Multiple support excitation

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May 28, 2013

Outline

Definitions

Consider the case of a structure where the supports are subjected to *assigned* displacements histories, $u_i = u_i(t)$.
To solve this problem, we start with augmenting the degrees of freedom with the support displacements.
We denote the superstructure *DOF* with $\mathbf{x}_T$, the support *DOF* with $\mathbf{x}_g$ and we have a global displacement vector $\mathbf{x}$,

$$\mathbf{x} = \begin{Bmatrix} \mathbf{x}_T \\ \mathbf{x}_g \end{Bmatrix}.$$

The Equation of Motion

Damping effects will be introduced at the end of our manipulations.
The equation of motion is

$$\begin{bmatrix}\mathbf{M} & \mathbf{M}_g \\ \mathbf{M}_g^T & \mathbf{M}_{gg}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{x}}_T \\ \ddot{\mathbf{x}}_g\end{Bmatrix}+\begin{bmatrix}\mathbf{K} & \mathbf{K}_g \\ \mathbf{K}_g^T & \mathbf{K}_{gg}\end{bmatrix}\begin{Bmatrix}\mathbf{x}_T \\ \mathbf{x}_g\end{Bmatrix}=\begin{Bmatrix}\mathbf{0} \\ \mathbf{p}_g\end{Bmatrix}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the usual structural matrices, while $\mathbf{M}_g$ and $\mathbf{M}_{gg}$ are, in the common case of a lumped mass model, zero matrices.

Static Components

We decompose the vector of displacements into two contributions, a static contribution and a dynamic contribution, attributing the *given* support displacements to the static contribution.

$$\begin{Bmatrix}\mathbf{x}_T\\ \mathbf{x}_g\end{Bmatrix} = \begin{Bmatrix}\mathbf{x}_s\\ \mathbf{x}_g\end{Bmatrix} + \begin{Bmatrix}\mathbf{x}\\ \mathbf{0}\end{Bmatrix}$$

where $\mathbf{x}$ is the usual *relative displacements* vector.

Determination of static components

Because the $\mathbf{x}_g$ are given, we can write two matricial equations that give us the static superstructure displacements and the forces we must apply to the supports,

$$\mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g = \mathbf{0}$$
$$\mathbf{K}_g^T\mathbf{x}_s + \mathbf{K}_{gg}\mathbf{x}_g = \mathbf{p}_g$$

From the first equation we have

$$\mathbf{x}_s = -\mathbf{K}^{-1}\mathbf{K}_g\mathbf{x}_g$$

and from the second we have

$$\mathbf{p}_g = (\mathbf{K}_{gg} - \mathbf{K}_g^T\mathbf{K}^{-1}\mathbf{K}_g)\mathbf{x}_g$$

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The support forces are zero when the structure is isostatic or the structure is subjected to a rigid motion.

Going back to the EOM

We need the first row of the two matrix equation of equilibrium,

$$\begin{bmatrix}\mathbf{M} & \mathbf{M}_g\\ \mathbf{M}_g^T & \mathbf{M}_{gg}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{x}}_T\\ \ddot{\mathbf{x}}_g\end{Bmatrix}+\begin{bmatrix}\mathbf{K} & \mathbf{K}_g\\ \mathbf{K}_g^T & \mathbf{K}_{gg}\end{bmatrix}\begin{Bmatrix}\mathbf{x}_T\\ \mathbf{x}_g\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{p}_g\end{Bmatrix}$$

substituting $\mathbf{x}_T = \mathbf{x}_s + \mathbf{x}$ in the first row

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{M}\ddot{\mathbf{x}}_s + \mathbf{M}_g\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} + \mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g = \mathbf{0}$$

by the equation of static equilibrium, $\mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g = \mathbf{0}$ we can simplify

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{M}\ddot{\mathbf{x}}_s+\mathbf{M}_g\ddot{\mathbf{x}}_g+\mathbf{K}\mathbf{x} = \mathbf{M}\ddot{\mathbf{x}}+(\mathbf{M}_g-\mathbf{M}\mathbf{K}^{-1}\mathbf{K}_g)\ddot{\mathbf{x}}_g+\mathbf{K}\mathbf{x} = \mathbf{0}.$$

Influence matrix

The equation of motion is

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{M}_g - \mathbf{M}\mathbf{K}^{-1}\mathbf{K}_g)\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} = \mathbf{0}.$$

We define the *influence matrix* $\mathbf{E}$ by

$$\mathbf{E} = -\mathbf{K}^{-1}\mathbf{K}_g,$$

and write, reintroducing the damping effects,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -(\mathbf{M}\mathbf{E} + \mathbf{M}_g)\ddot{\mathbf{x}}_g - (\mathbf{C}\mathbf{E} + \mathbf{C}_g)\dot{\mathbf{x}}_g$$

Simplification of the EOM

For a lumped mass model, $\mathbf{M}_g = \mathbf{0}$ and also the efficace forces due to damping are really small with respect to the inertial ones, and with this understanding we write

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{E}\ddot{\mathbf{x}}_g.$$

Significance of **E**

E can be understood as a collection of vectors $\mathbf{e}_i$, $i = 1, \ldots, N_g$ (N_g being the number of *DOF* associated with the support motion),

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_{N_g} \end{bmatrix}$$

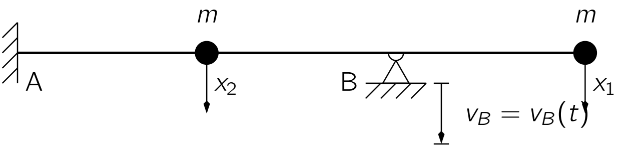
where the individual $\mathbf{e}_i$ collects the displacements in all the *DOF* of the superstructure due to imposing a unit displacement to the support *DOF* number i.

Significance of **E**

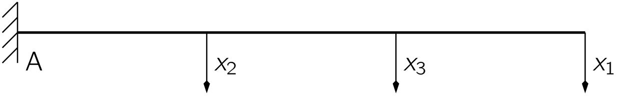
This understanding means that the influence matrix can be computed column by column,

- in the general case by releasing one support *DOF*, applying a unit force to the released *DOF*, computing all the displacements and scaling the displacements so that the support displacement component is made equal to 1,
- or in the case of an isostatic component by examining the instantaneous motion of the 1 *DOF* rigid system that we obtain by releasing one constraint.

EOM example



We want to determine the influence matrix **E** for the structure in the figure above, subjected to an assigned motion in B.



First step, put in evidence another degree of freedom x_3 corresponding to the assigned vertical motion of the support in B and compute, using e.g. the PVD, the flexibility matrix:

$$\mathbf{F} = \frac{L^3}{6EJ} \begin{bmatrix} 54.0000 & 8.0000 & 28.0000 \\ 8.0000 & 2.0000 & 5.0000 \\ 28.0000 & 5.0000 & 16.0000 \end{bmatrix}.$$

EOM example

The stiffness matrix is found by inversion,

$$\mathbf{K} = \frac{3EJ}{13L^3}\begin{bmatrix} +7.0000 & +12.0000 & -16.0000 \\ +12.0000 & +80.0000 & -46.0000 \\ -16.0000 & -46.0000 & +44.0000 \end{bmatrix}.$$

We are interested in the partitions $\mathbf{K}_{xx}$ and $\mathbf{K}_{xg}$:

$$\mathbf{K}_{xx} = \frac{3EJ}{13L^3}\begin{bmatrix} +7.0000 & +12.0000.0000 \\ +12.0000 & +80.0000.0000 \end{bmatrix}, \mathbf{K}_{xg} = \frac{3EJ}{13L^3}\begin{bmatrix} -16 \\ -46 \end{bmatrix}.$$

The influence matrix is

$$\mathbf{E} = -\mathbf{K}_{xx}^{-1}\mathbf{K}_{xg} = \frac{1}{16}\begin{bmatrix} 28.0000 \\ 5.0000 \end{bmatrix},$$

please compare **E** with the last column of the flexibility matrix, **F**.

Response analysis

Consider the vector of support accelerations,

$$\ddot{\mathbf{x}}_g = \left\{\ddot{x}_{gl}, \qquad l = 1, \ldots, N_g\right\}$$

and the effective load vector

$$\mathbf{p}_{eff} = -\mathbf{M}\mathbf{E}\ddot{\mathbf{x}}_g = -\sum_{l=1}^{N_g} \mathbf{M}\mathbf{e}_l \ddot{x}_{gl}(t).$$

We can write the modal equation of motion for mode number n

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = -\sum_{l=1}^{N_g} \Gamma_{nl}\ddot{x}_{gl}(t)$$

where

$$\Gamma_{nl} = \frac{\boldsymbol{\psi}_n^T \mathbf{M}\mathbf{e}_l}{M_n^*}$$

Response analysis, continued

The solution $q_n(t)$ is hence, with the notation of last lesson,

$$q_n(t) = \sum_{l=1}^{N_g} \Gamma_{nl} D_{nl}(t),$$

D_{nl} being the response function for ζ_n and ω_n due to the ground excitation $\ddot{x}_{gl}$.

Response analysis, continued

The total displacements $\mathbf{x}_T$ are given by two contributions, $\mathbf{x}_T = \mathbf{x}_s + \mathbf{x}$, the expression of the contributions are

$$\mathbf{x}_s = \mathbf{E}\mathbf{x}_g(t) = \sum_{l=1}^{N_g} \mathbf{e}_l x_{gl}(t),$$

$$\mathbf{x} = \sum_{n=1}^{N} \sum_{l=1}^{N_g} \boldsymbol{\psi}_n \Gamma_{nl} D_{nl}(t),$$

and finally we have

$$\mathbf{x}_T = \sum_{l=1}^{N_g} \mathbf{e}_l x_{gl}(t) + \sum_{n=1}^{N} \sum_{l=1}^{N_g} \boldsymbol{\psi}_n \Gamma_{nl} D_{nl}(t).$$

Forces

For a computer program, the easiest way to compute the nodal forces is

a) compute, element by element, the nodal displacements by $\mathbf{x}_T$ and $\mathbf{x}_g$,

b) use the element stiffness matrix compute nodal forces,

c) assemble element nodal loads into global nodal loads.

That said, let's see the analytical development...

Forces

The forces on superstructure nodes due to deformations are

$$\mathbf{f}_s = \sum_{n=1}^{N} \sum_{l=1}^{N_g} \Gamma_{nl} \mathbf{K} \boldsymbol{\psi}_n D_{nl}(t)$$

$$\mathbf{f}_s = \sum_{n=1}^{N} \sum_{l=1}^{N_g} (\Gamma_{nl} \mathbf{M} \boldsymbol{\psi}_n)(\omega_n^2 D_{nl}(t)) = \sum \sum r_{nl} A_{nl}(t)$$

the forces on support

$$\mathbf{f}_{gs} = \mathbf{K}_g^T \mathbf{x}_T + \mathbf{K}_{gg} \mathbf{x}_g = \mathbf{K}_g^T \mathbf{x} + \mathbf{p}_g$$

or, using $\mathbf{x}_s = \mathbf{E} \mathbf{x}_g$

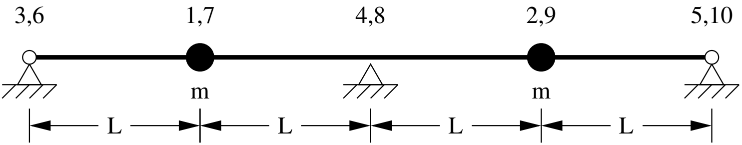
$$\mathbf{f}_{gs} = (\sum_{l=1}^{N_g} \mathbf{K}_g^T \mathbf{e}_l + \mathbf{K}_{gg,l}) x_{gl} + \sum_{n=1}^{N} \sum_{l=1}^{N_g} \Gamma_{nl} \mathbf{K}_g^T \boldsymbol{\psi}_n D_{nl}(t)$$

Forces

The structure response components must be computed considering the structure loaded by all the nodal forces,

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{f}_{gs} \end{Bmatrix}.$$

Example



The dynamic *DOF* are x_1 and x_2, vertical displacements of the two equal masses, x_3, x_4, x_5 are the imposed vertical displacements of the supports, $x_6, \ldots, x_{10}$ are the rotational degrees of freedom (removed by static condensation).

Example

The stiffness matrix for the 10x10 model is

$$\mathbf{K}_{10\times 10} = \frac{EJ}{L^3}\begin{bmatrix} 12 & -12 & 0 & 0 & 0 & 6L & 6L & 0 & 0 & 0 \\ -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 & 0 \\ 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 \\ 0 & 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L \\ 0 & 0 & 0 & -12 & 12 & 0 & 0 & 0 & -6L & -6L \\ 6L & -6L & 0 & 0 & 0 & 4L^2 & 2L^2 & 0 & 0 & 0 \\ 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 & 0 \\ 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 \\ 0 & 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 \\ 0 & 0 & 0 & 6L & -6L & 0 & 0 & 0 & 2L^2 & 4L^2 \end{bmatrix}$$

Example

The first product of the static condensation procedure is the linear mapping between translational and rotational degrees of freedom, given by

$$\vec{\boldsymbol{\phi}} = \frac{1}{56L}\begin{bmatrix} 71 & -90 & 24 & -6 & 1 \\ 26 & 12 & -48 & 12 & -2 \\ -7 & 42 & 0 & -42 & 7 \\ 2 & -12 & 48 & -12 & -26 \\ -1 & 6 & -24 & 90 & -71 \end{bmatrix}\vec{\mathbf{x}}.$$

Example

Following static condensation and reordering rows and columns, the partitioned stiffness matrices are

$$\mathbf{K} = \frac{EJ}{28L^3}\begin{bmatrix} 276 & 108 \\ 108 & 276 \end{bmatrix},$$

$$\mathbf{K}_\text{g} = \frac{EJ}{28L^3}\begin{bmatrix} -102 & -264 & -18 \\ -18 & -264 & -102 \end{bmatrix},$$

$$\mathbf{K}_\text{gg} = \frac{EJ}{28L^3}\begin{bmatrix} 45 & 72 & 3 \\ 72 & 384 & 72 \\ 3 & 72 & 45 \end{bmatrix}.$$

The influence matrix is

$$\mathbf{E} = \mathbf{K}^{-1}\mathbf{K}_\text{g} = \frac{1}{32}\begin{bmatrix} 13 & 22 & -3 \\ -3 & 22 & 13 \end{bmatrix}.$$

Example

The eigenvector matrix is

$$\mathbf{\Psi} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

the matrix of modal masses is

$$\mathbf{M}^\star = \mathbf{\Psi}^T \mathbf{M} \mathbf{\Psi} = m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

the matrix of the non normalized modal participation coefficients is

$$\mathbf{L} = \mathbf{\Psi}^T \mathbf{M} \mathbf{E} = m \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{16} & \frac{11}{8} & \frac{5}{16} \end{bmatrix}$$

and, finally, the matrix of modal participation factors,

$$\mathbf{\Gamma} = (\mathbf{M}^\star)^{-1} \mathbf{L} = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{5}{32} & \frac{11}{16} & \frac{5}{32} \end{bmatrix}$$

Example

Denoting with $D_{ij} = D_{ij}(t)$ the response function for mode i due to ground excitation $\ddot{x}_{gj}$, the response can be written

$$\mathbf{x} = \begin{pmatrix} \psi_{11}\left(-\frac{1}{4}D_{11}+\frac{1}{4}D_{13}\right)+\psi_{12}\left(\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22}\right) \\ \psi_{21}\left(-\frac{1}{4}D_{11}+\frac{1}{4}D_{13}\right)+\psi_{22}\left(\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4}D_{13}+\frac{1}{4}D_{11}+\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22} \\ -\frac{1}{4}D_{11}+\frac{1}{4}D_{13}+\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22} \end{pmatrix}.$$