1 Free Response

A 1 storey building can be considered a SDOF system. Its top is displaced by means of a hydraulic jack, the applied force is 90kN, and the measured displacement is $x_0 = 5.0\text{mm}$.

The applied force is istantaneously released, so that the building oscillates in free response, starting from initial conditions $x(0) = x_0, \quad \dot{x}(0) = 0$. Note that x_0 is a maximum, as the velocity at time $t = 0$ is equal to zero.

The maximum displacement after the first cycle of oscillation is measured, and it is found that $x_1 = 4.0\text{mm}$, at time $t = 1.40\text{s}$.

We want to determine the dynamical parameters of the system, and in particular its damping ratio.

1.1 Determination of the Dynamical Parameters

First, we can derive the elastic stiffness relating the applied force and the initial displacement,

$$k = \frac{F}{x_0} = \frac{90,000\text{N}}{0.005\text{m}} = 18.0\frac{\text{MN}}{\text{m}}.$$

Next, with the understanding that the damped period is $T_D = 1.4\text{s}$, we find the damped frequency,

$$\omega_D = \frac{2\pi}{T_D} = \frac{6.2832\text{rad}}{1.40\text{s}} = 4.488\frac{rad}{\text{s}}.$$

The logarithmic decrement equation, when written for two consecutive maxima of the response, is

$$\log(\frac{x_n}{x_{n+1}}) = 0.223143551314 = \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}.$$

Solving for ζ and substituting $\delta = \log 1.25$ gives

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = 3.54920237062\%.$$

As an alternative, one can use an iterative solution, starting with $\zeta_0 = 0$ and writing

$$\zeta_{i+1} = \left(\frac{\delta}{2\pi}\right)\sqrt{1-\zeta_i^2}.$$

Using this procedure, the successive approximations are

$$\begin{aligned}\zeta_1 &= 3.55143992107\% \\ \zeta_2 &= 3.54919954758\% \\ \zeta_3 &= 3.54920237420\% \\ \zeta_4 &= 3.54920237064\%\end{aligned}$$

Of course, from an engineering point of view the result $\zeta_1 = 3.55\%$ is *good enough.* The determination of the mass is left to the interested reader.

i	ω_i (rad/s)	ρ_i (μm)	ϑ_i (deg)	$\cos\vartheta_i$	$\sin\vartheta$
1	16.0	183.	15.0	0.966	0.259
2	25.0	368.	55.0	0.574	0.819

Table 1: Experimental data

2 Dynamic Testing

We want to measure the dynamical characteristics of a SDOF building system, i.e., its mass, its damping coefficient and its elastic stiffness.

To this purpose, we demonstrate that is sufficient to measure the steady-state response of the SDOF when subjected to a number of harmonic excitations with different frequencies.

The steady-state response is characterized by its amplitude,ρ and the phase delay, ϑ, as in $x_{SS}(t) = \rho\sin(\omega t - \vartheta)$.

E.g., we excite our stucture with a vibrodyne that exerts a harmonic force $p(t) = p_0 \sin\omega t$, with $p_0 = 2.224$kN, and measure the steady-state response parameters for two different input frequencies, as detailed in table 1.

2.1 Determination of the Dynamical Parameters

We start from the equation for steady-state response amplitude,

$$\rho = \frac{p_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

where we collect $(1-\beta^2)^2$ in the radicand in the right member,

$$\rho = \frac{p_0}{k}\frac{1}{1-\beta^2}\frac{1}{\sqrt{1+[2\zeta\beta/(1-\beta^2)]^2}}$$

but the equation for the phase angle, $\tan\vartheta = \frac{2\zeta\beta}{1-\beta^2}$, can be substituted in the radicand, so that, using simple trigonometric identities, we find that

$$\rho = \frac{p_0}{k}\frac{1}{1-\beta^2}\frac{1}{\sqrt{1+\tan^2\vartheta}} = \frac{p_0}{k}\frac{\cos\vartheta}{1-\beta^2}.$$

With $k(1-\beta^2) = k - k\frac{\omega^2}{k/m} = k - \omega^2 m$ and using a simple rearrangement, we have

$$k - \omega^2 m = \frac{p_0}{\rho}\cos\vartheta.$$

Substituting the data from table 1 into the previous equation for $i = 1, 2$ we can write, using matrix notation, a system of two algebraic equations in the unknown k and m,

$$\begin{bmatrix} 1 & -16^2 \\ 1 & -25^2 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = p_0 \begin{Bmatrix} \frac{0.966}{183\times10^{-6}} \\ \frac{0.574}{368\times10^{-6}} \end{Bmatrix},$$

that once solved gives us the values $k = 17.48\,\text{MN/m}$ and $m = 22415\,\text{kg}$, while the natural frequency is $\omega = \sqrt{k/m} = 27.924$rad/s.

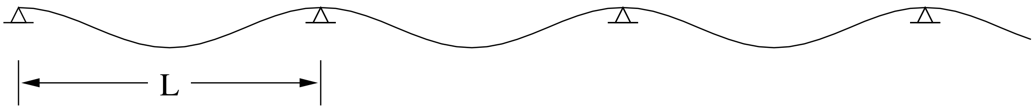

Figure 1: vertical profile of bridge surface

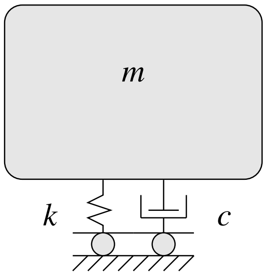

Figure 2: simplified model of the vehicle

Using the previously established relationship for $\cos\vartheta$, we can write $\cos\vartheta = k(1-\beta^2)\frac{\rho}{p_0}$, from the equation of the phase angle (see above), we can write $\cos\vartheta = \frac{1-\beta^2}{2\zeta\beta}\sin\vartheta$, and finally

$$\frac{\rho k}{p_0} = \frac{\sin\vartheta}{2\zeta\beta}, \quad \text{hence} \quad \zeta = \frac{p_0}{\rho k}\frac{\sin\vartheta}{2\beta},$$

and substituting the values for, e,g,, $i = 1$ gives $\zeta = 15.7\%$. Substituting the values for $i = 2$ offers a result that is equivalent from an engineering point of view.

3 Vibration Insulation, Displacements

A vehicle with mass $m = 1800$kg travels at constant velocity $v = 72$km/h over a very long bridge; the bridge has a constant span $L = 12$m and, due to viscous displacements, its surface is no more horizontal (see figure 1). The vertical profile of the bridge surface can be approximated by a trigonometric function,

$$y_g = y_{g0}\cos(\frac{2\pi x}{L}),$$

where $y_{g0} = \frac{\delta_{\max}}{2} = 3.0$cm, $\delta = 6.0$cm being the maximum deflection measured between the supports and the midspan.

The vehicle can be considered as a single mass, connected to the road surface by a suspension system composed by a spring and a viscous damper . The stiffness is $k = 225$kN/m, and the damping ratio is $\zeta = 40\%$.

It is required the maximum value of the *total* vertical displacement of the vehicle body at steady state.

3.1 Determination of the total steady state displacement

The point of contact between the suspension and the road, assuming a constant vehicle velocity, goes up and down with a period T that is equal to the time that the vehicle uses to go from one maximum to the successive maximum, that is the time it takes to travel $L = 12\text{m}$.

The vehicle velocity is

$$v = \frac{72000\text{m}}{3600\text{s}} = 20\text{m/s},$$

and the excitation period is hence

$$T = \frac{12\text{m}}{20\text{m/s}} = 0.6\text{s}.$$

The natural period of excitation of the suspension-vehicle system is

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{k/m}} = 0.562\text{s}$$

and the excitation frequency ratio is

$$\beta = \frac{T_n}{T} = 0.9366$$

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The transmittance ratio, TR, is defined as

$$TR = \frac{y_{\text{TOT}}}{y_{g0}} = \sqrt{\frac{1 + (2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}} = 1.647,$$

so that the maximum displacement is

$$y_{\text{TOT}} = 1.647 \times 3.0\text{cm} = 4.9371\text{cm}.$$

For $\zeta = 0.0$, TR is equal to?

4 Vibration Insulation, Transmitted Forces.

A rotating machine has a total mass $m = 90,000\text{kg}$; when it is in operation the machine transmits to its rigid support a harmonic force

$$p(t) = p_0 \sin(2\pi f_0 t), \quad \text{with} \quad p_0 = 2\text{kN and } f_0 = 40\text{Hz}.$$

Due to the excessive level of vibrations induced in the building in which the machine is housed, it is required that the transmitted force is reduced to a maximum value of 400N. This will be achieved by means of a suspension system that will consist of four equal springs of elastic constant k.

4.1 Maximum stiffness of the damping system

In this case the required maximum value of the transmissibility ratio is

$$TR = \frac{f_T}{p_o} = \frac{400\text{N}}{2000\text{N}} = 0.20,$$

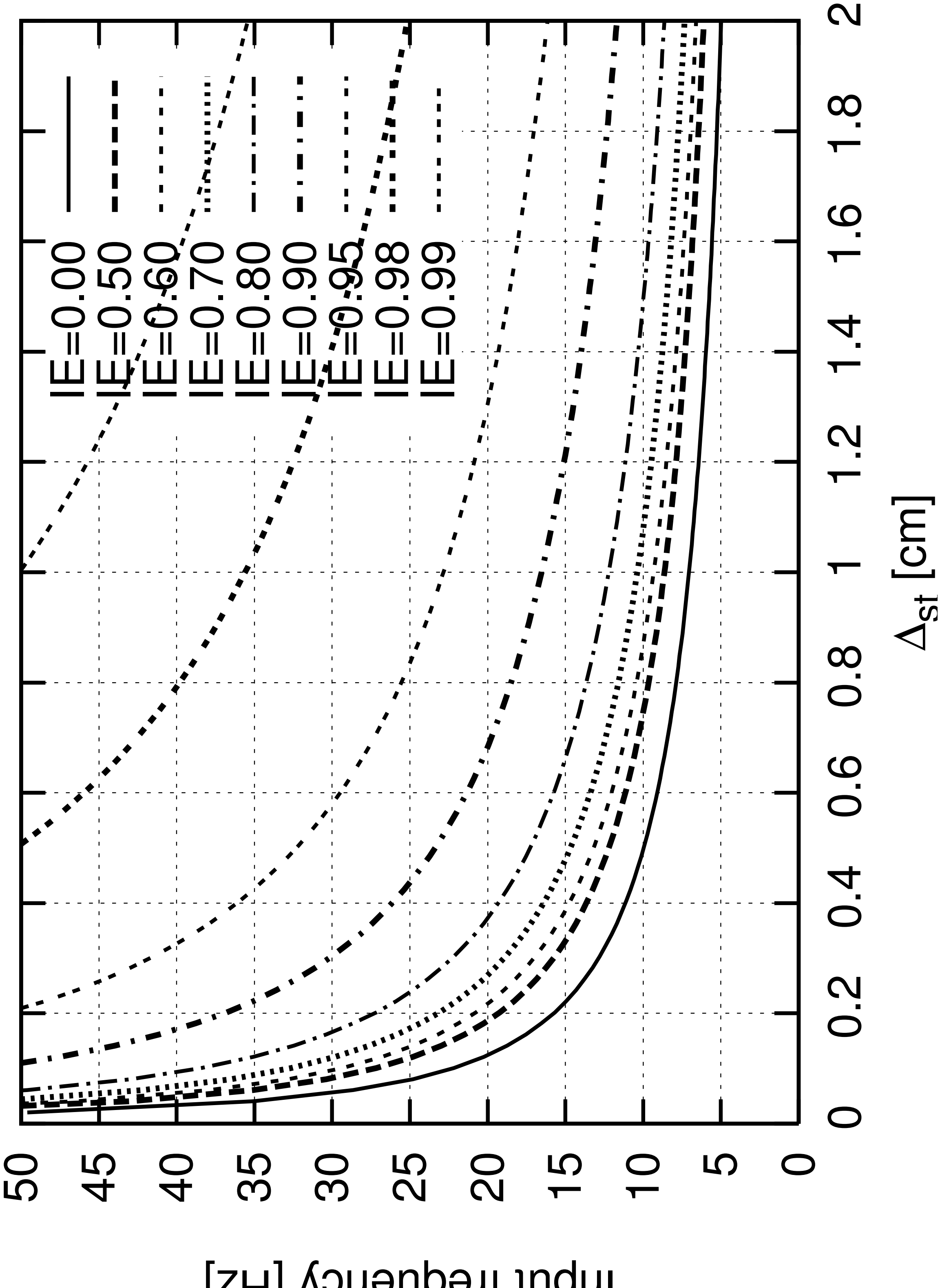

Figure 3: IE design chart

and the required insulation efficiency is

$$IE = 1 - TR = 0.80$$

From the design chart in figure 3, for an excitation frequency of 40Hz and $IE = 0.80$, we see the following requirement for the static displacement,

$$\Delta_{\text{static}} = W/k_{\text{total}} \geq 0.095\text{cm} = 0.00095\text{m}.$$

Solving for $k = k_{\text{total}}/4$,

$$k \leq \frac{90,000 \times 9.81}{4 \times 0.00095}\text{N/m} = 232.34\text{MN/m}.$$

Using a different approach, for an undamped system one can write

$$TR = \frac{1}{\beta^2 - 1}, \text{ hence } \beta = \sqrt{\frac{1 + TR}{TR}} = 2.45$$

deriving $\omega_n = (2\pi f_0)/2.0 = 102.64\text{rad/s}$, and

$$k = \frac{k_{\text{total}}}{4} = \frac{1}{4}m\omega_n^2 = \frac{90,000 \times 10,527.6}{4} = 236.87\frac{\text{MN}}{\text{m}}$$