Numerical Integration

Step-by-step Numerical Procedures

Giacomo Boffi

Dipartimento di Ingegneria Strutturale, Politecnico di Milano

April 9, 2013

Part I

Numerical Integration

Outline

Piecewise exact method

- We use the exact solution of the equation of motion for a system excited by a linearly varying force, so the source of all errors lies in the piecewise linearisation of the force function and in the approximation due to a local linear model.
- We will see that an appropriate time step can be decided in terms of the number of points required to accurately describe either the force or the response function.

Piecewise exact method

Step by Step Methods
Giacomo Boffi

For a generic time step of duration h, consider

- $\{x_0, \dot{x}_0\}$ the initial state vector,
- p_0 and p_1, the values of $p(t)$ at the start and the end of the integration step,
- the linearised force

$$p(\tau) = p_0 + \alpha\tau,\ 0 \le \tau \le h,\ \alpha = (p(h) - p(0))/h,$$

- the forced response

$$x = e^{-\zeta\omega\tau}(A\cos(\omega_D\tau) + B\sin(\omega_D\tau)) + (\alpha k\tau + kp_0 - \alpha c)/k^2,$$

  where k and c are the stiffness and damping of the SDOF system.

Piecewise exact method

Step by Step Methods
Giacomo Boffi

Evaluating the response x and the velocity $\dot{x}$ for $\tau = 0$ and equating to $\{x_0, \dot{x}_0\}$, writing $\Delta_{st} = p(0)/k$ and $\delta(\Delta_{st}) = (p(h) - p(0))/k$, one can find A and B

$$A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_D}$$

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}$$

substituting and evaluating for $\tau = h$ one finds the state vector at the end of the step.

Piecewise exact method

With

$$\mathcal{S}_{\zeta,h} = \sin(\omega_D h)\exp(-\zeta\omega h) \text{ and } \mathcal{C}_{\zeta,h} = \cos(\omega_D h)\exp(-\zeta\omega h)$$

and the previous definitions of Δ_{st} and $\delta(\Delta_{st})$, finally we can write

$$x(h) = A\,\mathcal{S}_{\zeta,h} + B\,\mathcal{C}_{\zeta,h} + (\Delta_{st} + \delta(\Delta_{st})) - \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h}$$

$$\dot{x}(h) = A(\omega_D\mathcal{C}_{\zeta,h} - \zeta\omega\mathcal{S}_{\zeta,h}) - B(\zeta\omega\mathcal{C}_{\zeta,h} + \omega_D\mathcal{S}_{\zeta,h}) + \frac{\delta(\Delta_{st})}{h}$$

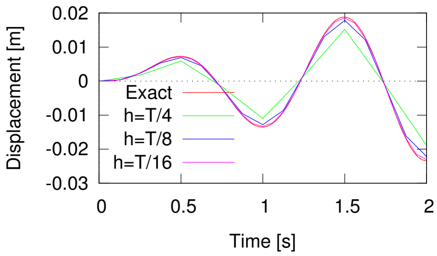
where

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}, \quad A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_D}.$$

Example

We have a damped system that is excited by a load in resonance with the system, we know the exact response and we want to compute a step-by-step approximation using different step lengths.

m=1000kg,
k=$4\pi^2$ 1000N/m,
$\omega=2\pi$,
ζ=0.05,
$p(t) = 4\pi^2 5\,\text{N}\,\sin(2\pi\,t)$



It is apparent that you have a very good approximation when the linearised loading is a very good approximation of the input function, let's say $h \le T/10$.

Central differences

To derive the Central Differences Method, we write the eq. of motion at time $\tau = 0$ and find the initial acceleration,

$$m\ddot{x}_0 + c\dot{x}_0 + kx_0 = p_0 \Rightarrow \ddot{x}_0 = \frac{1}{m}(p_0 - c\dot{x}_0 - kx_0)$$

On the other hand, the initial acceleration can be expressed in terms of finite differences,

$$\ddot{x}_0 = \frac{x_1 - 2x_0 + x_{-1}}{h^2} = \frac{1}{m}(p_0 - c\dot{x}_0 - kx_0)$$

solving for x_1

$$x_1 = 2x_0 - x_{-1} + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0)$$

Central differences

We have an expression for x_1, the displacement at the end of the step,

$$x_1 = 2x_0 - x_{-1} + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0),$$

but we have an additional unknown, x_{-1}... if we write the finite differences approximation to $\dot{x}_0$ we can find an approximation to x_{-1} in terms of the initial velocity $\dot{x}_0$ and the unknown x_1

$$\dot{x}_0 = \frac{x_1 - x_{-1}}{2h} \Rightarrow x_{-1} = x_1 - 2h\dot{x}_0$$

Substituting in the previous equation

$$x_1 = 2x_0 - x_1 + 2h\dot{x}_0 + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0),$$

and solving for x_1

$$x_1 = x_0 + h\dot{x}_0 + \frac{h^2}{2m}(p_0 - c\dot{x}_0 - kx_0)$$

Central differences

$$x_1 = x_0 + h\dot{x}_0 + \frac{h^2}{2m}(p_0 - c\dot{x}_0 - kx_0)$$

To start a new step, we need the value of $\dot{x}_1$, but we may approximate the mean velocity, again, by finite differences

$$\frac{\dot{x}_0 + \dot{x}_1}{2} = \frac{x_1 - x_0}{h} \Rightarrow \dot{x}_1 = \frac{2(x_1 - x_0)}{h} - \dot{x}_0$$

The method is very simple, but it is *conditionally stable*. The stability condition is defined with respect to the natural frequency, or the natural period, of the SDOF oscillator,

$$\omega_n h \leq 2 \Rightarrow h \leq \frac{T_n}{\pi} \approx 0.32T_n$$

For a SDOF this is not relevant because, as we have seen in our previous example, we need more points for response cycle to correctly represent the response.

Methods based on Integration

We will make use of an *hypothesis* on the variation of the acceleration during the time step and of analytical integration of acceleration and velocity to step forward from the initial to the final condition for each time step. In general, these methods are based on the two equations

$$\dot{x}_1 = \dot{x}_0 + \int_0^h \ddot{x}(\tau)\, d\tau,$$
$$x_1 = x_0 + \int_0^h \dot{x}(\tau)\, d\tau,$$

which express the final velocity and the final displacement in terms of the initial values x_0 and $\dot{x}_0$ and some definite integrals that depend on the *assumed* variation of the acceleration during the time step.

Integration Methods

Depending on the different assumption we can make on the variation of velocity, different integration methods can be derived.
We will see

- the constant acceleration method,
- the linear acceleration method,
- the family of methods known as *Newmark Beta Methods*, that comprises the previous methods as particular cases.

Constant Acceleration

Here we assume that the acceleration is constant during each time step, equal to the mean value of the initial and final values:

$$\ddot{x}(\tau) = \ddot{x}_0 + \Delta\ddot{x}/2,$$

where $\Delta\ddot{x} = \ddot{x}_1 - \ddot{x}_0$, hence

$$\begin{aligned}
\dot{x}_1 &= \dot{x}_0 + \int_0^h (\ddot{x}_0 + \Delta\ddot{x}/2)\, d\tau \\
&\Rightarrow \Delta\dot{x} = \ddot{x}_0 h + \Delta\ddot{x} h/2 \\
x_1 &= x_0 + \int_0^h (\dot{x}_0 + (\ddot{x}_0 + \Delta\ddot{x}/2)\tau) d\tau \\
&\Rightarrow \Delta x = \dot{x}_0 h + (\ddot{x}_0) h^2/2 + \Delta\ddot{x} h^2/4
\end{aligned}$$

Constant acceleration

Taking into account the two equations on the right of the previous slide, and solving for $\Delta\dot{x}$ and $\Delta\ddot{x}$ in terms of Δx, we have

$$\Delta\dot{x} = \frac{2\Delta x - 2h\dot{x}_0}{h}, \qquad \Delta\ddot{x} = \frac{4\Delta x - 4h\dot{x}_0 - 2\ddot{x}_0 h^2}{h^2}.$$

We have two equations and three unknowns... Assuming that the system characteristics are constant during a single step, we can write the equation of motion at times $\tau = h$ and $\tau = 0$, subtract member by member and write the *incremental equation of motion*

$$m\Delta\ddot{x} + c\Delta\dot{x} + k\Delta x = \Delta p,$$

that is a third equation that relates our unknowns.

Constant acceleration

Substituting the above expressions for $\Delta\dot{x}$ and $\Delta\ddot{x}$ in the incremental eq. of motion and solving for Δx gives, finally,

$$\Delta x = \frac{\tilde{p}}{\tilde{k}}, \qquad \Delta\dot{x} = \frac{2\Delta x - 2h\dot{x}_0}{h}$$

where

$$\tilde{k} = k + \frac{2c}{h} + \frac{4m}{h^2}$$

$$\tilde{p} = \Delta p + 2c\dot{x}_0 + m(2\ddot{x}_0 + \frac{4}{h}\dot{x}_0)$$

While it is possible to compute the final acceleration in terms of Δx, to achieve a better accuracy it is usually computed solving the equation of equilibrium written at the end of the time step.

Constant Acceleration

Two further remarks

1. The method is *unconditionally stable*
2. The effective stiffness, disregarding damping, is $\tilde{k} \approx k + 4m/h^2$.

Dividing both members of the above equation by k it is

$$\frac{\tilde{k}}{k} = 1 + \frac{4}{\omega_n^2 h^2} = 1 + \frac{4}{(2\pi/T_n)^2 h^2} = \frac{T_n^2}{\pi^2 h^2},$$

The number n_T of time steps in a period T_n is related to the time step duration, $n_T = T_n/h$, solving for h and substituting in our last equation, we have

$$\frac{\tilde{k}}{k} \approx 1 + \frac{n_T^2}{\pi^2}$$

For, e.g., $n_T = 2\pi$ it is $\tilde{k}/k \approx 1 + 4$, the mass contribution to the effective stiffness is four times the elastic stiffness and the 80% of the total.

Linear Acceleration

We assume that the acceleration is linear, i.e.

$$\ddot{x}(t) = \ddot{x}_0 + \Delta\ddot{x}\frac{\tau}{h}$$

hence

$$\Delta\dot{x} = \ddot{x}_0 h + \Delta\ddot{x} h/2, \quad \Delta x = \dot{x}_0 h + \ddot{x}_0 h^2/2 + \Delta\ddot{x} h^2/6$$

Following a derivation similar to what we have seen in the case of constant acceleration, we can write, again,

$$\Delta x = \left(k + 3\frac{c}{h} + 6\frac{m}{h^2}\right)^{-1}\left[\Delta p + c(\ddot{x}_0\frac{h}{2} + 3\dot{x}_0) + m(3\ddot{x}_0 + 6\frac{\dot{x}_0}{h})\right]$$

$$\Delta\dot{x} = \Delta x\frac{3}{h} - 3\dot{x}_0 - \ddot{x}_0\frac{h}{2}$$

Linear Acceleration

The linear acceleration method is *conditionally stable*, the stability condition being

$$\frac{h}{T} \le \frac{\sqrt{3}}{\pi} \approx 0.55$$

When dealing with SDOF systems, this condition is never of concern, as we need a shorter step to accurately describe the response of the oscillator, let's say $h \le 0.12T\ldots$
When stability is not a concern, the accuracy of the linear acceleration method is far superior to the accuracy of the constant acceleration method, so that this is the method of choice for the analysis of SDOF systems.

Newmark Beta Methods

The constant and linear acceleration methods are just two members of the family of Newmark Beta methods, where we write

$$\Delta\dot{x} = (1-\gamma)h\ddot{x}_0 + \gamma h\ddot{x}_1$$
$$\Delta x = h\dot{x}_0 + (\frac{1}{2}-\beta)h^2\ddot{x}_0 + \beta h^2\ddot{x}_1$$

The factor γ weights the influence of the initial and final accelerations on the velocity increment, while β has a similar role with respect to the displacement increment.

Newmark Beta Methods

Using $\gamma \ne 1/2$ leads to numerical damping, so when analysing SDOF systems, one uses $\gamma = 1/2$ (numerical damping may be desirable when dealing with MDOF systems).
Using $\beta = \frac{1}{4}$ leads to the constant acceleration method, while $\beta = \frac{1}{6}$ leads to the linear acceleration method. In the context of MDOF analysis, it's worth knowing what is the minimum β that leads to an unconditionally stable behaviour.

Newmark Beta Methods

The general format for the solution of the incremental equation of motion using the Newmark Beta Method can be written as follows:

$$\Delta x = \frac{\Delta\tilde{p}}{\tilde{k}}$$

$$\Delta v = \frac{\gamma}{\beta}\frac{\Delta x}{h} - \frac{\gamma}{\beta}v_0 + h\left(1-\frac{\gamma}{2\beta}\right)a_0$$

with

$$\tilde{k} = k + \frac{\gamma}{\beta}\frac{c}{h} + \frac{1}{\beta}\frac{m}{h^2}$$

$$\Delta\tilde{p} = \Delta p + \left(h\left(\frac{\gamma}{2\beta}-1\right)c + \frac{1}{2\beta}m\right)a_0 + \left(\frac{\gamma}{\beta}c + \frac{1}{\beta}\frac{m}{h}\right)v_0$$

Non Linear Systems

A convenient procedure for integrating the response of a non linear system is based on the incremental formulation of the equation of motion, where for the stiffness and the damping were taken values representative of their variation during the time step: in line of principle, the mean values of stiffness and damping during the time step, or, as this is usually not possible, their initial values, k_0 and c_0.
The Newton-Raphson method can be used to reduce the unbalanced forces at the end of the step.

Non Linear Systems

Usually we use the modified Newton-Raphson method, characterised by not updating the system stiffness at each iteration. In pseudo-code, referring for example to the Newmark Beta Method

```
x1,v1,f1 = x0,v0,f0 % initialisation; gb=gamma/beta
Dr = DpTilde
loop:
   Dx = Dr/kTilde
   x2 = x1 + Dx
   v2 = gb*Dx/h + (1-gb)*v1 + (1-gb/2)*h*a0
   x_pl = update_u_pl(...)
   f2 = k*(x2-x_pl)
   % important
   Df = (f2-f1) + (kTilde-k_ini)*Dx
   Dr = Dr - Df
   x1, v1, f1 = x2, v2, f2
   if ( tol(...) < req_tol ) BREAK loop
```

Exercise

A system has a mass $m = 1000$kg, a stiffness $k = 40000$N/m and a viscous damping whose ratio to the critical damping is $\zeta = 0.03$.
The spring is elastoplastic, with a yielding force of 2500N.
The load is an half-sine impulse, with duration 0.3s and maximum value of 6000N.
Use the constant acceleration method to integrate the response, with $h = 0.05$s and, successively, $h = 0.02$s .
Note that the stiffness is either 0 or k, write down the expression for the effective stiffness and loading in the incremental formulation, write a spreadsheet or a program to make the computations.

Part II

Rigid Assemblages

Outline

Introductory Remarks

Until now our *SDOF*'s were described as composed by a single mass connected to a fixed reference by means of a spring and a damper.
While the mass-spring is a useful representation, many different, more complex systems can be studied as *SDOF* systems, either exactly or under some simplifying assumption.

1. *SDOF* rigid body assemblages, where flexibility is concentrated in a number of springs and dampers, can be studied, e.g., using the Principle of Virtual Displacements and the D'Alembert Principle.
2. simple structural systems can be studied, in an approximate manner, assuming a fixed pattern of displacements, whose amplitude (the single degree of freedom) varies with time.

Further Remarks on Rigid Assemblages

Today we restrict our consideration to plane, 2-D systems.
In rigid body assemblages the limitation to a single shape of displacement is a consequence of the configuration of the system, i.e., the disposition of supports and internal hinges.
When the equation of motion is written in terms of a single parameter and its time derivatives, the terms that figure as coefficients in the equation of motion can be regarded as the *generalised* properties of the assemblage: generalised mass, damping and stiffness on left hand, generalised loading on right hand.

$$m^\star\ddot{x} + c^\star\dot{x} + k^\star x = p^\star(t)$$

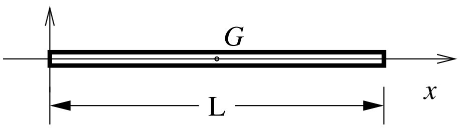
Final Remarks on Generalised *SDOF* Systems

From the previous comments, it should be apparent that everything we have seen regarding the behaviour and the integration of the equation of motion of proper *SDOF* systems applies to rigid body assemblages (we will see that it applies also to *SDOF* models of flexible systems), provided that we have the means for determining the *generalised* properties of the dynamical systems under investigation.

Assemblages of Rigid Bodies

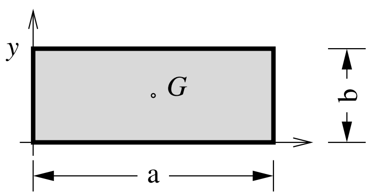
- planar, or bidimensional, rigid bodies, constrained to move in a plane,
- the flexibility is *concentrated* in discrete elements, springs and dampers,
- rigid bodies are connected to a fixed reference and to each other by means of springs, dampers and smooth, bilateral constraints (read hinges, double pendulums and rollers),
- inertial forces are distributed forces, acting on each material point of each rigid body, their resultant can be described by
  - a force applied to the centre of mass of the body, proportional to acceleration vector and total mass $M = \int \mathrm{d}m$
  - a couple, proportional to angular acceleration and the moment of inertia J of the rigid body, $J = \int (x^2 + y^2)\mathrm{d}m$.

Rigid Bar



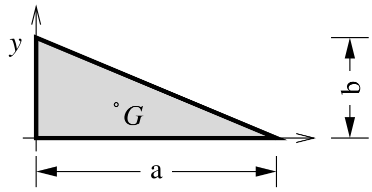
Unit mass	$\bar{m} = \text{constant},$
Length	$L,$
Centre of Mass	$x_G = L/2,$
Total Mass	$m = \bar{m}L,$
Moment of Inertia	$J = m\frac{L^2}{12} = \bar{m}\frac{L^3}{12}$

Rigid Rectangle



Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/2, \quad y_G = b/2$
Total Mass	$m = \gamma ab,$
Moment of Inertia	$J = m\frac{a^2 + b^2}{12} = \gamma\frac{a^3 b + ab^3}{12}$

Rigid Triangle



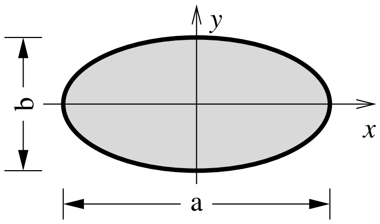
For a right triangle.

Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/3, \quad y_G = b/3$
Total Mass	$m = \gamma ab/2,$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{18} = \gamma\dfrac{a^3b+ab^3}{36}$

Step by Step Methods
Giacomo Boffi
Introductory Remarks
Assemblage of Rigid Bodies

Rigid Oval

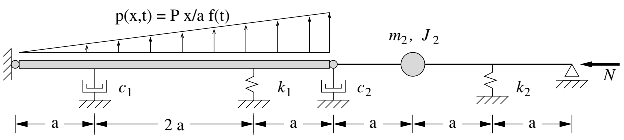
When $a = b = D = 2R$ the oval is a circle.



Unit mass	$\gamma = \text{constant},$
Axes	$a,\ b$
Centre of Mass	$x_G = y_G = 0$
Total Mass	$m = \gamma\dfrac{\pi ab}{4},$
Moment of Inertia	$J = m\dfrac{L^2}{12} = \bar{m}\dfrac{L^3}{12}$

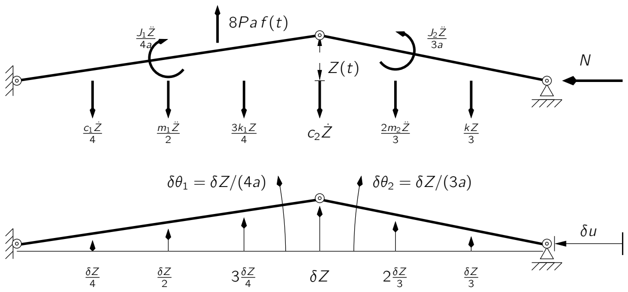
Step by Step Methods
Giacomo Boffi
Introductory Remarks
Assemblage of Rigid Bodies

trabacolo1



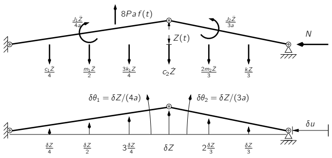
The mass of the left bar is $m_1 = \bar{m}\,4a$ and its moment of inertia is $J_1 = m_1\frac{(4a)^2}{12} = 4a^2m_1/3$.
The maximum value of the external load is
$P_{\max} = P\,4a/a = 4P$ and the resultant of triangular load is
$R = 4P \times 4a/2 = 8Pa$

Forces and Virtual Displacements



$$u = 7a - 4a\cos\theta_1 - 3a\cos\theta_2, \quad \delta u = 4a\sin\theta_1\delta\theta_1 + 3a\sin\theta_2\delta\theta_2$$
$$\delta\theta_1 = \delta Z/(4a), \quad \delta\theta_2 = \delta Z/(3a)$$
$$\sin\theta_1 \approx Z/(4a), \quad \sin\theta_2 \approx Z/(3a)$$
$$\delta u = \left(\frac{1}{4a} + \frac{1}{3a}\right) Z\,\delta Z = \frac{7}{12a} Z\,\delta Z$$

Principle of Virtual Displacements



The virtual work of the InertialDampingElasticExternal forces:

$$\delta W_I = -m_1\frac{\ddot{Z}}{2}\frac{\delta Z}{2} - J_1\frac{\ddot{Z}}{4a}\frac{\delta Z}{4a} - m_2\frac{2\ddot{Z}}{3}\frac{2\delta Z}{3} - J_2\frac{\ddot{Z}}{3a}\frac{\delta Z}{3a}$$
$$= -\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z}\,\delta Z$$

$$\delta W_D = -c_1\frac{\dot{Z}}{4}\frac{\delta Z}{4} - -c_2\dot{Z}\,\delta Z = -(c_2 + c_1/16)\,\dot{Z}\,\delta Z$$

$$\delta W_S = -k_1\frac{3Z}{4}\frac{3\delta Z}{4} - k_2\frac{Z}{3}\frac{\delta Z}{3} = -\left(\frac{9k_1}{16} + \frac{k_2}{9}\right) Z\,\delta Z$$

$$\delta W_{Ext} = 8Pa\,f(t)\frac{2\delta Z}{3} + N\frac{7}{12a} Z\,\delta Z$$

Principle of Virtual Displacements

For a rigid body in condition of equilibrium the total virtual work must be equal to zero

$$\delta W_I + \delta W_D + \delta W_S + \delta W_{Ext} = 0$$

Substituting our expressions of the virtual work contributions and simplifying δZ, the equation of equilibrium is

$$\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z} +$$
$$+ (c_2 + c_1/16)\,\dot{Z} + \left(\frac{9k_1}{16} + \frac{k_2}{9}\right) Z =$$
$$8Pa\,f(t)\frac{2}{3} + N\frac{7}{12a} Z$$

Principle of Virtual Displacements

Collecting Z and its time derivatives give us

$$m^\star \ddot{Z} + c^\star \dot{Z} + k^\star Z = p^\star f(t)$$

introducing the so called *generalised properties*, in our example it is

$$m^\star = \frac{1}{4}m_1 + \frac{4}{9}9m_2 + \frac{1}{16a^2}J_1 + \frac{1}{9a^2}J_2,$$
$$c^\star = \frac{1}{16}c_1 + c_2,$$
$$k^\star = \frac{9}{16}k_1 + \frac{1}{9}k_2 - \frac{7}{12a}N,$$
$$p^\star = \frac{16}{3}Pa.$$

It is worth writing down the expression of $k^\star$:

$$k^\star = \frac{9k_1}{16} + \frac{k_2}{9} - \frac{7}{12a}N$$

Geometrical stiffness