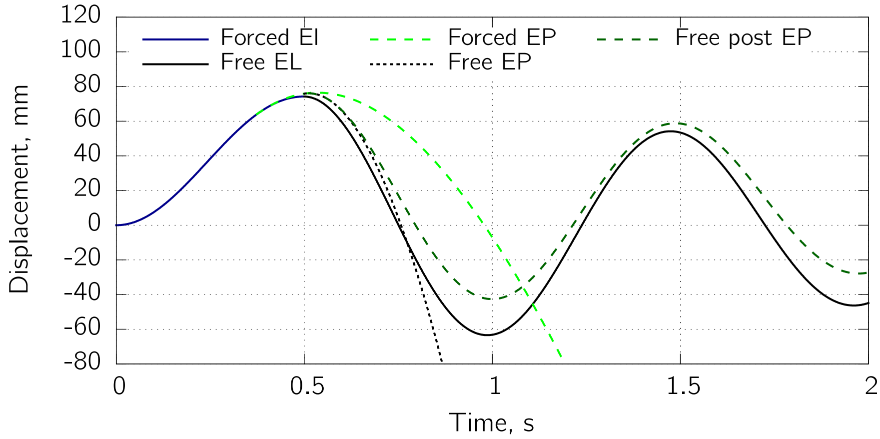
$$x_1(t) = exp^{-0.323t}(-73.935 \cos 6.447t - 15.766 \sin 6.447t) + 12.069.$$

Numerical Integration

The numerical integration was accomplished using two python programs, *one to compute the elastic response* and another *one to compute the elastoplastic response*.

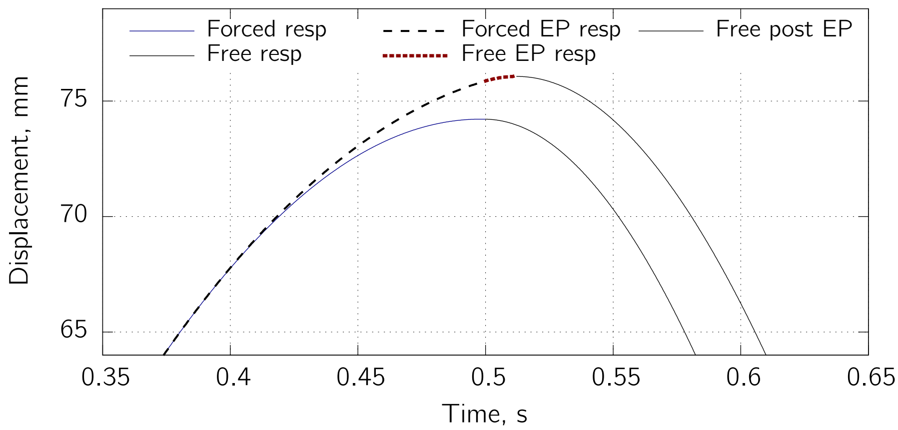
Presentation of the results

First, a graph that presents all of the exact results for both the indefinitely elastic behaviour and the elastic-perfectly plastic behaviour.



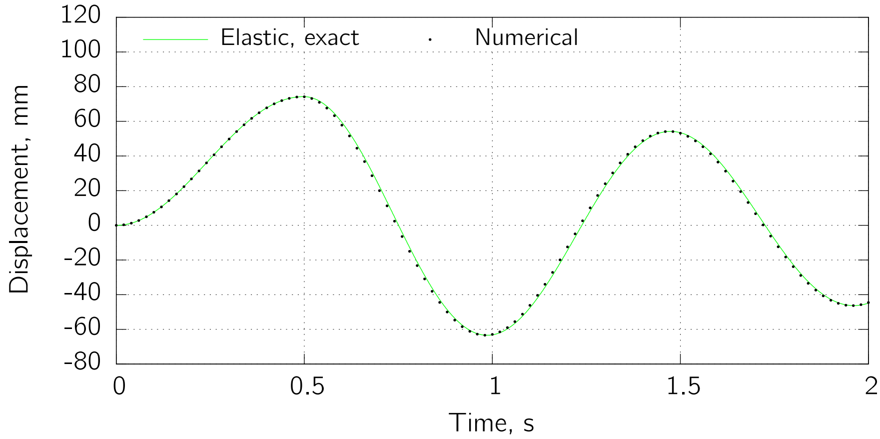
The EP responses are plotted also outside their interval of definition, so that it is possible to appreciate the asymptotic behaviour (linear) of the EP responses.

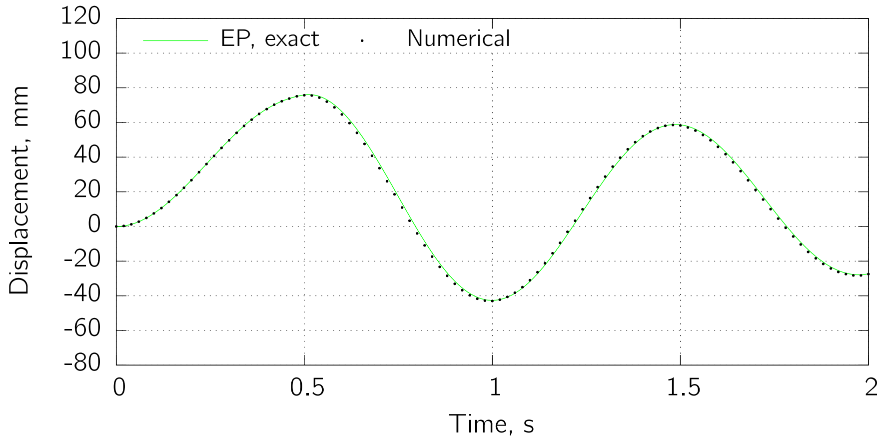
Here it is a *zoom* of the time interval around t_0, the minimum displacement being x_y, so that one can appreciate the transition between the different EP phases.



Below, the comparison between the exact responses and the results of the linear acceleration numerical procedure, with $h = 0.02\,\mathrm{s}$.

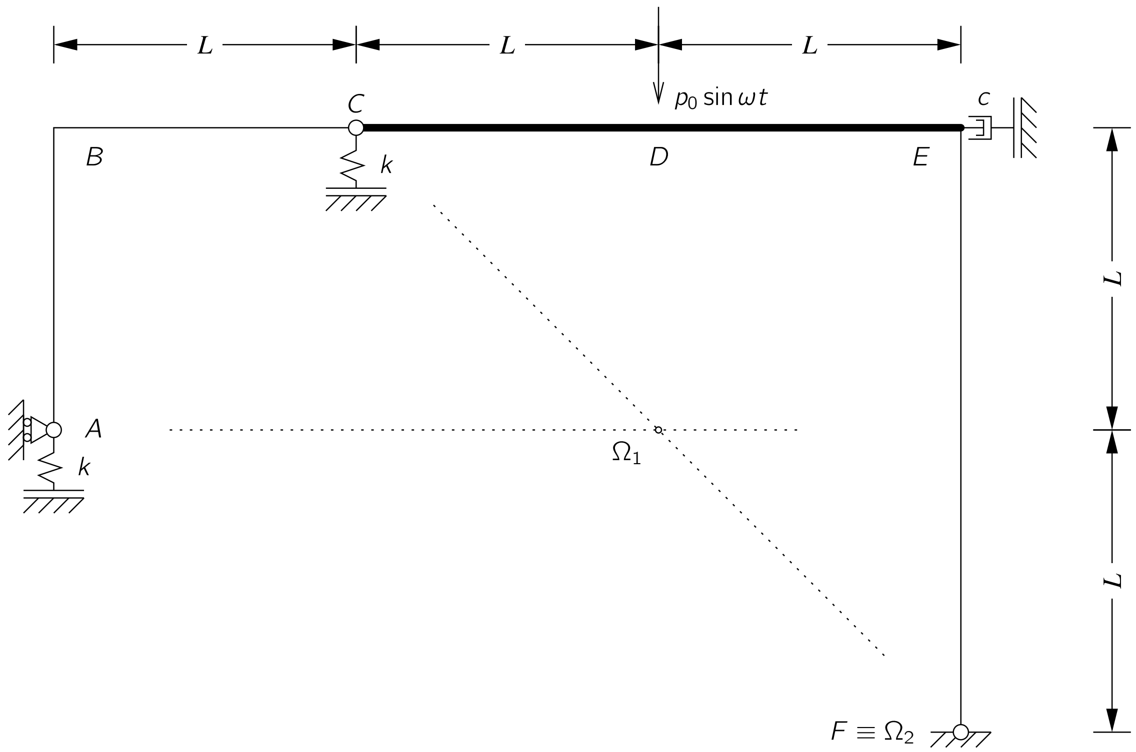
The continuous lines in green are obtained merging the diverse phases of the elastic and EP responses, the black dots are placed at the points $(t_n,\ x_n)$ obtained numerically.





The analytical and the numerical results are indeed in good agreement.

4 Generalized Coordinates (rigid bodies)



The articulated system in figure, composed by

- two rigid bars, (1) ABC and (2) CDEF,
- three fixed constraints, (1) a vertical roller in A, (2) an internal hinge in C and (3) a hinge in F,
- three deformable constraints, (1) a vertical spring in A, its stiffness k, (2) a vertical spring in C, its stiffness k and (3) a horizontal dash pot in E, its damping coefficient c,

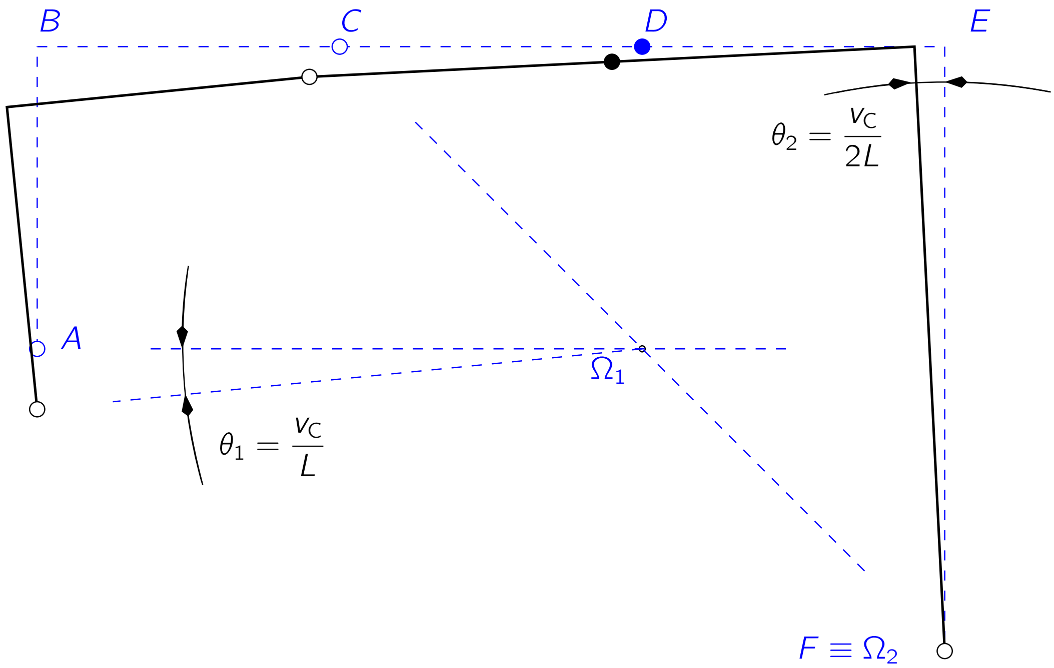
is excited by a vertical harmonic force applied in D, $p(t) = p_0 \sin \omega t$.

The horizontal part of the bar CDEF has a constant unit mass $\overline{m}$, with $\overline{m}\, L = m$; all the other parts of the system are massless.

Using v_c (the vertical displacement of C) as the generalized coordinate

1. compute the generalized parameters m^*, c^* and k^*,
2. compute the generalized loading $p^*(t)$ and
3. write the equation of dynamic equilibrium.

4.1 Solution



To write the equation of equilibrium we need the displacements for a small v_c, analyzing the figure we can write

	u/vc	v/v_C	θ/v_c
A	0	2	—
C	1	1	—
D	1	1/2	$1/2L$
E	1	0	—

The spring forces are $f_A = 2kv_c$ and $f_C = kv_c$, the dash pot force is $f_E = c\dot{v}_c$, the horizontal component of the inertial force is $f_x = 2m\ddot{v}_c$, the vertical component is $f_y = m\ddot{v}_c$ and the inertial couple is $w = 2m\frac{(2L)^2}{12}\frac{\ddot{v}_c}{2L} = \frac{1}{3}mL\ddot{v}_c$.

The equation of the virtual works, for a virtual rigid displacement and changing all the signs,

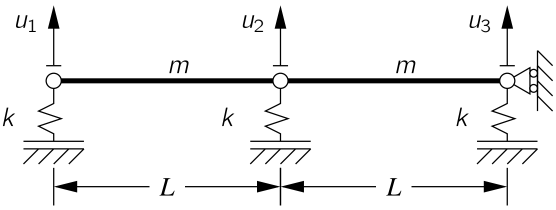
$$2m\ddot{v}_c\delta v_c + m\ddot{v}_c\frac{\delta v_c}{2} + \frac{1}{3}mL\ddot{v}_c\frac{\delta v_c}{2L} +$$
$$c\dot{v}_c\delta v_c + 2kv_c 2\delta v_c + kv_c\delta v_c - p_0 \sin\omega t\frac{\delta v_c}{2} = 0$$

or

$$\frac{8}{3}m\ddot{v}_c + c\dot{v}_c + 5kv_c = \frac{1}{2}p_0 \sin\omega t$$

5 Rayleigh quotient

The undamped 3 *DOF* system in figure is composed of 2 identical rigid bars, their masses equal to m, and three identical vertical springs, their stiffnesses equal to k. Use the free coordinates indicated in the figure.



Starting with a trial shape $\boldsymbol{\phi} = \begin{Bmatrix}1 & 1 & 1\end{Bmatrix}^T$ (i.e., $u_1 = u_2 = u_3 = Z_0 \sin \omega t$) give the successive Rayleigh estimates R_{00}, R_{01} and R_{11} of ω^2.

Hints:

— the bars have a not negligible rotational inertia, $J_i = mL^2/12$, that you should take into account,

— the free coordinates are not referred to the centers of mass of the bars, hence the mass matrix is non-diagonal,

— the simplest way to write the inertial forces on the nodes is using the matrix notation, $f_I = \boldsymbol{M}\,\ddot{\boldsymbol{u}}$, where the mass matrix's coefficients can be deduced comparing an explicit derivation of the kinetic energy T in terms of $\dot{u}_i$, m and J to the matrix expression $T = \frac{1}{2}\dot{\boldsymbol{u}}^T\boldsymbol{M}\,\dot{\boldsymbol{u}} = \frac{1}{2}\left(m_{11}\,\dot{u}_1^2 + \cdots + (m_{12} + m_{21})\,\dot{u}_1\dot{u}_2 + \cdots\right)$, where $m_{ij} = m_{ji}$.

5.1 Solution

The mass and the stiffness matrices are

$$\boldsymbol{M} = \frac{m}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}, \qquad \boldsymbol{K} = k\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The displacements and the velocities can be written (with $\boldsymbol{\phi} = \begin{Bmatrix}1 & 1 & 1\end{Bmatrix}$)

$$\boldsymbol{u} = \begin{Bmatrix}1\\1\\1\end{Bmatrix} Z_0 \sin \omega t, \qquad \dot{\boldsymbol{u}} = \omega\begin{Bmatrix}1\\1\\1\end{Bmatrix} Z_0 \cos \omega t.$$

The (double of the) maximum value of the deformation energy is

$$2V_{\max} = \boldsymbol{u}^T\boldsymbol{K}\,\boldsymbol{u} = 3kZ_0^2,$$

the (double of the) maximum value of the kinetic energy is

$$2T_{\max} = \dot{\boldsymbol{u}}^T\boldsymbol{M}\,\dot{\boldsymbol{u}} = \omega^2\,2mZ_0^2,$$

equating the maximum values

$$3kZ_0^2 = \omega^2\,2mZ_0^2 \quad \Rightarrow \quad \omega^2 = \frac{3}{2}\frac{k}{m} = 1.50\frac{k}{m}.$$

A better approximation to $2V_{\max}$ is given by the work done by the inertial forces

$$\boldsymbol{f}_{\mathrm{I}} = -\omega^2 \boldsymbol{M}\,\boldsymbol{\phi} Z_0 \sin\omega t = -\omega^2 m \begin{Bmatrix} 1/2 \\ 1 \\ 1/2 \end{Bmatrix} Z_0 \sin\omega t$$

and the displacements produced by these forces,

$$\bar{\boldsymbol{u}} = \boldsymbol{K}^{-1}\boldsymbol{f}_{\mathrm{I}} = -\omega^2 \frac{m}{k} \begin{Bmatrix} 1/2 \\ 1 \\ 1/2 \end{Bmatrix} Z_0 \sin\omega t.$$

The (double of the) maximum value of the deformation energy is now

$$2V_{\max} = \boldsymbol{f}_{\mathrm{I}}^T \bar{\boldsymbol{u}} = \omega^4 \frac{3}{2}\frac{m^2}{k} Z_0^2$$

and, equating to the maximum value of the kinetic energy,

$$\omega^4 \frac{3}{2}\frac{m^2}{k} Z_0^2 = \omega^2\, 2m Z_0^2 \quad \Rightarrow \quad \omega^2 = \frac{4}{3}\frac{k}{m} = 1.3333\frac{k}{m}.$$

A better approximation to the kinetic energy can be found using

$$\dot{\bar{\boldsymbol{u}}} = -\omega^3 \frac{m}{k} \begin{Bmatrix} 1/2 \\ 1 \\ 1/2 \end{Bmatrix} Z_0 \cos\omega t,$$

$$2T_{\max} = \dot{\bar{\boldsymbol{u}}}^T \boldsymbol{M}\, \dot{\bar{\boldsymbol{u}}} = \omega^6 \frac{7}{6}\frac{m^3}{k^2} Z_0^2$$

equating the maximum values

$$\omega^4 \frac{3}{2}\frac{m^2}{k} Z_0^2 = \omega^6 \frac{7}{6}\frac{m^3}{k^2} Z_0^2 \quad \Rightarrow \quad \omega^2 = \frac{9}{7}\frac{k}{m} = 1.2857\frac{k}{m}.$$

In the next problem we will find that

$$\omega_1^2 = (3 - \sqrt{3})\frac{k}{m} = 1.2679\frac{k}{m}$$

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6 3 DOF System

With reference to the system of problem 5, using the position $\omega_0^2 = \dfrac{k}{m}$

1. compute the three eigenvalues of the system and the corresponding eigenvectors,
2. normalize the eigenvectors with respect to the mass matrix $\boldsymbol{M}$ (it must be $\boldsymbol{\Psi}^T\,\boldsymbol{M}\,\boldsymbol{\Psi} = m$).

Considering that the system is at rest for $t = 0$ and is then loaded by a load vector $\boldsymbol{p}(t)$,

$$\boldsymbol{p}(t) = \frac{kL}{500} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \sin(2\omega_0 t),$$

3. write the three *modal* equations of motion,
4. integrate the modal equations of motion and write the three equations of modal displacement, $q_i = q_i(t)$,
5. find the analytical expression of $u_3 = u_3(t)$, showing your intermediate results and
6. plot u_3 in the interval $0 \le \omega_0\, t \le 10$.

6.1 Solution

The mass and the stiffness matrices, as in problem 5, are given by

$$\boldsymbol{M} = \frac{m}{6} \begin{bmatrix} 2.0000 & 1.0000 & 0.0000 \\ 1.0000 & 4.0000 & 1.0000 \\ 0.0000 & 1.0000 & 2.0000 \end{bmatrix}, \qquad \boldsymbol{K} = k \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix},$$

the equation of free vibrations is

$$\begin{bmatrix} 1-\frac{1}{3}\Omega^2 & \frac{-1}{6}\Omega^2 & 0 \\ \frac{-1}{6}\Omega^2 & 1-\frac{2}{3}\Omega^2 & \frac{-1}{6}\Omega^2 \\ 0 & \frac{-1}{6}\Omega^2 & 1-\frac{1}{3}\Omega^2 \end{bmatrix} \boldsymbol{\psi} = \boldsymbol{0}, \qquad \text{with } \Omega^2 = \frac{\omega^2}{k/m},$$

the equation of frequencies is

$$(\Omega^2)^3 - 9(\Omega^2)^2 + 24(\Omega^2) - 18 = (\Omega^2 - 3)\,\left((\Omega^2)^2 - 6(\Omega^2) + 6\right) = 0$$

and the roots are

$$\Omega_1^2 = 3 - \sqrt{3}, \quad \Omega_2^2 = 3, \quad \Omega^2 = 3 + \sqrt{3}.$$

Substituting Ω_i in the equation of frequencies, solving with $\psi_{1j} = 1$ for ψ_{2j} and ψ_{3j} and, finally, normalizing with respect to $\boldsymbol{M}$ gives the *normalized* eigenvector matrix,

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.36602540 & +1.22474487 & -1.36602540 \\ +1.00000000 & +0.00000000 & +1.00000000 \\ +0.36602540 & -1.22474487 & -1.36602540 \end{bmatrix}.$$

The particular integral is $\boldsymbol{\xi}(t) = \boldsymbol{x} \sin 2\omega_0 t$, substituting in the equation of motion

$$(\boldsymbol{K} - 4\omega_0^2 \boldsymbol{M})\boldsymbol{x} \sin 2\omega_0 t = \frac{kL}{500} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \sin 2\omega_0 t,$$

hence, removing the time dependency and collecting the unit stiffness k in the left member it is

$$k(\frac{\boldsymbol{K}}{k} - 4\frac{\boldsymbol{M}}{m})\,\boldsymbol{x} = k\begin{bmatrix}1-\frac{4}{3} & -\frac{2}{3} & 0\\ -\frac{2}{3} & 1-\frac{8}{3} & -\frac{2}{3}\\ 0 & -\frac{2}{3} & 1-\frac{4}{3}\end{bmatrix}\boldsymbol{x} = k\frac{L}{500}\begin{Bmatrix}0\\1\\0\end{Bmatrix}$$

simplifying k

$$\boldsymbol{x} = \frac{L}{500}\begin{bmatrix}1-\frac{4}{3} & -\frac{2}{3} & 0\\ -\frac{2}{3} & 1-\frac{8}{3} & -\frac{2}{3}\\ 0 & -\frac{2}{3} & 1-\frac{4}{3}\end{bmatrix}^{-1}\begin{Bmatrix}0\\1\\0\end{Bmatrix} = \frac{L}{500}\begin{Bmatrix}-2\\+1\\-2\end{Bmatrix}.$$

In modal coordinates (remember that $\boldsymbol{M}^\star \equiv m\boldsymbol{I}$, hence $\boldsymbol{M}^{\star-1} = \boldsymbol{I}/m$), the steady-state response is

$$\boldsymbol{q}_{\text{s-s}} = \frac{1}{m}\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x} = \frac{L}{500}\begin{Bmatrix}-0.36602540\\+0.00000000\\+1.36602540\end{Bmatrix}.$$

The initial rest conditions, in nodal coordinates, are

$$\boldsymbol{u}_0 = -\boldsymbol{\xi}(0) = \boldsymbol{0}, \qquad \dot{\boldsymbol{u}}_0 = -\dot{\boldsymbol{\xi}}(0) = -2\,\omega_0\frac{L}{500}\begin{Bmatrix}-2\\1\\-2\end{Bmatrix}$$

and in modal coordinates we have

$$\boldsymbol{q}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{q}}_0 = -2\omega_0\frac{1}{m}\boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{x} = \omega_0\frac{L}{500}\begin{Bmatrix}+0.73205081\\-0.00000000\\-2.73205081\end{Bmatrix}$$

For each mode the cosine coefficient is $a_i = 0$ and the sine coefficient is

$b_i = \dfrac{\dot{q}_{0,i}}{\omega_i}$, with $\omega_i = \omega_0\begin{bmatrix}1.1260 & 1.7321 & 2.1753\end{bmatrix}$ it is

$b_1 = \dfrac{L}{500}\dfrac{0.732}{1.126} = 0.650115\dfrac{L}{500}, \quad b_2 = 0, \quad b_3 = \cdots = -1.255926\dfrac{L}{500}.$

The modal responses can now be written

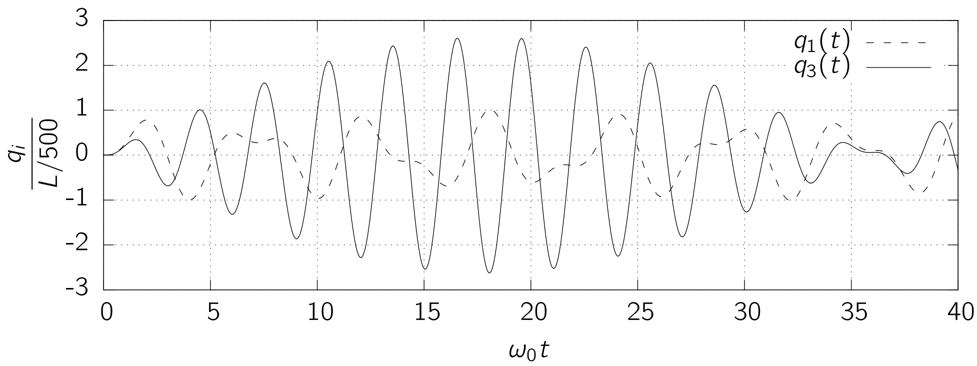
$$\begin{aligned}\frac{q_1(t)}{L/500} &= +0.650115\,\sin 1.1260\,\omega_0 t - 0.366025\,\sin 2\,\omega_0 t,\\ \frac{q_2(t)}{L/500} &= 0,\\ \frac{q_3(t)}{L/500} &= --\,1.25592606\,\sin 2.1753\,\omega_0 t + 1.366025\,\sin 2\,\omega_0 t.\end{aligned}$$

Our last point is to write the response for $u_3(t) = \sum \psi_{3i}\,q_i(t)$ (note that the sum of the steady state contributions to the modal responses is by definition equal to $-2\,\sin 2\,\omega_0 t$):

$$\begin{aligned}\frac{u_3(t)}{L/500} &= 0.36602540\cdot 0.650115\,\sin 1.1260\,\omega_0 t + 1.36602540\cdot 1.25592606\,\sin 2.1753\,\omega_0 t - 2\,\sin 2\,\omega_0 t\\ &= 0.2379587\,\sin 1.1260\,\omega_0 t + 1.715627\,\sin 2.1753\,\omega_0 t - 2\,\sin 2\,\omega_0 t\end{aligned}$$

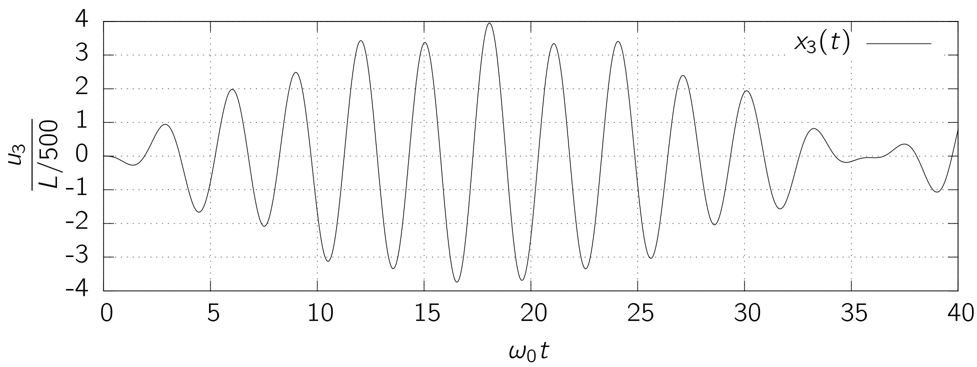
Presentation of the Results

In the figure below, the modal responses q_1 and q_3, note the beating between the two sine components in q_3, due to the two frequencies being close to each other.



It is worth to say that $q_2 \equiv 0$ because a symmetric loading vector clearly cannot excite an antisymmetric mode.

In the figure below, the displacements $u_3(t)$,



it is apparent that the response of u_3, in this case and for this particular loading frequency, is dominated by the response of the third mode.

Peace of mind

The response of the system to the specified loading was computed also numerically, using the following, short `Python` script

```
from scipy import mat, sin, zeros

K = mat('1 0 0;0 1 0;0 0 1')
M = mat('2 1 0;1 4 1;0 1 2')/6.0
r = mat('0;1;0')
```

```
def load(t): return r*sin(2.0*t) #

h = 0.05 ; duration = 40.0

# linear acceleration coefficients
Flex = (K+6*M/h/h).I ; A = 3.0*M ; V = 6.0*M/h

# initial state
t = 0
x  = mat(zeros((3,1))) ; v = mat(zeros((3,1))) ; p = load(0.0)
MI = M.I               ; a = MI*(p - K*x)

while t < duration+h/2:
    print "%12.9f" % t, ' '.join(["%12.9f" % x_i for x_i in x])
    t  = t + h
    dp = load(t) - p
    dx = Flex*(dp + A*a + V*v)
    dv = 3.0*dx/h - 3*v - a*h/2.0
    # update
    x = x + dx ; v = v + dv ; p = p + dp ; a = MI*(p-K*x)
```

The output of the program (here the first lines)

```
 0.000000000  0.000000000  0.000000000  0.000000000
 0.050000000 -0.000041493  0.000083091 -0.000041493
 0.100000000 -0.000330912  0.000663273 -0.000330912
```

was saved into the file `num.dat` and finally plotted by the following script,

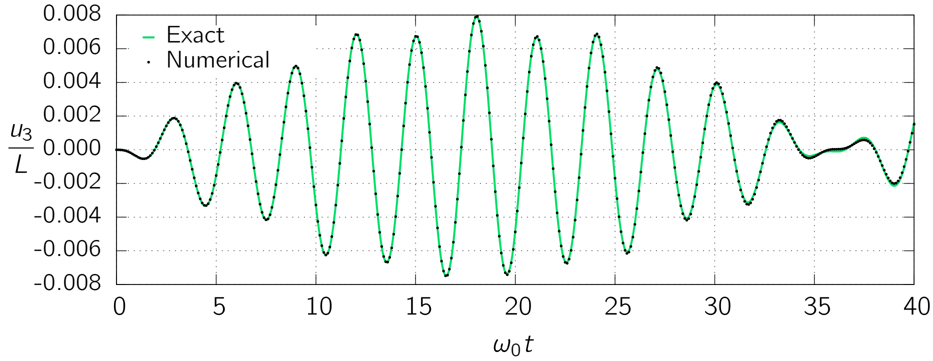
```
w1 = 1.126     ; w2 =sqrt(3) ; w3 = 2.1753     ; w = 2.0
x1 = 0.2379587 ; x2 = 0      ; x3 = 1.71562701 ; xs = -2
u3(x) = x1*sin(w1*x) + x3*sin(w3*x) + xs*sin(w*x)

set output "x3_exact+num.tex"
set xrange [0:40]
plot u3(x)/500 lt 1 lw 3 lc rgb '#00dd66' t "\\K Exact", \
     "num.dat" u 1:($4/500) every 2 lt 0 lw 1 t "\\K Numerical"

set output "x3_short.tex"
set xrange [0:6]
plot u3(x)/500 lt 1 lw 3 lc rgb '#00dd66' t "\\K Exact", \
     "num.dat" u 1:($4/500) every 1 lt 0 lw 1 t "\\K Numerical"
```

aside with the analytical solution

Here it is the graph over 40 adimensional time units, as wellas in the previous plots (note that i haven't normilezed with respect to 500 in this case)



the solutions, obtained by completely different methods, are in very good agreement.

If you have plotted the response over a shorter time interval (losing the appreciation for the quasi-resonant behaviour) here it is the same response function and the same data plotted over 6 adimansional time units.