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Multi Degrees of Freedom Systems
MDOF's

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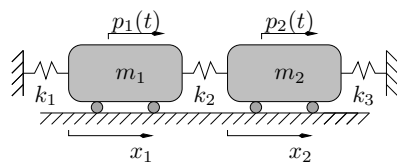Linear Operators**
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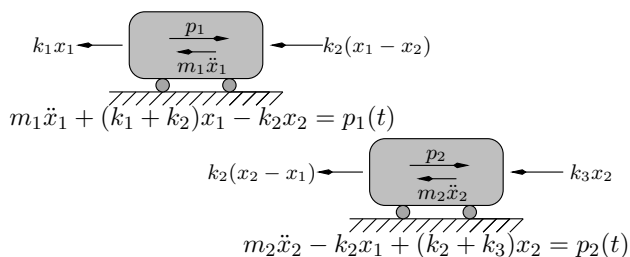
Examples

Consider an undamped system with two masses and two degrees of freedom.

We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally. write an equation of dynamic equilibrium for each mass.

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t)$$

$$m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t)$$

The equation of motion of a 2DOF system

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With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

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Introducing the loading vector $\boldsymbol{p}$, the vector of inertial forces $\boldsymbol{f}_I$ and the vector of elastic forces $\boldsymbol{f}_S$,

$$\boldsymbol{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \boldsymbol{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \boldsymbol{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\boldsymbol{f}_I + \boldsymbol{f}_S = \boldsymbol{p}(t).$$

$\boldsymbol{f}_S = \boldsymbol{K}\,\boldsymbol{x}$

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It is possible to write the linear relationship between $\boldsymbol{f}_S$ and the vector of displacements $\boldsymbol{x} = \{x_1 x_2\}^T$ in terms of a matrix product.
In our example it is

$$\boldsymbol{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{K}\,\boldsymbol{x}$$

introducing the stiffness matrix $\boldsymbol{K}$.
The stiffness matrix $\boldsymbol{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\boldsymbol{K}$ is hence a *square matrix* $\underset{\mathrm{ndof} \times \mathrm{ndof}}{\boldsymbol{K}}$

$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}$

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Analogously, introducing the mass matrix $\boldsymbol{M}$ that, for our example, is

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\boldsymbol{f}_I = \boldsymbol{M}\,\ddot{\boldsymbol{x}}.$$

Also the mass matrix $\boldsymbol{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

Matrix Equation

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Finally it is possible to write the equation of motion in matrix format:

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t).$$

Of course, we can consider the damping forces too, taking into account the velocity vector $\dot{\boldsymbol{x}}$, introducing a *damping matrix* $\boldsymbol{C}$ and writing

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t),$$

however it is now more productive to keep our attention on undamped systems.

Properties of $\boldsymbol{K}$

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- $\boldsymbol{K}$ is symmetrical, because the elastic force that acts on mass i due to an unit displacement of mass j, $f_{S,i} = k_{ij}$ is equal to the force on mass j due to unit diplacement of mass i, $f_{S,j} = k_{ji}$ in virtue of *Betti's theorem*.
- The strain energy V for a discrete system can be written

$$V = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{f}_S = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{K}\,\boldsymbol{x},$$

because the strain energy is positive it follows that $\boldsymbol{K}$ is a positive definite matrix.

Properties of $\boldsymbol{M}$

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Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive, as well as the stiffness matrix is symmetrical and definite positive.

En passant, take note that the kinetic energy for a discrete system is

$$T = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{M}\, \dot{\boldsymbol{x}}.$$

Generalisation of previous results

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The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with one exception.

For a general structural system, $\boldsymbol{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy could be zero.

The problem

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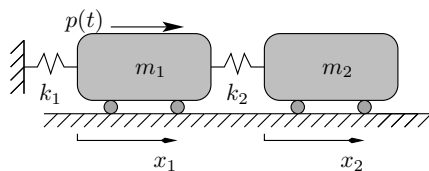The Homogeneous Problem

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Graphical statement of the problem

$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

The equations of motion

$$m_1\ddot{x}_1 + k_1 x_1 + k_2\,(x_1 - x_2) = p_0 \sin \omega t,$$
$$m_2\ddot{x}_2 + k_2\,(x_2 - x_1) = 0.$$

... but we prefer the matrix notation ...

The steady state solution

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because using the matrix notation we can follow the same steps we used to find the steady-state response of a *SDOF* system. First, the equation of motion

$$m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ddot{x} + k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} x = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \sin \omega t$$

substituting $x(t) = \xi \sin \omega t$ and simplifying $\sin \omega t$, dividing by k, with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

$$\left(\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \xi = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \xi = \Delta_{st} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

solving for ξ/Δ_{st} gives

$$\frac{\xi}{\Delta_{st}} = \frac{\begin{bmatrix} 1 - \beta^2 & 1 \\ 1 & 3 - 2\beta^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}{(\beta^2 - \frac{1}{2})(\beta^2 - 2)} = \frac{\begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix}}{(\beta^2 - \frac{1}{2})(\beta^2 - 2)}.$$

The solution, graphically

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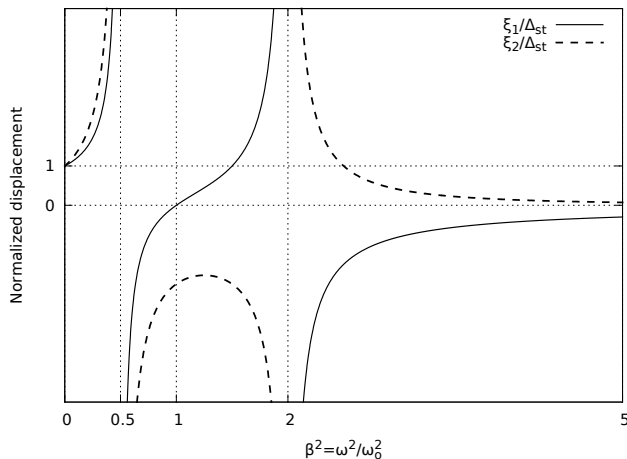**Properties of
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steady-state response for a 2 dof system, harmonic load

Homogeneous equation of motion

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To understand the behaviour of a *MDOF* system, we start writing the homogeneous equation of motion,

$$M \ddot{x} + K x = 0.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* ψ:

$$x(t) = \psi(A \sin \omega t + B \cos \omega t).$$

Substituting in the equation of motion, we have

$$\left(K - \omega^2 M \right) \psi (A \sin \omega t + B \cos \omega t) = 0$$

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The previous equation must hold for every value of t, so it can be simplified removing the time dependency:

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M} \right) \boldsymbol{\psi} = \boldsymbol{0}.$$

This is a homogeneous linear equation, with unknowns ψ_i and the coefficients that depends on the parameter ω^2.

Speaking of homogeneous systems, we know that there is always a trivial solution, $\boldsymbol{\psi} = \boldsymbol{0}$, and that different non-zero solutions are available when the determinant of the matrix of coefficients is equal to zero,

$$\det \left(\boldsymbol{K} - \omega^2 \boldsymbol{M} \right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation (the *equation of frequencies*) is verified.

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For a system with N degrees of freedom the expansion of $\det \left(\boldsymbol{K} - \omega^2 \boldsymbol{M} \right)$ is an algebraic polynomial of degree N in ω^2, whose roots, ω_i^2, $i = 1, \dots, N$ are all real and greater than zero if both $\boldsymbol{K}$ and $\boldsymbol{M}$ are positive definite matrices, condition that is always satisfied by stable structural systems. Substituting one of the N roots ω_i^2 in the characteristic equation,

$$\left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M} \right) \boldsymbol{\psi}_i = \boldsymbol{0}$$

the resulting system of $N - 1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue ω_i^2.

A common choice for the normalisation of the eigenvectors is *normalisation with respect to the mass matrix*, $\boldsymbol{\psi}_i^T \boldsymbol{M} \boldsymbol{\psi}_i = 1$

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The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\boldsymbol{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\boldsymbol{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \boldsymbol{x}_0, \qquad \dot{\boldsymbol{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\boldsymbol{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji} \omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \ldots, N.$$

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Orthogonality - 1

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\boldsymbol{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{M}\,\boldsymbol{\psi}_r$$
$$\boldsymbol{K}\,\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{M}\,\boldsymbol{\psi}_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M}\,\boldsymbol{\psi}_r$$
$$\boldsymbol{\psi}_r^T \boldsymbol{K}\,\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \boldsymbol{M}\,\boldsymbol{\psi}_s$$

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Orthogonality - 2

The term $\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r\right)^T = \boldsymbol{\psi}_r^T \boldsymbol{K}^T \boldsymbol{\psi}_s$$

but $\boldsymbol{K}$ is symmetrical, $\boldsymbol{K}^T = \boldsymbol{K}$ and we have

$$\boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{K}\,\boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^T \boldsymbol{M}\,\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \boldsymbol{M}\,\boldsymbol{\psi}_s.$$

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Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^T \boldsymbol{K} \, \boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^T \boldsymbol{M} \, \boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^T \boldsymbol{K} \, \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \boldsymbol{M} \, \boldsymbol{\psi}_s$$

subtracting member by member we find that

$$\left(\omega_r^2 - \omega_s^2\right) \boldsymbol{\psi}_r^T \boldsymbol{M} \, \boldsymbol{\psi}_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\boldsymbol{\psi}_r^T \boldsymbol{M} \, \boldsymbol{\psi}_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 4

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The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \boldsymbol{K} \, \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \boldsymbol{M} \, \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \boldsymbol{M} \, \boldsymbol{\psi}_i$$

and

$$\boldsymbol{\psi}_i^T \boldsymbol{K} \, \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

Eigenvectors are a base

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The eigenvectors are linearly independent, so for every vector $\boldsymbol{x}$ we can write

$$\boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j.$$

The coefficients are readily given by premultiplication of $\boldsymbol{x}$ by $\boldsymbol{\psi}_i^T \boldsymbol{M}$, because

$$\boldsymbol{\psi}_i^T \boldsymbol{M} \, \boldsymbol{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_i^T \boldsymbol{M} \, \boldsymbol{\psi}_j q_j = \boldsymbol{\psi}_i^T \boldsymbol{M} \, \boldsymbol{\psi}_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\boldsymbol{\psi}_j^T \boldsymbol{M} \, \boldsymbol{x}}{M_j}.$$

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Generalising our results for the displacement vector to the
acceleration vector, we can write

$$\boldsymbol{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\boldsymbol{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \psi_{ij} \ddot{q}_j(t).$$

Introducing $\boldsymbol{q}(t)$, the vector of *modal coordinates* and $\boldsymbol{\Psi}$, the
eigenvector matrix, whose columns are the eigenvectors,

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}\,\boldsymbol{q}(t), \qquad \ddot{\boldsymbol{x}}(t) = \boldsymbol{\Psi}\,\ddot{\boldsymbol{q}}(t).$$

EoM in Modal Coordinates...

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Substituting the last two equations in the equation of motion,

$$\boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T \boldsymbol{M}\,\boldsymbol{\Psi}\,\ddot{\boldsymbol{q}} + \boldsymbol{\Psi}^T \boldsymbol{K}\,\boldsymbol{\Psi}\,\boldsymbol{q} = \boldsymbol{\Psi}^T \boldsymbol{p}(t)$$

introducing the so called *starred* matrices we can finally write

$$\boldsymbol{M}^\star\,\ddot{\boldsymbol{q}} + \boldsymbol{K}^\star\,\boldsymbol{q} = \boldsymbol{p}^\star(t)$$

where $\boldsymbol{p}^\star(t) = \boldsymbol{\Psi}^T \boldsymbol{p}(t)$, and the scalar equation are

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j.$$

... are N independent equations!

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We must examine the structure of the starred symbols.
The generic element, with indexes i and j, of the *starred*
matrices can be expressed in terms of single eigenvectors,

$$\begin{aligned} m_{ij}^\star &= \boldsymbol{\psi}_i^T \boldsymbol{M}\,\boldsymbol{\psi}_j &&= \delta_{ij} M_i, \\ k_{ij}^\star &= \boldsymbol{\psi}_i^T \boldsymbol{K}\,\boldsymbol{\psi}_j &&= \omega_i^2 \delta_{ij} M_i. \end{aligned}$$

where δ_{ij} is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p_i^\star = \boldsymbol{\psi}_i^T \boldsymbol{p}(t)$ we
have a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \dots, N$$

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Initial Conditions Revisited

The initial displacements can be written in modal coordinates,

$$x_0 = \Psi\, q_0$$

and premultiplying both members by $\Psi^T M$ we have the following relationship:

$$\Psi^T M\, x_0 = \Psi^T M\, \Psi\, q_0 = M^\star q_0.$$

Premultiplying by the inverse of $M^\star$ and taking into account that $M^\star$ is diagonal,

$$q_0 = (M^\star)^{-1}\, \Psi^T M\, x_0 \quad \Rightarrow \quad q_{i0} = \frac{\psi_i^T M\, x_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\psi_i^T M\, \dot{x}_0}{M_i}$$

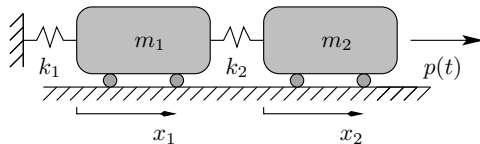---

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2 DOF System

2 DOF System

$$k_1 = k, \quad k_2 = 2k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin\omega t.$$

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \; p(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin\omega t,$$

$$M = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; K = k \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

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Equation of frequencies

The equation of frequencies is

$$\left\| K - \omega^2 M \right\| = \begin{Vmatrix} 3k - 2\omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{Vmatrix} = 0.$$

Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m}\frac{7 - \sqrt{33}}{4} \qquad \omega_2^2 = \frac{k}{m}\frac{7 + \sqrt{33}}{4}$$

$$\omega_1^2 = 0.31386\frac{k}{m} \qquad \omega_2^2 = 3.18614\frac{k}{m}$$

Eigenvectors

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2 DOF System

Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\left(3 - 2 \times 0.31386\right)\psi_{11} - 2k\psi_{21} = 0$$

while substituting ω_2^2 gives

$$k\left(3 - 2 \times 3.18614\right)\psi_{12} - 2k\psi_{22} = 0.$$

Solving with the arbitrary assignment $\psi_{21} = \psi_{22} = 1$ gives the *unnormalized* eigenvectors,

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} +0.84307 \\ +1.00000 \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -0.59307 \\ +1.00000 \end{Bmatrix}.$$

Normalization

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2 DOF System

We compute first M_1 and M_2,

$$M_1 = \boldsymbol{\psi}_1^T \boldsymbol{M} \boldsymbol{\psi}_1$$

$$= \left\{0.84307, \quad 1\right\} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix}$$

$$= \left\{1.68614m, \quad m\right\} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} = 2.42153m$$

$$M_2 = 1.70346m$$

the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{2.42153}, \qquad \alpha_2 = \sqrt{1.70346}.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.54177 & -0.45440 \\ +0.64262 & +0.76618 \end{bmatrix}$$

Modal Loadings

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The modal loading is

$$\boldsymbol{p}^\star(t) = \boldsymbol{\Psi}^T \boldsymbol{p}(t)$$

$$= p_0 \begin{bmatrix} +0.54177 & +0.64262 \\ -0.45440 & +0.76618 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \sin\omega t$$

$$= p_0 \begin{Bmatrix} +0.64262 \\ +0.76618 \end{Bmatrix} \sin\omega t$$

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Substituting its modal expansion for x into the equation of motion and premultiplying by $\mathbf{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} m\ddot{q}_1 + 0.31386k\, q_1 = +0.64262\, p_0 \sin\omega t \\ m\ddot{q}_2 + 3.18614k\, q_2 = +0.76618\, p_0 \sin\omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by m both equations, we have

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = +0.64262\, \dfrac{p_0}{m} \sin\omega t \\ \ddot{q}_2 + \omega_2^2 q_2 = +0.76618\, \dfrac{p_0}{m} \sin\omega t \end{cases}$$

Particular Integral

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We set
$$\xi_1 = C_1 \sin\omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin\omega t$$
and substitute in the first modal EoM:
$$C_1 \left(\omega_1^2 - \omega^2\right) \sin\omega t = \frac{p_1^\star}{m} \sin\omega t$$
solving for C_1
$$C_1 = \frac{p_1^\star}{m} \frac{1}{\omega_1^2 - \omega^2}$$
with $\omega_1^2 = K_1/m \;\Rightarrow\; m = K_1/\omega_1^2$:
$$C_1 = \frac{p_1^\star}{K_1} \frac{\omega_1^2}{\omega_1^2 - \omega^2} = \Delta_{\text{st}}^{(1)} \frac{1}{1 - \beta_1^2} \quad \text{with } \Delta_{\text{st}}^{(1)} = \frac{p_1^\star}{K_1} = 2.047\frac{p_0}{k} \text{ and } \beta_1 = \frac{\omega}{\omega_1}$$
of course
$$C_2 = \Delta_{\text{st}}^{(2)} \frac{1}{1 - \beta_2^2} \quad \text{with } \Delta_{\text{st}}^{(2)} = \frac{p_2^\star}{K_2} = 0.2404\frac{p_0}{k} \text{ and } \beta_2 = \frac{\omega}{\omega_2}$$

Integrals

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The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin\omega_1 t + B_1 \cos\omega_1 t + \Delta_{\text{st}}^{(1)} \dfrac{\sin\omega t}{1 - \beta_1^2} \\ q_2(t) = A_2 \sin\omega_2 t + B_2 \cos\omega_2 t + \Delta_{\text{st}}^{(2)} \dfrac{\sin\omega t}{1 - \beta_2^2} \end{cases}$$

for a system initially at rest

$$\begin{cases} q_1(t) = \Delta_{\text{st}}^{(1)} \dfrac{1}{1 - \beta_1^2} \left(\sin\omega t - \beta_1 \sin\omega_1 t\right) \\ q_2(t) = \Delta_{\text{st}}^{(2)} \dfrac{1}{1 - \beta_2^2} \left(\sin\omega t - \beta_2 \sin\omega_2 t\right) \end{cases}$$

we are interested in structural degrees of freedom, too... disregarding transient

$$\begin{cases} x_1(t) = \left(\psi_{11}\dfrac{\Delta_{\text{st}}^{(1)}}{1 - \beta_1^2} + \psi_{12}\dfrac{\Delta_{\text{st}}^{(2)}}{1 - \beta_2^2}\right)\sin\omega t = \left(\dfrac{1.10926}{1 - \beta_1^2} - \dfrac{0.109271}{1 - \beta_2^2}\right)\dfrac{p_0}{k}\sin\omega t \\ x_2(t) = \left(\psi_{21}\dfrac{\Delta_{\text{st}}^{(1)}}{1 - \beta_1^2} + \psi_{22}\dfrac{\Delta_{\text{st}}^{(2)}}{1 - \beta_2^2}\right)\sin\omega t = \left(\dfrac{1.31575}{1 - \beta_1^2} + \dfrac{0.184245}{1 - \beta_2^2}\right)\dfrac{p_0}{k}\sin\omega t \end{cases}$$

The response in modal coordinates

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To have a feeling of the response in modal coordinates, let's say
that the frequency of the load is $\omega = 2\omega_0$.
This implies that $\beta_1 = \frac{\omega}{\omega_1} = \frac{2.0}{0.31386} = 6.37226$ and
$\beta_2 = \frac{\omega}{\omega_2} = \frac{2.0}{3.18614} = 0.62771$.
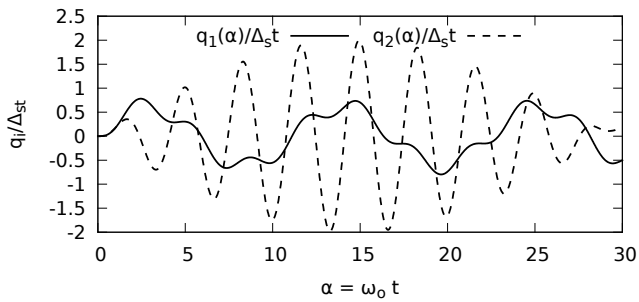
In the graph above, the responses are plotted against an
adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates
are adimensionalised with respect to $\Delta_{\text{st}} = \frac{p_0}{k}$

The response in structural coordinates

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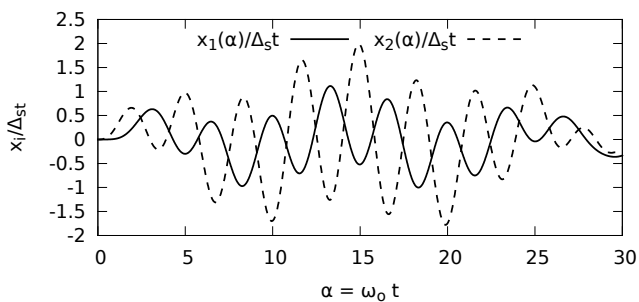
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Using the same normalisation factors, here are the response
functions in terms of $x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$: