Continuous Systems, Infinite Degrees of Freedom

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Outline

Intro

Discrete models

Until now, structures were discretized, maybe lumping their masses in the *dynamical degrees of freedom* or maybe to use the *FEM* to derive a stiffness matrix, to be subjected to static condensation in the occurence of lumped masses or, on the contrary, to be used *as is*.
Multistory buildings are ecellent examples of structures for which a few dynamical degrees of freedom can describe the dynamical response.

Intro

Continuous models

For different type of structures (e.g., bridges, chimneys), a lumped mass model is not the first option. While a *FE* model is always appropriate, there is no apparent way of lumping the structural masses in a way that is obviously correct, and a great number of degrees of freeedom must be retained in the dynamic analysis.
An alternative to detailed *FE* models is deriving the equation of motion, in terms of partial derivatives differential equation, for the continuous systems.

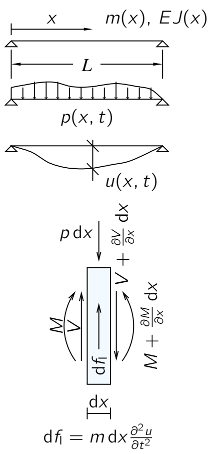
Continuous Systems

There are many different continuous systems whose dynamics are approachable with the instruments of classical mechanics,

- taught strings,
- axially loaded rods,
- beams in flexure,
- plates and shells,
- 3D solids.

In the following, we will focus our interest on beams in flexure.

EoM for an undamped beam



At the left, a straight beam with characteristic depending on position x: $m = m(x)$ and $EJ = EJ(x)$; with the signs conventions for displacements, accelerations, forces and bending moments reported left, the equation of vertical equilibrium for an infinitesimal slice of beam is

$$V - (V + \frac{\partial V}{\partial x}\,dx) + m\,dx\frac{\partial^2 u}{\partial t^2} - p(x,t)\,dx = 0.$$

Rearranging and simplifying dx,

$$\frac{\partial V}{\partial x} = m\frac{\partial^2 u}{\partial t^2} - p(x,t).$$

The rotational equilibrium, neglecting rotational inertia and simplifying dx is

$$\frac{\partial M}{\partial x} = V.$$

Equation of motion, 2

Deriving with respect to x both members of the rotational equilibrium equation, it is

$$\frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2}$$

Substituting in the equation of vertical equilibrium and rearranging

$$m(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 M}{\partial x^2} = p(x,t)$$

Using the moment-curvature relationship,

$$M = -EJ\frac{\partial^2 u}{\partial x^2}$$

and substituting in the equation above we have the equation of dynamic equilibrium

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t).$$

Effective Earthquake Loading

If our continuous structure is subjected to earthquake excitation, we will write, as usual, $u_{\text{tot}} = u(x,t) + u_{\text{g}}(t)$ and, consequently,

$$\ddot{u}_{\text{tot}} = \ddot{u}(x,t) + \ddot{u}_{\text{g}}(t)$$

and, using the usual considerations,

$$p_{\text{eff}}(x,t) = -m(x)\ddot{u}_{\text{g}}(t).$$

In p_{eff} we have a separation of variables: in the case of earthquake excitation all the considerations we have done on expressing the response in terms of static modal responses and pseudo/acceleration response will be applicable.
Only a word of caution, in every case we must consider the component of earthquake acceleration *parallel* to the transverse motion of the beam.

Free Vibrations

For free vibrations, $p(x,t) \equiv 0$ and the equation of equilibrium is

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 u}{\partial x^2}\right] = 0.$$

Using separation of variables, with the following notations,

$$u(x,t) = q(t)\phi(x),\ \frac{\partial u}{\partial t} = \dot{q}\phi,\ \frac{\partial u}{\partial x} = q\phi'$$

etc for higher order derivatives, we have

$$m(x)\ddot{q}(t)\phi(x) + q(x)\left[EJ(x)\phi''\right]'' = 0.$$

Free Vibrations, 2

Dividing both terms in

$$m(x)\ddot{q}(t)\phi(x) + q(t)\left[EJ(x)\phi''(x)\right]'' = 0.$$

by $m(x)u(x,t) = m(x)q(t)\phi(x)$ and rearranging, we have

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{[EJ(x)\phi''(x)]''}{m(x)\phi(x)}.$$

The left member is a function of time only, the right member a function of position only, and they are equal... this is possible if and only if both terms are constant, let's name this constant ω^2 and write

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{[EJ(x)\phi''(x)]''}{m(x)\phi(x)} = \omega^2,$$

Free Vibrations, 3

From the previous equations we can derive the following two

$$\ddot{q} + \omega^2 q = 0$$
$$\left[EJ(x)\phi''(x)\right]'' = \omega^2 m(x)\phi(x)$$

From the first, $\ddot{q} + \omega^2 q = 0$, it is apparent that free vibration shapes $\phi(x)$ will be modulated by a trig function

$$q(t) = A\sin\omega t + B\cos\omega t.$$

To find something about ω's and ϕ's (that is, the eigenvalues and the *eigenfunctions* of our problem), we have to introduce an important simplification.

Eigenpairs of a uniform beam

With $EJ = \text{const.}$ and $m = \text{const.}$, we have from the second equation in previous slide,

$$EJ\phi^{\text{IV}} - \omega^2 m\phi = 0,$$

with $\beta^4 = \frac{\omega^2 m}{EJ}$ it is

$$\phi^{\text{IV}} - \beta^4\phi = 0$$

a differential equation of 4$^{\text{th}}$ order with constant coefficients.
Substituting $\phi = \exp st$ and simplfying,

$$s^4 - \beta^4 = 0,$$

the roots of the associated polynomial are

$$s_1 = \beta,\ s_2 = -\beta,\ s_3 = i\beta,\ s_4 = -i\beta$$

and the general integral is

$$\phi(x) = \mathcal{A}\sin\beta x + \mathcal{B}\cos\beta x + \mathcal{C}\sinh\beta x + \mathcal{D}\cosh\beta x$$

Constants of Integration

For a uniform beam in free vibration, the general integral is

$$\phi(x) = \mathcal{A}\sin\beta x + \mathcal{B}\cos\beta x + \mathcal{C}\sinh\beta x + \mathcal{D}\cosh\beta x$$

In this expression we see 5 parameters, the 4 constants of integration and the wave number β (further consideration shows that the constants can be arbitrarily scaled).
In general for the transverse motion of a segment of beam supported at the extremes we can write exactly 4 equations expressing boundary conditions, either from kinematc or static considerations.
All these boundary conditions

- lead to linear, homogeneous equation where
- the coefficients of the equations depend on the parameter β.

Eigenvalues and eigenfunctions

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Imposing the boundary conditions give a homogeneous linear system with coefficients depending on β, hence:

- a non trivial solution is possible only for particular values of β, for which the determinant of the matrix of cofficients is equal to zero and
- the constants are known within a proportionality factor.

In the case of *MDOF* systems, the determinantal equation is an algebraic equation of order N, giving exactly N eigenvalues, now the equation to be solved is a trascendental equation (examples from the next slide), with an infinity of solutions.

Simply supported beam

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Consider a simply supported uniform beam of lenght L, displacements at both ends must be zero, as well as the bending moments:

$$\phi(0) = \mathcal{B}+\mathcal{D} = 0, \qquad \phi(L) = 0,$$
$$-EJ\phi''(0) = \beta^2 EJ(\mathcal{B}-\mathcal{D}) = 0, \qquad -EJ\phi''(L) = 0.$$

The conditions for the left support require that $\mathcal{B}=\mathcal{D}=0$
Now, we can write the equations for the right support as

$$\phi(L) = \mathcal{A}\sin\beta L + \mathcal{C}\sinh\beta L = 0$$
$$-EJ\phi''(L) = \beta^2 EJ(\mathcal{A}\sin\beta L - \mathcal{C}\sinh\beta L) = 0$$

or

$$\begin{bmatrix} +\sin\beta L & +\sinh\beta L \\ +\sin\beta L & -\sinh\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{C} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Simply supported beam, 2

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Example

For the simply supported beam we have

$$\begin{bmatrix} +\sin\beta L & +\sinh\beta L \\ +\sin\beta L & -\sinh\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{C} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The determinant is $-2\sin\beta L\sinh\beta L$, equating to zero with the understanding that $\sinh\beta L \neq 0$ if $\beta \neq 0$ results in

$$\sin\beta L = 0.$$

All positive β solutions are given by

$$\beta L = n\pi$$

with $n = 1, \ldots, \infty$. We have an infinity of eigenvalues,

$$\beta_n = \frac{n\pi}{L} \text{ and } \omega_n = \beta^2\sqrt{\frac{EJ}{m}} = n^2\pi^2\sqrt{\frac{EJ}{mL^4}}$$

and of eigenfunctions

$$\phi_1 = \sin\frac{\pi x}{L},\ \phi_2 = \sin\frac{2\pi x}{L},\ \phi_3 = \sin\frac{3\pi x}{L},\ \cdots$$

Cantilever beam

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For $x = 0$, we have zero displacement and zero rotation

$$\phi(0) = \mathcal{B}+\mathcal{D} = 0, \qquad \phi'(0) = \beta(\mathcal{A}+\mathcal{C}) = 0,$$

for $x = L$, both bending moment and shear must be zero

$$-EJ\phi''(L) = 0, \qquad -EJ\phi'''(L) = 0.$$

Substituting the expression of the general integral, with $\mathcal{D} = -\mathcal{B}$, $\mathcal{C} = -\mathcal{A}$ from the left end equations, in the right end equations and simplifying

$$\begin{bmatrix} \sinh\beta L + \sin\beta L & \cosh\beta L + \cos\beta L \\ \cosh\beta L + \cos\beta L & \sinh\beta L - \sin\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{B} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Cantilever beam, 2

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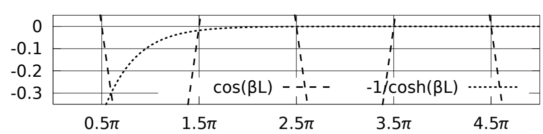
Imposing a zero determinant results in

$$\begin{aligned}(\cosh^2\beta L - \sinh^2\beta L) + \\ + (\sin^2\beta L + \cos^2\beta L) + 2\cos\beta L\cosh\beta L = \\ = 2(1+\cos\beta L\cosh\beta L) = 0\end{aligned}$$

Rearranging,it is $\cos\beta L = -(\cosh\beta L)^{-1}$; plotting these functions on the same graph gives insight on the roots

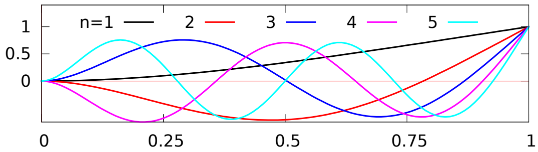


it is $\beta_1 L = 1.8751$ and $\beta_2 L = 4.6941$, while for $n > 2$ a good approximation is $\beta_n L \approx \frac{2n-1}{2}\pi = n\pi - \frac{\pi}{2}$.

Cantilever beam, 3

Eigenfunctions are given by

$$\phi_n(x) = C_n\left[(\cosh\beta_n x - \cos\beta_n x) - \frac{\cosh\beta_n L + \cos\beta_n L}{\sinh\beta_n L + \sin\beta_n L}(\sinh\beta_n x - \sin\beta_n x)\right]$$



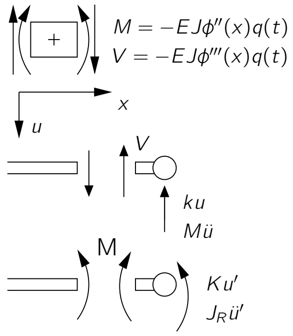
Above, in abscissas x/L and in ordinates $\phi_n(x)$ for the first 5 modes of vibration of the cantilever beam.

n	1	2	3	4	5
$\beta_n L$	1.8751	4.6941	7.8548	10.9962	$\approx 4.5\pi$
$\omega\sqrt{\frac{mL^4}{EJ}}$	3.516	22.031	61.70	120.9	$\cdots$

Other Boundary Conditions

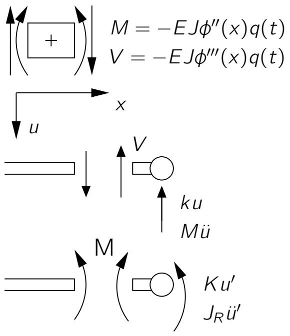
Boundary conditions can be expressed also by the relation between displacements and forces.

The shear in the beam is equal and opposite *a*) to the spring reaction or *b*) to the inertial force, so we can write, for a spring constant $k = \alpha EJ/L^3$



$$-EJ\phi'''(\beta L)q(t) + k\phi(\beta L)q(t) = 0$$
$$-EJ\phi'''(\beta L) + \alpha\frac{EJ}{L^3}\phi(\beta L) = 0$$
$$-L^3\phi'''(\beta L) + \alpha\phi(\beta L) = 0$$
$$-(\beta L)^3(-\mathcal{A}\cos\beta L + \cdots) + \alpha\phi(\beta L) = 0$$

Other Boundary Conditions



Consider now an inertial force

$$M\ddot{u} = -\omega^2 M\phi(x)q(t)$$

(by $\ddot{q} = -\omega^2 q$), with $M = \gamma mL$ the equation of equilibrium is

$$-EJ\phi'''(\beta L)q(t) + M\phi(\beta L)\ddot{q}(t) = 0$$
$$-EJ\phi'''(\beta L)q(t) - \omega^2\gamma mL\phi(\beta L)q(t) = 0$$
$$-EJ\phi'''(\beta L) - \omega^2\gamma mL\phi(\beta L) = 0$$

by $\omega^2 = \beta^4 EJ/m$

$$-EJ\phi'''(\beta L) - \beta^4\frac{EJ}{m}\gamma mL\phi(\beta L) = 0$$
$$-L^3\phi'''(\beta L) - (\beta L)^4\gamma\phi(\beta L) = 0$$
$$-(\beta L)^3(-\mathcal{A}\cos\beta L + \cdots) - (\beta L)^4\gamma\phi(\beta L) = 0$$

Similar considerations apply to equilibrium of bending moment and applied couple.

Mode Orthogonality

We will demonstrate mode orhogonality for a restricted set of boundary conditions, i.e., disregarding elastic supports and supported masses. In the beginning we have, for $n = r$,

$$\left[EJ(x)\phi_r''(x)\right]'' = \omega_r^2 m(x)\phi_r(x)$$

premultiplying both members by $\phi_s(x)$ and integrating over the length of the beam gives

$$\int_0^L \phi_s(x)\left[EJ(x)\phi_r''(x)\right]''\,\mathrm{d}x = \omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x$$

Mode Orthogonality, 2

The left member can be integrated by parts, two times, as in

$$\int_0^L \phi_s(x)\left[EJ(x)\phi_r''(x)\right]''\,\mathrm{d}x =$$
$$\left[\phi_s(x)\left[EJ(x)\phi_r''(x)\right]'\right]_0^L - \left[\phi_s'(x)EJ(x)\phi_r''(x)\right]_0^L +$$
$$\int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x$$

but the terms in brackets are always zero, the first being the product of end displacement by end shear, the second the product of end rotation by bending moment, and for fixed constraints or free end one of the two terms must be zero. So it is

$$\int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x = \omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x.$$

Mode Orthogonality, 3

We write the last equation exchanging the roles of r and s and subtract from the original,

$$\int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x - \int_0^L \phi_r''(x)EJ(x)\phi_s''(x)\,\mathrm{d}x =$$
$$\omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x - \omega_s^2\int_0^L \phi_r(x)m(x)\phi_s(x)\,\mathrm{d}x.$$

This obviously can be simplyfied giving

$$(\omega_r^2 - \omega_s^2)\int_0^L \phi_r(x)m(x)\phi_s(x)\,\mathrm{d}x = 0$$

implying that, for $\omega_r^2 \neq \omega_s^2$ the modes are orthogonal with respect to the mass distribution and the bending stiffness distribution.

Forced dynamic response

With $u(x,t) = \sum_1^\infty \phi_m(x) q_m(t)$, the equation of motion can be written

$$\sum_1^\infty m(x)\phi_m(x)\ddot q_m(t) + \sum_1^\infty \left[EJ(x)\phi_m''(x)\right]'' q_m(t) = p(x,t)$$

premultiplying by ϕ_n and integrating each sum and the loading term

$$\sum_1^\infty \int_0^L \phi_n(x) m(x) \phi_m(x) \ddot q_m(t)\,dx +$$
$$\sum_1^\infty \int_0^L \phi_n(x)\left[EJ(x)\phi_m''(x)\right]'' q_m(t)\,dx = \int_0^L \phi_n(x) p(x,t)\,dx$$

Forced dynamic response, 2

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By the orthogonality of the eigenfunctions this can be simplyfied to

$$m_n \ddot q_n(t) + k_n q_n(t) = p_n(t), \qquad n = 1, 2, \ldots, \infty$$

with

$$m_n = \int_0^L \phi_n m \phi_n\,dx, \qquad k_n = \int_0^L \phi_n \left[EJ\phi_n''\right]''\,dx,$$
$$\text{and} \qquad p_n(t) = \int_0^L \phi_n p(x,t)\,dx.$$

For free ends and/or fixed supports, $k_n = \int_0^L \phi_n'' EJ \phi_n''\,dx$.

Earthquake response

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Consider an effective earthquake load, $p(x,t) = m(x)\ddot u_g(t)$, with

$$\mathcal{L}_n = \int_0^L \phi_n(x) m(x)\,dx, \qquad \Gamma_n = \frac{\mathcal{L}_n}{m_n},$$

the modal equation of motion can be written (with an obvious generalisation)

$$\ddot q_n + 2\omega_n\zeta_n \dot q_n + \omega_n^2 q = -\Gamma_n \ddot u_g(t)$$

and the modal response can be written, also for the case of continuous structures, as the product of the modal partecipation factor and the deformation response,

$$q_n(t) = \Gamma_n D_n(t).$$

Earthquake response, 2

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Modal contributions can be computed directly, e.g .

$$u_n(x,t) = \Gamma_n \phi_n(x) D_n(t),$$
$$M_n(x,t) = -\Gamma_n EJ(x)\phi_n''(x) D_n(t),$$

or can be computed from the equivalent static forces,

$$f_s(x,t) = \left[EJ(x) u(x,t)''\right]''.$$

Earthquake response, 3

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The modal contributions to equiv. static forces are

$$f_{sn}(x,t) = \Gamma_n \left[EJ(x)\phi_n(x)''\right]'' D_n(t),$$

that, because it is

$$\left[EJ(x)\phi''(x)\right]'' = \omega^2 m(x)\phi(x)$$

can be written in terms of the mass distribution and of the pseudo-acceleration response $A_n(t) = \omega_n^2 D_n(t)$

$$f_{sn}(x,t) = \Gamma_n m(x)\phi_n(x)\omega_n^2 D_n(t) = \Gamma_n m(x)\phi_n(x) A_n(t).$$

Earthquake response, 4

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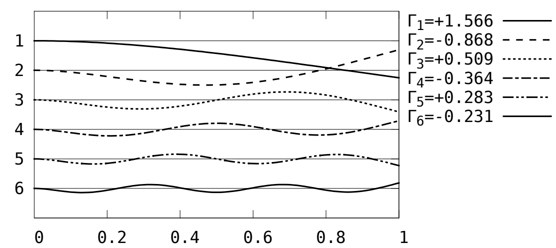
Example

The effective load is proportional to the mass distribution, and we can do a modal mass decomposition in the same way that we had for *MDOF* systems,

$$m(x) = \sum r_n(x) = \sum \Gamma_n m(x)\phi_n(x)$$



Above, the modal mass decomposition $r_n = \Gamma_n m \phi_n$,for the first six modes of a uniform cantilever, in abscissa x/L.

EQ example, cantilever

For a cantilever, it is possible to derive explicitly some response quantities,

$$V(x),\qquad V_b,\qquad M(x),\qquad M_b,$$

that is, the shear force and the base shear force, the bending moment and the base bending moment.

$$V_n^{st}(x)=\int_x^L r_n(s)\,ds,\qquad V_b^{st}=\int_0^L r_n(s)\,ds=\Gamma_n\mathcal{L}_n=M_n^\star,$$
$$M_n^{st}(x)=\int_x^L r_n(s)(s-x)\,ds,\quad M_b^{st}=\int_0^L s\,r_n(s)\,ds=M_n^\star h_n^\star.$$

$M_n^\star$ is the *partecipating modal mass* and expresses the partecipation of the different modes to the base shear, it is $\sum M_n^\star=\int_0^L m(x)\,dx$.
$M_n^\star h_n^\star$ expresses the modal partecipation to base moment, $h_n^\star$ is the height where the partecipating modal mass $M_n^\star$ must be placed so that its effects on the base are the same of the static modal forces effects, or $M_n^\star$ is the resultant of s.m.f. and $h_n^\star$ is the position of this resultant.

EQ example, cantilever, 2

Starting with the definition of total mass and operating a chain of substitutions,

$$\begin{aligned}M_{tot}&=\int_0^L m(x)\,dx=\sum\int_0^L r_n(x)\,dx\\&=\sum\int_0^L\Gamma_n m(x)\phi_n(x)\,dx=\sum\Gamma_n\int_0^L m(x)\phi_n(x)\,dx\\&=\sum\Gamma_n\mathcal{L}_n=\sum M_n^\star,\end{aligned}$$

we have demonstrated that the sum of the partecipating modal mass is equal to the total mass.
The demonstration that $M_{b,tot}=\sum M_n^\star h_n^\star$ is similar and is left as an exercise.

EQ example, cantilever, 3

For the first 6 modes of a uniform cantilever,

n	$\mathcal{L}_n$	m_n	Γ_n	$V_{b,n}$	h_n	$M_{b,n}$
1	0.391496	0.250	1.565984	0.613076	0.726477	0.445386
2	-0.216968	0.250	-0.867872	0.188300	0.209171	0.039387
3	0.127213	0.250	0.508851	0.064732	0.127410	0.008248
4	-0.090949	0.250	-0.363796	0.033087	0.090943	0.003009
5	0.070735	0.250	0.282942	0.020014	0.070736	0.001416
6	-0.057875	0.250	-0.231498	0.013398	0.057875	0.000775
7	0.048971	0.250	0.195883	0.009593	0.048971	0.000470
8	-0.042441	0.250	-0.169765	0.007205	0.042442	0.000306

The convergence for M_b is faster than for V_b, because the latter is proportional to a higher derivative of displacements.

Continuous Systems

an example

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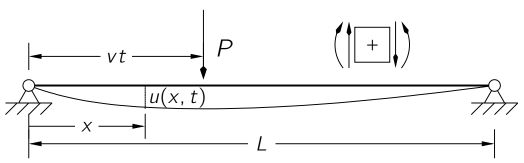
Problem statement



A uniform beam ($m(x)=m$, $EJ(x)=EJ$) of lenght L is loaded by a moving load P, moving with constant velocity, $v(t)=v$, in the interval $0\le t\le t_0={}^L/_v=t_0$.
Using the sign conventions indicated above, compute and plot the midspan displacement $u({}^L/_2,t)$ and the midspan bending moment $M_b({}^L/_2,t)$ as functions of time in the interval $0\le t\le t_0$ for different values of the velocity.

NB: the beam is at rest for $t=0$.

Equation of motion

For an uniform beam, the equation of dynamic equilibrium is

$$m\,\frac{\partial^2u(x,t)}{\partial t^2}+EJ\,\frac{\partial^4u(x,t)}{\partial x^4}=p(x,t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x,t)=P\,\delta(x-vt).$$

The Dirac's delta is a *generalized* function of one variable, defined by

$$\delta(x-x_0)\equiv 0\qquad\text{and}\qquad\int f(x)\delta(x-x_0)\,dx=f(x_0).$$

Note that the Dirac distribution and the Kronecker's symbol δ_{ij} are two different things.

Equation of motion

The solution will be computed by separation of variables

$$u(x, t) = q(t)\phi(x)$$

and modal analysis,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\phi_n(x) = \sin\beta_n x, \qquad \beta_n = \frac{n\pi}{L},$$
$$m_n = \frac{mL}{2}, \qquad \omega_n^2 = \beta_n^4\frac{EJ}{m} = n^4\pi^4\frac{EJ}{mL^4}.$$

Orthogonality relationships

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For an uniform beam, the orthogonality relationships are

$$m\int_0^L \phi_n(x)\phi_m(x)\,\mathrm{d}x = m_n\delta_{nm},$$
$$EJ\int_0^L \phi_n(x)\phi_m^{\mathrm{IV}}(x)\,\mathrm{d}x = k_n\delta_{nm} = m_n\omega_n^2\delta_{nm}.$$

in the equations above δ is the Kroneker's δ symbol, a completely different thing from Dirac's δ distribution.

Decoupling the EOM

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Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of $u(x, t)$:

$$m\sum_{m=1}^{\infty}\ddot{q}_m\phi_m + EJ\sum_{m=1}^{\infty}q_m\phi_m^{\mathrm{IV}} = P\,\delta(x - vt),$$

2. every term is multiplied by ϕ_n and integrated over the lenght of the beam

$$m\int_0^L \phi_n\sum_{m=1}^{\infty}\ddot{q}_m\phi_m\,\mathrm{d}x + EJ\int_0^L\phi_n\sum_{m=1}^{\infty}q_m\phi_m^{\mathrm{IV}}\,\mathrm{d}x =$$
$$P\int_0^L \phi_n\delta(x - vt), \qquad n = 1, \ldots, \infty$$

3. we use the ortogonality relationships and the definition of δ,

$$m_n\ddot{q}(t) + k_n q(t) = P\,\phi_n(vt) = P\,\sin\frac{n\pi\,vt}{L}, \qquad n = 1, \ldots, \infty.$$

Solutions

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Considering that the initial conditions are nil for all the modal equations, with $\overline{\omega}_n = n\pi v/L$ and $\beta_n = \overline{\omega}_n/\omega_n$ the individual solutions are given by

$$q_n(t) = \frac{P}{k_n}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n\sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}$$

With $k_n = m_n\omega_n^2 = \frac{mL}{2}\,n^4\pi^4\frac{EJ}{mL^4} = n^4\pi^4\frac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2PL^3}{n^4\pi^4 EJ}\frac{1}{1-\beta_n^2}\left(\sin\overline{\omega}_n t - \beta_n\sin\omega_n t\right), \quad 0 \le t \le \frac{L}{v}.$$

It is apparent that for $\beta_n^2 = 1$ there is resonance.

Solutions

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The critical velocity $v_{\mathrm{cr},n}$ for mode n is given by $\beta_n = 1$, substituting $\omega_n = n^2\omega_1$ we have ${}^{n\pi v_{\mathrm{cr},n}/L}/{}_{n^2\omega_1} = 1$ that gives $v_{\mathrm{cr},n} = {}^{n\omega_1 L}/{}_{\pi} = n\,v_{\mathrm{cr},1} = n\,v_{\mathrm{cr}}$, where $v_{\mathrm{cr}} = {}^{\omega_1 L}/{}_{\pi}$.
With the position $v = \kappa v_{\mathrm{cr}}$ it is

$$\overline{\omega}_n = \kappa n\omega_1 \text{ and } \beta_n = n\kappa\omega_1/n^2\omega_1 = {}^{\kappa}/{}_{n}.$$

The solution can be rewritten as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(\frac{\kappa}{n}\omega_n t) - \frac{\kappa}{n}\sin\omega_n t\right),$$
$$\text{for } 0 \le t \le \frac{L}{v}.$$

Adimensional time

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Introducing an adimensional time coordinate ξ with $t = t_0\xi$, noting that $\omega_n = n^2\omega_1$ we can write the argument of the first sine as follows:

$$\frac{\kappa}{n}\omega_n t = \kappa n\omega_1\xi t_0 = n\xi t_0\kappa v_{\mathrm{cr}}\pi/L = n\pi\xi \times (vt_0)/L = n\pi\xi.$$

In a similar way we have $\omega_n t = n^2\pi\xi/\kappa$.
Substituting in the equation of the modal responses the new expressions for the sine arguments, it is

$$q_n(\xi) = \frac{2PL^3}{\pi^4 EJ}\frac{1}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right)$$

for $0 \le \xi \le 1$.

Adimensional time IS adimensional position

If we denote with $\mathbb{X}(t)$ the position of the load at time t, it is $\mathbb{X}(t) = vt = \xi L$, or $\xi = \mathbb{X}/L$ and the expression $u(x, \xi) = \sum q_n(\xi)\phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in $\mathbb{X} = \xi L$.

Analytical expressions of u and M_b

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The displacement and the bending moment are given by

$$u(x,\xi) = \frac{2PL^3}{\pi^4 EJ}\sum_{n=1}^{\infty}\frac{\sin(n\pi\frac{x}{L})}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right),$$

$$M_b(x,\xi) = -EJ\frac{\partial^2 u(x,\xi)}{\partial x^2} = $$

$$= \frac{2PL}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi\frac{x}{L})}{n^2-\kappa^2}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right).$$

Midspan deflection and bending moment

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The maximum values of the midspan deflection and bending moment are obtained when P is placed at midspan,

$$u_{\text{stat}} = \frac{PL^3}{48EJ}, \qquad M_{\text{b stat}} = \frac{PL}{4}.$$

It is convenient to normalize the responses with respect to these maxima to have an appreciation of the dynamical effects.

Midspan deflection and bending moment

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The normalized midspan displacement $\eta(\xi) = u(L/2, \xi)/u_{\text{stat}}$ has the expression

$$\eta(\xi) = \frac{96}{\pi^4}\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^2(n^2-\kappa^2)}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right),$$

where $sin(n\pi/2) = 1, 0, -1, 0, 1, \dots$ for $n = 1, 2, 3, 4, 5, \dots$
Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^2-\kappa^2}\left(\sin(n\pi\xi) - \frac{\kappa}{n}\sin(\frac{n^2}{\kappa}\pi\xi)\right).$$

Partial sums with N terms will be denoted in the following by $\eta_N(\xi)$ and $\mu_N(\xi)$.

Error estimates

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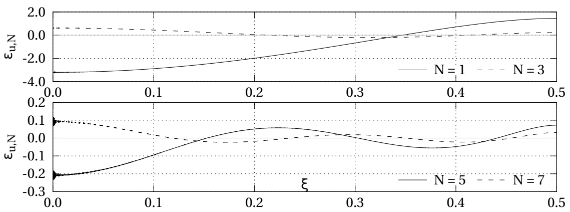
The normalized midspan statical displacement for a load P placed at $\mathbb{X} = \xi L$ is $\eta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ for $0 \le \xi \le 1/2$ and we can define a percent error function (using $\kappa = 10^{-6}$ to obtain a good approximation to the static response)

$$\epsilon_{u,N}(\xi) = 100\left(1 - \frac{\eta_N(\xi)|_{\kappa=10^{-6}}}{\eta_{\text{stat}}(\xi)}\right) \qquad \text{for } 0 \le \xi \le 1/2,$$



With 5 terms the approximation is in the order of $1/1000$.

Error estimates

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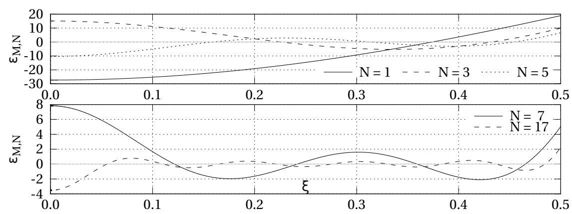
Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\text{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100\left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)}\right)$$

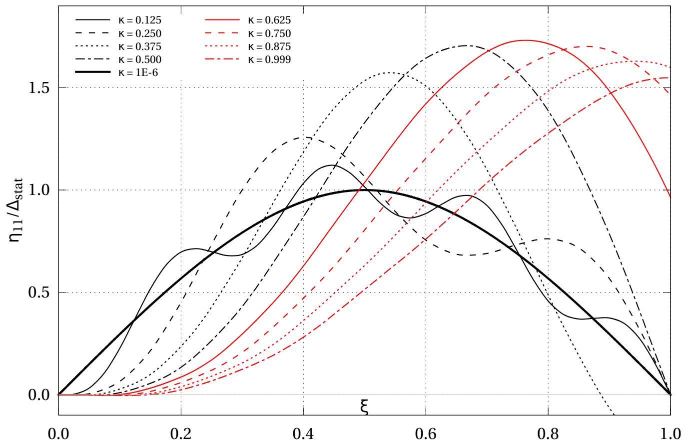


With 17 terms the approximation is in the order of 4%. As usual, worse convergence for internal forces.

The plots

Finally, we plot the normalized displacement and the normalized bending moment different values of the velocity (i.e., for different values of κ).
Note that for the displacement I used $N = 11$ while for the bending moment I used $N = 25$.

Displacements



Bending moments