Seismic Excitation

The most important quantity related to earthquake excitation is the ground acceleration. Ground acceleration can be recorded with an accelerometer, basically a SDOF oscillator, with a damping ratio $\zeta \approx 70\%$, whose displacements are proportional to ground accelerations up to a given frequency. Instrument records of strong ground motion first became available in the '30s, the first record of a destructive ground motion being the 1940 records of El Centro earthquake.

Historically, most of the strong motion records were recorded for a few earthquakes, in California and Japan, in different places and different locations (in the free field, on building foundations, on different building storeys etc), while a lesser number of records were available for different areas.

In more recent years, many national research agencies installed and operated networks of strong motion accelerometers, so that the availability of strong motion records, recorded in different geographic areas and under different local conditions is constantly improving. Moreover, in many countries the building codes require that important constructions must be equipped with accelerometers, further increasing the number of available records.

Seismic Excitation Samples

A number of different strong motion records, recorded at different sites and due to different earthquakes, are plotted with the same scale, both in time and in acceleration. Appreciate the large variability in terms of amplitudes, duration and frequency content of the different records. We need a method to categorize this variability.

Detailed Sample

Above, the acceleration recorded at El Centro during the Imperial Valley 1940 e.q., along with the velocity and displacements obtained by numerical integration.

For meaningful results, the initial conditions of integration and the removal of linear trends from the acceleration record are of capital importance (read: don’t try this at home).
Strong motion being a very irregular motion, a high number of samples is required to accurately describe it. Modern digital instruments record the acceleration at a rate of 200 samples per second and minimize the need for sophisticated correction of the accelerations before the time integration.

**T\(_n\)** and \(\zeta\) dependency

Leftmost column, fixed \(\zeta = 0.02\) and \(T\_n = 0.5, 1.0, 2.0\) s. Although the ground motion is irregular, the responses have a similarity, each one having a period close to \(T\_n\).

Centre column, fixed \(T\_n = 2.0\) s and \(\zeta = 0.0, 0.02, 0.05\). For a fixed period the shapes are similar while the maximum response values depend on \(\zeta\).

Displacement response functions for the El Centro 1940 NS acceleration record.

**Response Spectrum**

Introduced by M.A. Biot in 1932, popularised by G.W. Housner, the concept of response spectrum is fundamental to characterise e.q. response.

The response spectrum is a plot of the peak values of a response quantity, say the displacement response function, computed for different values of \(T\_n\) and the same \(\zeta\), versus natural period \(T\_n\).

A graph where several such plots, obtained for different values of \(\zeta\), representative of different damping ratios that characterise different structures, are plotted close to each other represents the e.q. characteristics from the point of view of peak structural response (wait later slides for examples).

**About Ground Motion**

In the following, we consider so called free-field records, that is recorded on ground free surface in a position that is deemed free from effects induced by building response.

Tough accelerations vary with time in a very irregular manner, the variation is fully known, and for an individual record we can write the equation of motion in terms of the displacement response function \(D(t)\).

\[ D + 2\zeta\omega D + \omega^2 D = -u(t) \]

Clearly, the displacement response function, for assigned \(u(t)\) depends on \(\zeta\) and \(\omega\) only.

Of course, due to the irregular nature of ground excitation the response must be evaluated numerically.

Our first step will be to explore the dependency of \(D\) on \(\omega\) (or rather \(T\_n\) as it is usual in earthquake engineering) and \(\zeta\).

**Pseudo Acceleration**

From deformation response, we can compute the equivalent static force

\[ F(t) = ma\varepsilon(t) = ma(t) \]

where \(A(t)\) is the pseudo acceleration,

\[ A(t) = \omega\varepsilon(t) = (2\pi)^2D(t)/T\_n^2 \]

Note, one more time, that \(\varepsilon\) is proportional to \(A(t)\) and not to the acceleration \(D(t)\).

Left, pseudo accelerations computed for varying \(T\_n\). Compare with previous page’s figure. The relative magnitudes are reverted: for \(T\_n = 0.5\) s we have a maximum force and a minimum displacement, while for \(T\_n = 2.0\) s the force is minimum and the displacement maximum.

**Computing the DRS**

For a fixed values of \(\zeta\) (usually one of 0% 0.5% 1% 2% 3% 5% 7% 10% 15% and 20%) and for variable values of \(T\_n\) (usually ranging from 0.01 s to 20 s)

1. the displacement response function is numerically integrated,
2. the peak value is individuated,
3. the peak value is plotted.
**Pseudo Spectra**

Only the Deformation Response Spectrum (DRS) is required to fully characterise the peaks of deformations and equivalent static forces. It is however useful to study also the pseudo acceleration (PARS) and pseudo velocity (PVRS) spectra, as they are useful in understanding excitation intrinsic characteristics, in constructing design spectra and to connect dynamics and building codes.

We have already introduced $A(t)$, consider now the quantity

$$V(t) = \omega_0 D(t) = \frac{2\pi}{T} D(t)$$

that is, the pseudo velocity.

The peak value of $V$ is connected with the maximum strain energy,

$$E_s,0 = \frac{1}{2} m V^2_0$$

being $E_s,0 = \frac{1}{2} D_0 m \omega^2 D_0$. Once again, $V \neq \dot{x}$, the relative velocity.

**Combined $D - V - A$ spectrum**

In the following, we will use the symbols $D$, $V$ and $A$ to represent the values of the DRS, PVRS and PARS spectra, respectively, with

$$V = \omega_0 D, \quad A = \omega_0^2 D$$

While $D$, $V$ and $A$ represent the same information, nonetheless it is useful to maintain a distinction as they are connected to different response quantities, the maximum deformation, the maximum strain energy and the maximum equivalent static force.

Moreover, it is possible to plot all three spectra on the same logarithmic plot, giving what is regarded as a fundamental insight into the ground motion characteristics.

**Constant A**

Consider a plane with axes $\log T_n$ and $\log V$, and the locus of this plane where $A$ is constant, $A = \hat{A}$.

It is

$$A = 2\pi V / T_n = \hat{A}$$

taking the logarithm

$$\log \frac{2\pi}{\hat{A}} = \log V - \log T_n$$

or

$$\log V = \log T_n + \log \frac{2\pi}{\hat{A}}$$

In the log-log plane straight lines at $45^\circ$ are characterised by a constant value of $A$.

**Constant D**

In the same plane with axes $\log T_n$ and $\log V$ we seek the locus where $D$ is constant, $D = \hat{D}$:

$$D = T_n V / 2\pi = \hat{D}$$

taking the logarithm

$$\log 2\pi \hat{D} = \log V + \log T_n$$

or

$$\log V = \log 2\pi \hat{D} - \log T_n$$

In the log-log plane straight lines at $-45^\circ$ are characterised by a constant value of $D$. 

**Example of Construction, 1**

Deformation spectrum, pseudo velocity and pseudo acceleration spectra for El Centro 1940 NS, $\zeta = 2\%$. 
Example of Construction, 2

**Peak Structural Response**

The peak deformation $u_b$ is given by

$$u_b = D$$

and the peak of the equivalent static force $f_{S,0}$ is given by

$$f_{S,0} = k u_b = m u_a^2 u_b = k D = mA$$

It is required to know the peak of the base bending moment for the structure on the right, when subjected to the NS component of the El Centro 1940 record. The mass is $m = 2360$ kg, the stiffness is $k = 36.84$ kN/m, the natural period of vibration is computed as $T_0 = 1.59$ s. The damping ratio is assumed to be $\zeta = 5\%$.

On the graph of the relevant $D - V - A$ spectrum, for $T_0 = 1.59$ s, we find the value $A = 0.20$ g. The equivalent static force is $f_{S,0} = 2360 \times 0.20 \times 9.81 \text{ m/s}^2 = 4.63$ kN and the peak base bending moment is $M_{b,0} = 4.63 \times 12 \times 0.305 = 16.93$ kNm

**Idealised Response Spectrum, 1**

First step in the construction of an idealised $D - V - A$ response spectrum is to make a tripartite plot with all three ordinate axes normalised with respect to $u_{D,0}$, $u_{V,0}$ and $u_{A,0}$.

**Example of $D - V - A$ spectrum**

Combined $D - V - A$ response spectrum, El Centro 1940 NS record, $\zeta = 0.02$.

**Response Spectrum Characteristics**

For intermediate values of $T_0$, it is apparent that

- $A > u_{A,0}$, $V > u_{V,0}$ and $D > u_{D,0}$;
- $u_{max}$ constant for each value of $\zeta$;
- there is a clear dependency on $\zeta$.

**Example of Use of the Tripartite Plots**

Combined $D - V - A$ response spectrum, El Centro 1940 NS record, for $0 \leq \zeta \leq 5\%$ and full range of periods.

**Idealised Response Spectrum, 2**

Next,

- draw the $(\zeta = 5\%$ spectrum;
- individuate the intervals where $a) A \approx u_{D,0}$, $b) A \approx 2 u_{A,0}$, $c) V \approx 2 u_{V,0}$, $d) D \approx 2 u_{D,0}$, $e) D \approx u_{D,0}$ and
- individuate approximate amplification factors, $\lambda_A$, $\lambda_V$ and $\lambda_D$;
- connect the constant value intervals with straight lines.
**Idealised Response Spectrum**

Our procedure results look good in the log-log graph, but should we represent the same piecewise lineisation in a lin-lin graph it will be apparent that’s a rather crude approximation. This consideration is however not particularly important, because we are not going to use the idealised spectrum in itself, but as a guide to help developing design spectra. Finally, consider that the positions of the points $T_s$, $T_i$, and $T_f$ and the amplifications factors $a_s$, $a_i$, and $a_f$ are not equal for spectra of different earthquakes recorded at different sites, they depend in complex and not fully determined ways on different parameters, for example the focal distance and the focal mechanism and, very important, the local soil characteristics, showing in the whole a large variability.

**Elastic Design Spectra**

A design spectrum is usually specified as an idealized response spectrum, as a set of connected straight lines on the log-log $D - V$ – $A$ plot, and has not, in contrast with a response spectrum, a jagged appearance. Note that straight lines on a log-log graph map on straight or curved lines on conventional $T - n$ – $A$ plots. The requirements of a design spectrum are manifold, but mostly important a design spectrum must be an envelope of possible peak values.

**Derivation of an Elastic Design Spectra**

Riddel and Newmark (1979)

Riddel and Newmark collected a large set of records for similar sites in Southern California, computed the normalised response spectra for $z = 5\%$ and, finally computed the mean value and the standard deviation of the peak response distribution.

In the graph, the summary of their research: the mean and mean+1σ spectra for 5% damping ratio. In the same graph, you can also (dashed) an idealised spectrum representation of the mean spectrum.

**Elastic Design Spectra**

On the right, the $u_{0.9}$ normalised $A$ response spectra for 3 different earthquakes NS records, recorded at the same El Centro site. Clearly, it is not possible to infer the jagged appearance of the 1968 spectra from the 1940’s and 1956’s ones. For design purposes, however, it is not necessary to know in advance and in detail the next quake’s response spectra as it suffices to know some sort of an upper bound on spectral ordinates, that is a Design Spectrum.

**Idealised Elastic Design Spectra**

It is common practice to subdivide the design $D - V$ – $A$ elastic spectrum in 7 segments and use 4 key vibration periods, together with given amplification factors, to draw the required idealised design spectrum.

The key periods $T_s = 0.03\, s$ and $T_a = 0.125\, s$ define the segment where $A$ rises from 1 to $\alpha_A$. The key periods $T_s = 10\, s$ and $T_s = 33\, s$ define the segment where $D$ decreases from $A_0$ to 1. The key periods $T_s$ and $T_a$ instead, follows from applying the given amplification factors to pseudo accelerations, pseudo velocities and deformation.
Example Data

<table>
<thead>
<tr>
<th>ζ (%)</th>
<th>Median (50th percentile)</th>
<th>Median+1σ (84th percentile)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.21 2.31 1.82</td>
<td>4.38 3.38 2.73</td>
</tr>
<tr>
<td>2</td>
<td>2.74 2.03 1.63</td>
<td>3.66 2.92 2.42</td>
</tr>
<tr>
<td>5</td>
<td>2.12 1.65 1.39</td>
<td>2.71 2.30 2.01</td>
</tr>
<tr>
<td>10</td>
<td>1.64 1.37 1.20</td>
<td>1.99 1.84 1.69</td>
</tr>
<tr>
<td>20</td>
<td>1.17 1.08 1.01</td>
<td>1.26 1.37 1.38</td>
</tr>
</tbody>
</table>


Comparison of Design and Response Spectra, 1

In the figure, the response spectrum for the 1940 El Centro NS acceleration record, computed for ζ = 5%, and the corresponding design spectrum, with amplifications corresponding to median values of the ordinates.

The spectrum was constructed from the real value of ω₀ = 0.319 g and estimated values of ω₀ = ζω₀ / (ζ² + 1) = 15.3 inch/s and ω₀ = ζω₀ / (ζ² + 1) = 11.5 inch, estimated values that are significantly higher than the effective values.

There is a good concordance in the acceleration controlled part of the design spectrum, but spectral velocities and deformations are not very good, due to rather poor estimates of the relevant ground motion peak quantities.

Comparison of Design and Response Spectra, 2

In this second slide the design spectra are two, the median and the median + 1σ versions, both based on exact peak values of the ground motion.

While the median spectrum is ok, in a median position with respect to the ordinates of the elastic response spectrum, the presumed envelope spectrum does effectively a good job, maxing out most of the spikes present in the elastic response spectrum.

Differences between Response and Design Spectra

The response spectrum is a description, in terms of its peak effects, of a particular ground motion.

The design spectrum is a specification, valid for a site or a class of sites, of design seismic forces.

If a site falls in two different classifications, e.g., the site is near to a seismic fault associated with low magnitude earthquakes and it is distant from a fault associated with high magnitude earthquakes, with the understanding that the frequency contents of the two classes of events are quite dissimilar the design spectrum should be derived from the superposition of the two design spectra.

Procedure Summary

1. For the site in case, get an estimate of ω₀, ω₁, and ω₀, from an analysis of relevant data or assuming it from literature.
2. In the tripartite log-log graph, draw a line for each of the shaking parameters.
3. For a selected value of ζ, amplify the shaking parameters by an appropriate amplification factor and draw a line for each amplified parameter.
4. Draw vertical lines from the key periods to individuate the connection ramps.
5. Draw the idealised design spectrum.

Earthquake Response of Inelastic Systems

Giacomo Boffi
Dipartimento di Ingegneria Civile e Ambientale, Politecnico di Milano

June 11, 2013
Motivation

If you know the peak ground acceleration associated with the design earthquake, you can derive elastic design spectra and then, from the ordinates of the pseudo-acceleration spectrum, derive equivalent static forces to be used in the member design procedure. However, in the almost totality of cases the structural engineer does not design the anti-seismic structures considering the ordinates of the elastic spectrum of the maximum earthquake, the preferred procedure is to reduce these ordinates by factors that can be as high as 6 or 8. This, of course, leads to a large reduction in the cost of the structure.

A period of return of 500 years means that in a much larger interval, say 50,000 years, you expect say 100 earthquakes that are no smaller (in the sense of some metrics, e.g., the peak ground acceleration) than the design earthquake.

If we design for forces smaller than the forces likely to occur during a large earthquake, our structures will be damaged, or even destroyed.

The reasoning behind such design procedure is that, for the unlikely occurrence of a large earthquake, a large damage in the construction is acceptable as far as no human lives are taken in a complete structural collapse and that, in the mean, the costs for repairing a damaged building are not disproportionate to its value.

What to do?

To ascertain the amount of acceptable reduction of earthquake loads it is necessary to study

- the behaviour of structural members and systems subjected to cyclic loading outside the elastic range, to understand the amount of plastic deformation and cumulated plastic deformation that can be sustained before collapse and
- the global structural behaviour for inelastic response, so that we can relate the reduction in design ordinates to the increase in members’ plastic deformation.

The first part of this agenda pertains to Earthquake Engineering proper, the second part is across EE and Dynamics of Structures, and today’s subject.

Cyclic behaviour

Investigation of the cyclic behaviour of structural members, sub-assemblages and scaled or real sized building model, either in labs or via numerical simulations, constitutes the bulk of EE.

What is important, at the moment, is understanding of how different these behaviours can be, due to different materials or structural configurations, with instability playing an important role.

We will see 3 different diagrams, force vs deformation, for a clamped steel beam subjected to flexure, a reinforced concrete sub-assemblage an a masonry wall.
A more complex behaviour may be represented with an elasto-perfectly plastic (e-p) bilinear idealisation, see figure, where two important requirements are obeyed:

1. the initial stiffness of the idealised e-p system is the same as the real system, so that the natural frequencies of vibration for small deformation are equal,
2. the yielding strength is chosen so that the sum of stored and dissipated energy in the e-p system is the same as the energy stored and dissipated in the real system.

In perfect plasticity, when yielding at unloading, the force $f_y$ is the yielding force, the displacement $x_y$ is the yield deformation.

In the right part of the figure, you can see that at unloading ($dx = 0$) the stiffness is equal to the initial stiffness, and we have $f_y = k(x - x_{pl})$, where $x_{pl}$ is the total plastic deformation.

### Definitions

For a given seismic excitation, we give the following definitions:

- **equivalent system**: a linear system with the same characteristics $(\omega_n, \zeta)$ of the non-linear system.
- **normalised yield strength**, $f_y^*$, is the ratio of the yield strength to the peak force of the equivalent system:
  
  $$ f_y^* = \min \left\{ \frac{f_y}{f_y}, \frac{2\omega_n x_y}{\omega_n^2 x_y} \right\} $$

  It is $f_y^* \leq 1$ because for $f_y \geq f_y$ there is no yielding, and in such case we define $f_y^* = 1$.

- **yield strength reduction factor**, $R_y$, it comes handy to define $R_y$, as the reciprocal of $f_y^*$:
  
  $$ R_y = \frac{1}{f_y^*} = \max \left\{ 1, \frac{f_y}{f_y}, \frac{x_y}{2\omega_n x_y} \right\} $$

- **normalised spring force**, $f_S^*$, is the ratio of the e-p spring force to the yield strength:
  
  $$ f_S^* = \frac{f_S}{R_y} $$

### Normalising the force

For an e-p system, the equation of motion (EOM) is:

$$ mx + cx + f_S(x, x) = -mu(t) $$

with $f_S$ as shown in a previous slide. The EOM must be integrated numerically to determine the time history of the e-p response, $x(t)$.

For a given excitation $x(t)$, the response depends on 3 parameters, $\omega_n = \sqrt{k/m}$, $\zeta = c/(2\omega_n m)$ and $x_y$.

If we divide the EOM by $m$, recalling our definition of the normalised spring force, the last term is:

$$ f_S = \frac{1}{m} \frac{f_S}{f_y^*} = \frac{1}{m} k x_y f_y^* = \omega_n^2 x_y f_y^* $$

and we can write:

$$ \ddot{x} + 2\zeta \omega_n x + \omega_n^2 x_y f_y^*(x, x) = -u(t) $$

### Normalising the displacements

With the position $x(t) = \mu(t) x_y$, substituting in the EOM and dividing all terms by $x_y$, it is:

$$ \ddot{x} + 2\omega_n \zeta \mu + \omega_n^2 \mu = \frac{\omega_n^2 x_y}{\omega_n^2} \frac{x_y}{x_y} \dot{\mu} $$

It is now apparent that the input function for the ductility response is the acceleration ratio: doubling the ground acceleration or halving the yield strength leads exactly the same response $\mu(t)$ and the same peak value $\mu$.

The equivalent acceleration can be expressed in terms of the normalised yield strength $f_y^*$:

$$ \ddot{x}_y = \frac{f_y}{m} \frac{x_y}{x_y} \ddot{f_y} = \frac{f_y}{m} k x_y = \ddot{f_y} \omega_n^2 x_y $$

and recognising that $x_y$ depends only on $\zeta$ and $\omega_n$, we conclude that, for given $x_y(t)$ and $f_y^*(\mu, \mu)$ the ductility response depends only on $\zeta$, $\omega_n$, $f_y^*$. 
Elastic response, required parameters

In the figure above, the elastic response of an undamped, \( T_n = 0.5 \) s system to the NS component of the El Centro 1940 ground motion (all our examples will be based on this input motion).

Top, the deformations, bottom the elastic force normalised with respect to weight, from the latter peak value we know that all e-p systems with \( f_y < 1.37w \) will experience plastic deformations during the EC1940NS ground motion.

Inelastic response, \( \bar{f}_y = 1/8 \)

The force-deformation diagram for the first two excursions in plastic domain, the time points \( a, b, c, d, e, f \) and \( g \) are the same in all 4 graphs:

- until \( t = b \) we have an elastic behaviour,
- until \( t = c \) the velocity is positive and the system accumulates positive plastic deformations,
- until \( t = d \) the force is zero, the deformation is equal to the total plastic deformation,
- until \( t = f \) we have another plastic excursion, cumulating negative plastic deformations
- until at \( t = f \) we have an inversion of the velocity and an elastic reloading.

Ductility demand and capacity

We can say that, for a given value of the normalised yield strength \( \bar{f}_y \) or of the yield strength reduction factor \( R_y \), there is a ductility demand, a measure of the extension of the plastic behaviour that is required when we reduce the strength of the construction.

Corresponding to this ductility demand our structure must be designed so that there is a sufficient ductility capacity. Ductility capacity is, in the first instance, the ability of individual members to sustain the plastic deformation demand without collapsing, the designer must verify that the capacity is greater than the demand for all structural members that go non linear during the seismic excitation.

Response for different \( \bar{f}_y \)'s

<table>
<thead>
<tr>
<th>( \bar{f}_y )</th>
<th>( X_{0n} )</th>
<th>( X_{perm} )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>2.25</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.500</td>
<td>1.62</td>
<td>0.17</td>
<td>1.44</td>
</tr>
<tr>
<td>0.250</td>
<td>1.75</td>
<td>1.10</td>
<td>3.11</td>
</tr>
<tr>
<td>0.125</td>
<td>2.07</td>
<td>1.13</td>
<td>7.36</td>
</tr>
</tbody>
</table>

This table was computed for \( T_n = 0.5 \) s and \( \zeta = 5\% \) for the EC1940NS excitation. Elastic response was computed first, with peak response \( X_0 = 2.25 \) in and peak force \( f_0 = 0.919w \), later the computation was repeated for \( \bar{f}_y = 0.5, 0.25, 0.125 \).

In our example, all the peak values of the e-p responses are smaller than the elastic one, but this is not always true, and shouldn’t be generalised. The permanent displacements increase for decreasing yield strengths, and also this fact shouldn’t be generalised.

Last, the ductility ratios increase for decreasing yield strengths, for our example it is \( \mu \approx R_y \).

Effects of \( T_n \)

For EC1940NS, for \( \zeta = 0.05 \), for different values of \( T_n \) and for \( \bar{f}_y = 1.0, 0.5, 0.25, 0.125 \) the peak response \( X_0 \) of the equivalent system (in black) and the peak responses of the 3 inelastic systems has been computed.

There are two distinct zones: left there is a strong dependency on \( \bar{f}_y \), the peak responses grow with \( R_y \); right the 4 curves intersects with each other and there is no clear dependency on \( \bar{f}_y \).
Effects of \( T_y \)

With the same setup as before, here it is the ratio of the \( x_m \) to \( x_0 \), what is evident is the fact that, for large \( T_y \), this ratio is equal to 1... this is justified because, for large \( T_y \)'s, the mass is essentially at rest, and the deformation, either elastic or elasto-plastic, are equal and opposite to the ground displacement.

Also in the central part, where elastic spectrum ordinates are dominated by the ground velocity, there is a definite tendency for the \( x_m/x_0 \) ratio, that is \( x_m/x_0 \approx 1 \).

Response Spectrum for Yield States

The first step in an anti seismic design is to set an available ductility (based on materials, conception, details). In consequence, we desire to know the yield displacement \( \dot{u}_y \) or the yield force \( f_y \).

\[
f_y = k \dot{u}_y = m \ddot{u}_y
\]

for which the ductility demand imposed by the ground motion is not greater than the available ductility.

Computing \( D_y \)

On the left, for different \( T_y \)'s and \( \mu = 5\% \), the independent variable is in the ordinates, either \( \ddot{x}_m \) (left) or \( R_y \) (right) the strength reduction factor. Dash-dash lines are \( \ddot{x}_m/\dot{u}_y \), dash-dot is \( \ddot{x}_m/\dot{u}_y \).

\( \dot{u}_y \) and \( \ddot{x}_y \) are the peaks of positive and negative displacements of the inelastic system, the maximum of their ratios to \( \dot{u}_y \) is the ductility \( \mu \).

If we look at these graphs using \( \mu \) as the independent variable, it is possible that for a single value of \( \mu \) there are different values on the tick line: in this case, for security reasons, the designer must design for the higher value of \( \ddot{x}_m \).

Effects of \( T_y \)

With the same setup as before, in this graph are reported the values of the ductility factor \( \mu \).

The values of \( \mu \) are almost equal to \( R_y \) for large values of \( T_y \), and in the limit, for \( T_y \to \infty \), there is a strict equality. An even more interesting observation regard the interval \( T_c \leq T_y \leq T_r \), where the values of \( \mu \) oscillate near the value of \( R_y \).

On the other hand, the behaviour is completely different in the acceleration controlled zone, where \( \mu \) grows very fast, and the ductility demand is very high even for low values (0.5) of the yield strength reduction factor.

The results we have discussed are relative to one particular excitation, nevertheless research and experience confirmed that these propositions are true also for different earthquake records, taking into account the differences in the definition of spectral regions.

Example

For EC1940NS, \( z = 5\% \), the yield-strength response spectra for \( \mu = 1.0, 1.5, 2.0, 4.0, 8.0 \).

On the left, a lin-lin plot of the pseudo-acceleration normalized (and adimensionalised) with respect to \( g \), the acceleration of gravity.

On the right, a log-log tripartite plot of the same spectrum. Even a small value of \( \mu \) produces a significant reduction in the required strength.
Energy Dissipation

\[ f(t) \text{mix} \, dx + \int f(t) \, c \, dx + \int f(t) \, f_0(x) \, dx = -\int f(t) \, m \ddot{x} \, dx \]

This is an energy balance, between the input energy \( f \cdot \ddot{x} \) and the sum of the kinetic, damped, elastic and dissipated energy.

In every moment, the elastic energy \( E_E(t) = \frac{\ddot{x}(t)}{2m} \) so the yielded energy is

\[ E_Y = \int f_0(x, \dot{x}) \, dx - \frac{\ddot{x}(t)}{2m} \]

The damped energy can be written as a function of \( t \), as \( dx = \dot{x} \, dt \):

\[ E_D = \int c \, \dot{x}^2(t) \, dt \]

Yielding and Damping

For a system with \( m = 1 \) and

a) \( f = 1 \)

b) \( f = 0.25 \)

the energy contributions during the EC1940NS, \( T_n = 0.5s \) and \( \zeta = 5\% \).

In a), input energy is stored in kinetic-elastic energy during strong motion phases and is subsequently dissipated by damping.

In b), yielding energy is dissipated by means of some structural damage.

Energy Dissipation

Inelastic Design Spectra

Two possible approaches:

1. compute response spectra for constant ductility demand for many consistent records, compute response parameters statistics and derive inelastic design spectra from these statistics, as in the elastic design spectra procedures;
2. directly modify the elastic design spectra to account for the ductility demand values.

The first procedure is similar to what we have previously seen, so we will concentrate on the second procedure, that is much more used in practice.

\[ \bar{T}_y = f_y(\mu, T_n, \zeta) \]

We have seen that \( \bar{T}_y = f_y(\mu, T_n, \zeta) \) is a monotonically increasing function of \( \mu \) for fixed \( T_n \) and \( \zeta \).

Left the same spectra of the previous slides, plotted in a different format: \( \bar{T}_y \) vs \( T_n \) for different values of \( \mu \).

The implication of this figure is that an anti seismic design can be based on strength, ductility or a combination of the two.

For \( T_n = 1.0 \), the peak force for EC1940NS in an elastic sistem is \( f_0 = 0.919w \), so it is possible to design for \( \mu = 1.0 \), hence \( f_y = 0.919w \) or for an high value of ductility, \( \mu = 8.0 \), hence \( f_y = 0.120 \cdot 0.919w \) or, if such an high value of ductility cannot be easily reached, design for \( \mu = 4.0 \) and a yielding force of 0.195 times \( f_0 \).

Based on observations and energetic considerations, the plots of \( R_Y \) vs \( T_n \) for different \( \mu \) values can be approximated with straight lines in a log-log diagram, where the constant pieces are defined in terms of the key periods in \( D = V - A \) graphs.

\[ R_Y = \begin{cases} 1 & T_n < T_1 \\mu < T_n \\mu > T_n \end{cases} \]

The key period \( T_n \) is different from \( T_1 \), as we will see in the next slide; the constant pieces are joined with straight lines in the log-log diagram.
Construction of Design Spectrum

Start from a given elastic design spectrum, defined by the points a-b-c-d-e-f. Choose a value $\mu$ for the ductility demand.
Reduce all ordinates right of $T_0$ by the factor $\mu$, reduce the ordinates in the interval $T_0 < T < T_0$ by $\sqrt{2\mu - 1}$.
Draw the two lines $A = \frac{\mu_y}{\mu_T}$ and $A = \frac{\mu_y}{\mu_T}$, their intersection define the key point $T_e$.
Connect the point $(T_a, A = \bar{x}_0)$ and the point $(T_b, A = \frac{\mu_y}{\mu_T})$ with a straight line.
As we already know (at least in principle) the procedure to compute the elastic design spectra for a given site from the peak values of the ground motion, using this simple procedure it is possible to derive the inelastic design spectra for any ductility demand level.

Important Relationships

For different zones on the $T_n$ axis, the simple relationships we have previously defined can be made explicit using the equations that define $R_y$, in particular we want relate $u_m$ to $u_0$ and $f_y$ to $f_0$ for the elastoplastic system and the equivalent system.

1. region $T_n < T_a$, here it is $R_y = 1.0$ and consequently
   
   $u_m = \mu u_0$  
   $f_y = f_0$

2. region $T_b < T < T_c$, here it is $R_y = \sqrt{2\mu - 1}$ and
   
   $u_m = \mu \sqrt{2\mu - 1} u_0$  
   $f_y = f_0 / \sqrt{2\mu - 1}$

3. region $T < T_c$, here it is $R_y = \mu$ and
   
   $u_m = u_0$  
   $f_y = f_0 / \mu$

Similar equations can be established also for the inclined connection segments in the $R_y$ vs $T_n$ diagram.

Application: design of a SDOF system

- Decide the available ductility level $\mu$ (type of structure, materials, details etc).
- Preliminary design, $m$, $k$, $\xi$, $\omega_0$, $T_n$.
- From an inelastic design spectrum, for known values of $\xi$, $T_0$ and $\mu$ read $A_y$.
- The design yield strength is
  
  $f_y = m A_y$.
- The design peak deformation, $u_m = \mu D_y / R_y$, is
  
  $u_m = \frac{\mu A_y}{R_y(\mu, T_n) u_0^2}$.

Example

One storey frame, weight $w$, period $T_n = 0.25$ s, damping ratio is $\xi = 5\%$, peak ground acceleration is $x_0 = 0.5$ g. Find design forces for
1) system remains elastic, 2) $\mu = 4$ and 3) $\mu = 8$.
In the figure, a reference elastic spectrum for $x_0 = 1$ g, $A_y(0.25) = 2.71$ g; for $x_0 = 0.5$ g it is $f_0 = 1.355 w$.

For $T_n = 0.25$ s, $R_y = \sqrt{2\mu - 1}$, hence

\[
\frac{f_y}{w} = 1.355 w \sqrt{2\mu - 1}, \quad u_m = \mu \frac{A_y}{R_y(\mu, T_n) u_0^2} = \mu \frac{1.355 g T_n^2}{4 \pi^2}.
\]

- $\mu = 1$: $f_y = 1.355 w$, $u_m = 2.104$ cm,
- $\mu = 4$: $f_y = 0.512 w$, $u_m = 3.182$ cm,
- $\mu = 8$: $f_y = 0.350 w$, $u_m = 4.347$ cm.