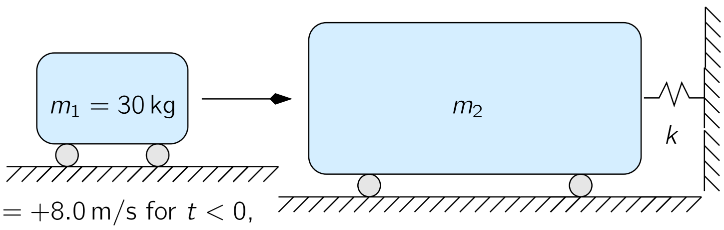
Dynamics of Structures 2012-2013

1st home assignment, exercises'solutions

Contents

1 Impact (elastic rebound)



$v_1 = +8.0\,\text{m/s}$ for $t < 0$,
$v_1 = -4.8\,\text{m/s}$ for $t > 0$.

A free body, mass $m_1 = 30\,\text{kg}$ and velocity $v = 8\,\text{m}\,\text{s}^{-1}$, impacts an undamped SDOF system, mass m_2 and stiffness k.

The impact is, hypothetically, a perfect elastic impact, meaning that *also* the energy is conserved during the impact and that the duration of the contact is infinitesimal.

After the impact the free body has a negative velocity, $v = -4.8\,\text{m}$ and the amplitude of the harmonic motion of the SDOF is $x_{2,\text{max}} = 32\,\text{mm}$.

Determine the mass and the stiffness of the SDOF.

Solution

In the text, it is stated that the energy is conserved and that the duration of contact is negligible, so we can say that the kinetic energy is conserved across the moment of the impact,

$$\frac{1}{2}\,30\,\text{kg}\,(+8.0\,\text{m}\,\text{s}^{-1})^2 = \frac{1}{2}\,30\,\text{kg}\,(-4.8\,\text{m}\,\text{s}^{-1})^2 + \frac{1}{2}\,m_2\,v_2^2 \quad \Rightarrow$$
$$\Rightarrow \quad m_2\,v_2^2 = 1228.8\,\text{kg}\,\text{m}^2\,\text{s}^{-2}.$$

Of course, the momentum is also conserved,

$$30\,\text{kg}\,(+8.0\,\text{m}\,\text{s}^{-1}) = 30\,\text{kg}\,(-4.8\,\text{m}\,\text{s}^{-1}) + m_2\,v_2 \quad \Rightarrow$$
$$\Rightarrow \quad m_2\,v_2 = 384.0\,\text{kg}\,\text{m}\,\text{s}^{-1},$$

and substituting in the previous equation you have

$$v_2 = \frac{1228.8}{384.0}\text{m}\,\text{s}^{-1} = 3.2\,\text{m}\,\text{s}^{-1}.$$

From $m_2\,v_2 = 384.0\,\text{kg}\,\text{m}\,\text{s}^{-1}$ you have

$$m_2 = \frac{384.0}{3.2}\text{kg} = 120\,\text{kg}.$$

The stiffness can be derived observing that the initial conditions for the SDOF are

$$x(0) = 0, \qquad \dot{x}(0) = v_2,$$

hence it is

$$x(t) = \frac{v_2}{\omega}\,\sin\omega t.$$

Knowing that the maximum displacement amounts to 32 mm, you have

$$\frac{3200\,\text{mm}\,\text{s}^{-1}}{\omega} = 32\,\text{mm}, \quad \Rightarrow \quad \omega = 100\,\text{rad}\,\text{s}^{-1}$$

and

$$k = m_2\,\omega^2 = 1\,200\,000\,\text{N}\,\text{m}^{-1} = 1.2\,\text{kN/millim}$$

2 Dynamical Testing

You want to determine the mass m, the stiffness k and the damping ratio ζ of a small, one storey building that can be modeled as a single degree of freedom system.

A series of 4 dynamical test is performed, loading the building with a vibrodyne and measuring the amplitude ρ and the phase difference θ of the steady state motion (note that the number of measurements taken is greater than what is strictly requested, as it is recognized that there are sources of uncertainties in the experimental setup).

In each test the load amplitude is $p_0 = 1600\,\mathrm{N}$, while the excitation frequencies ω_n (with $n = 1, \ldots, 4$) are different.

The measurements are summarized in the following table

n	$f_n/1\mathrm{Hz}$	$\rho_n/1\mathrm{\mu m}$	$\theta_n/1°$
1	6.0	44.29	12.0
2	7.0	93.44	30.7
3	8.0	119.25	131.6
4	9.0	42.54	162.4

Give your best estimate of m, ζ and k.

2.1 Solution

From the steady-state amplitude for a harmonic excitation,

$$\rho = \frac{p_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

collecting $(1-\beta^2)^2$ in the radicand, taking it outside the square rooot and rearranging it is

$$\rho = \frac{p_0}{k}\frac{1}{1-\beta^2}\frac{1}{\sqrt{1 + \left(2\zeta\beta/1-\beta^2\right)^2}}.$$

Squaring the expression of the phase angle

$$\tan\theta = \frac{2\zeta\beta}{1-\beta^2}$$

and expressing the square tangent in terms of the sole cosine, it is

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} - 1 = \left(\frac{2\zeta\beta}{1-\beta^2}\right)^2 \quad\Rightarrow\quad \cos^2\theta = \frac{1}{1 + \left(\frac{2\zeta\beta}{1-\beta^2}\right)^2}.$$

Comparing with the last term in the second equation of this solution, you can write

$$\rho = \frac{p_0}{k}\frac{\cos\theta}{1-\beta^2} \quad\Rightarrow\quad k(1-\beta^2) = k + \omega^2\frac{k}{\omega_\mathrm{n}^2} = k - \omega^2 m = \frac{p_0\ \cos\theta}{\rho}$$

For each one of the 4 dynamic tests you can now write

$$k - \omega_i^2 m = \frac{p_0\ \cos\theta_i}{\rho_i}, \qquad i = 1, 2, 3, 4$$

so you can write 4 linear equations for the two unknowns k and m.

When you have a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with more equations than unknowns, it is usually solved under the hypotesis that the *best solution* is the solution that minimises the sum of the squares of the residuals $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$.

A detailed discussion of the least squares minimization procedure is to be found in an any algebra or statistic textbook. For what it's worth, I gave a summary of it in http://stru.polimi.it/people/boffi/dati_2011/ha01/solutions.pdf

That said, substituting the quantities measured during the tests in the previous equations, you have

$$\begin{bmatrix} 1 & -1421.22 \\ 1 & -1934.44 \\ 1 & -2526.62 \\ 1 & -3197.75 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = \begin{Bmatrix} +35336106.6 \\ +14723497.8 \\ -08908024.7 \\ -35851082.9 \end{Bmatrix}$$

and the equation that gives the *best solution* in terms of least squares is

$$\begin{bmatrix} 4.0000 & -9080.0360 \\ -9080.0360 & 22371361.4832 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = \begin{Bmatrix} 5300496.7973 \\ 58447799899.0092 \end{Bmatrix},$$

from which you eventually have

$$m = 40\,054.5\,\mathrm{kg}, \qquad k = 92.25\,\mathrm{MN\,m^{-1}}.$$

We can derive different expressions for $\cos\theta$, from the second eq. of this solution,

$$\cos\theta = (1-\beta^2)\frac{\rho\,k}{p_0}$$

and from the eq. of the phase

$$\cos\theta = (1-\beta^2)\frac{\sin\theta}{2\zeta\beta}.$$

Equating the two right members and solving for ζ gives you

$$\zeta = \frac{p_0\sin\theta}{\rho}\,\frac{1}{2\beta\,k} = \frac{p_0\sin\theta}{\rho}\,\frac{1}{c_{\mathrm{cr}}\omega}$$

and hence

$$c = \frac{p_0\sin\theta_i}{\rho_i\,\omega_i}.$$

Applying the least squares procedure,

$$4c = p_0\sum\frac{\sin\theta_i}{\rho_i\,\omega_i} = (199233 + 198765 + 199607 + 201112)\mathrm{N\,s\,m^{-1}}$$

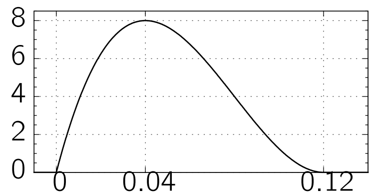
or

$$c = 199\,679.6\,\mathrm{N\,s\,m^{-1}}.$$

3 Numerical Integration

A single degree of freedom system, with a mass $m = 3\,\text{kg}$, a stiffness $k = 1200\,\text{N}\,\text{m}^{-1}$ and a damping ratio $\zeta = 0.10$ is at rest when it is subjected to an external force $p(t)$:

$$p(t) = \begin{cases} p_0 \dfrac{(t_0 - t)^2 t}{t_0^3} & \text{for } 0.0 \le t \le t_0, \\ 0.0 & \text{otherwise.} \end{cases}$$



where $t_0 = 0.12\,\text{s}$ and $p_0 = 54\,\text{N}$.

(1) Give the analytical solution of the equation of motion in the interval $0 \le t \le 10t_0$. (2) Integrate the equation of motion numerically, using the algorithm of constant acceleration with a time step $h = t_0/30$ in the same interval. (3) Plot your results (both the exact response and the numerical solution) in a meaningful manner. (4) [Optional] Repeat the numerical integration assuming an elasto-plastic spring with a yield strength $f_y = 6\,\text{N}$ and plot your results.

Solution

We start by determining all the characteristics of the SDOF system that weren't given in the text,

$$\begin{aligned} \omega_\text{n} &= \sqrt{\frac{k}{m}} = 20\,\text{rad}\,\text{s}^{-1}, \\ \omega_\text{D} &= \omega_\text{n}\sqrt{1-\zeta^2} = 19.90\,\text{rad}\,\text{s}^{-1}, \\ c &= 2\,\zeta\,m\,\omega_\text{n} = 12\,\text{N}/(\text{m/s}). \end{aligned}$$

Analytical solution

The particular integral for the given loading can be expressed as a polynomial of third degree,

$$\xi(t) = \frac{P_0 t^3 + P_1 t_0 t^2 + P_2 t_0^2 t + P_3 t_0^3}{t_0^3},$$

substituting in the equation of motion and multiplying by t_0^3 both members,

$$\begin{aligned} & P_0 k t^3 + (t_0 P_1 k + 3P_0 c)t^2 + \\ & \quad + (P_2 t_0^2 k + 2t_0 P_1 c + 6P_0 m)t + (P_3 t_0^3 k + P_2 t_0^2 c + 2t_0 P_1 m) = \\ & \qquad = p_0\,(t^3 - 2t_0 t^2 + t_0^2 t). \end{aligned}$$

Equating the coefficients of each power of t,

$$\begin{aligned}
P_0 &= \frac{p_0}{k} = \frac{9}{200}\mathrm{m} = 0.045\,\mathrm{m},\\
P_1 &= \frac{-3P_0 c - 2t_0 p_0}{t_0 k} = -\frac{81}{800}\mathrm{m} = -0.101\,250\,\mathrm{m},\\
P_2 &= \frac{-6P_0 m - 2t_0 P_1 c + t_0^2 p_0}{t_0^2 k} = \frac{3}{200}\mathrm{m} = 0.015\,\mathrm{m},\\
P_3 &= \frac{-2t_0 P_1 m - t_0^2 P_2 c}{t_0^3 k} = \frac{217}{6400}\mathrm{m} = 0.033\,906\,250\,\mathrm{m}.
\end{aligned}$$

Hence. with the position $\upsilon = t/t_0$, it is

$$\begin{aligned}
\frac{\xi(t)}{1\,\mathrm{m}} &= \frac{9}{200}\upsilon^3 - \frac{81}{800}\upsilon^2 + \frac{3}{200}\upsilon + \frac{217}{6400},\\
\frac{\dot\xi(t)}{1\,\mathrm{m\,s^{-1}}} &= \left(\frac{27}{200}\upsilon^2 - \frac{81}{400}\upsilon + \frac{3}{200}\right)\frac{1}{t_0}
\end{aligned}$$

and we can write

$$\frac{x(t)}{1\,\mathrm{m}} = -\frac{0.033\,906\,250\,\cos\omega_\mathrm{D}t + 0.009\,689\,193\,\sin\omega_\mathrm{D}t}{\exp\zeta\omega_\mathrm{n}t} + \frac{\xi(t)}{1\,\mathrm{m}}.$$

The state at the end of the excitation is

$$x(t_0) = 0.006\,890\,095\,\mathrm{m}, \qquad \dot x(t_0) = 0.007\,823\,390\,\mathrm{m\,s^{-1}}.$$

and using $\tau = t - t_0$ as the new time coordinate it is

$$\frac{x(t)}{1\,\mathrm{m}} = \frac{0.006\,890\,095\,\cos\omega_\mathrm{D}\tau + 0.001\,085\,621\,\sin\omega_\mathrm{D}\tau}{\exp\zeta\omega_\mathrm{n}\tau}.$$

Numerical Integration

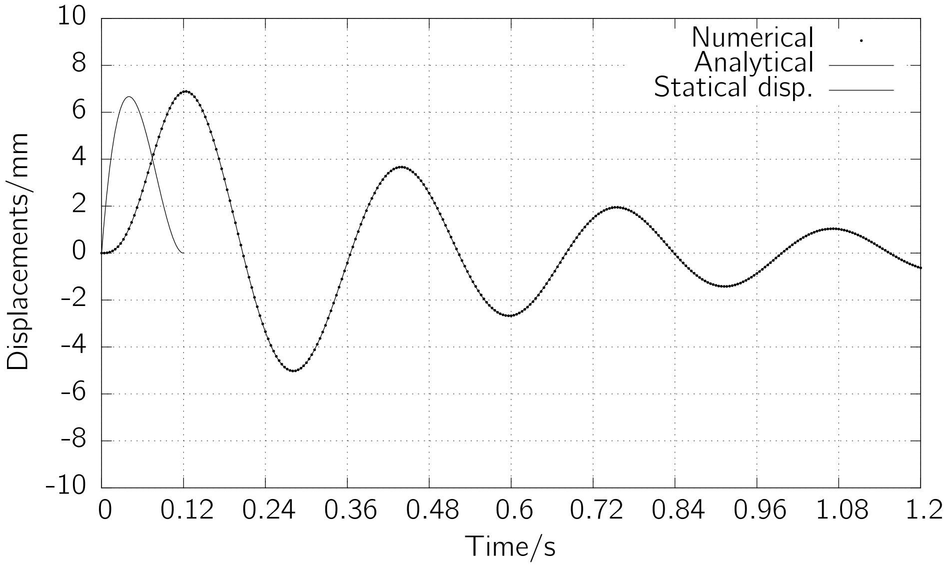
Here it is the small `Python` program that computed the response,

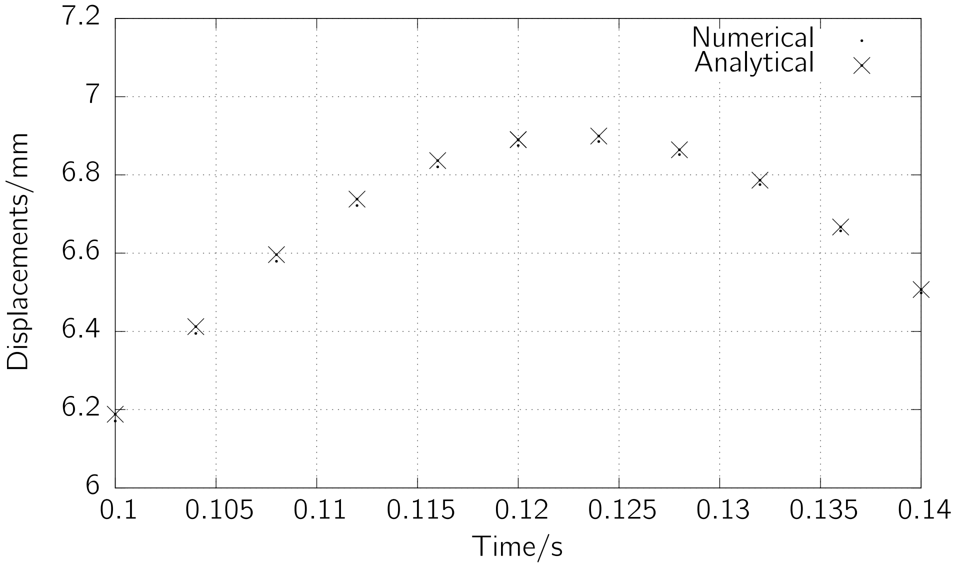
```
from math import sqrt

def p(t):
    "returns the applied load"
    return 0 if t>t0 else p0*(t0-t)**2*t/t0**3

m = 3.0                        # mass, kg
k = 1200.0                     # stiffness, N/m
z = 0.10                       # damping ratio
p0 = 54.0                      # loading factor, N
t0 = 0.12                      # load duration, s
wn = sqrt(k/m)                 # natural freq., rad/s
c = 2*wn*z*m                   # damping, N/(m/s)
h = t0/30.                     # time step, s
dur = t0*10.0+h/2.             # duration, s
K = k + 2*c/h + 4*m/h/h        # modified stiff., N/m
V = 2*c + 4*m/h                # damping correction, N/(m/s)
A = 2*m                        # inertial correction, kg

T0 = 0.0 ; X0 = 0.0 ; V0 = 0.0 ; P0 = p(T0)
while T0<dur:
    A0 = (P0-V0*c-X0*k)/m
```

```
print T0, X0
T1 = T0+h
P1 = p(T1)
dP = P1-P0 + V*V0 + A*A0
dX = dP/K
X1 = X0+dX
dV = 2*dX/h -2*V0
V1 = V0+dV
X0 = X1 ; V0 = V1 ; T0 = T1 ; P0 = P1
```

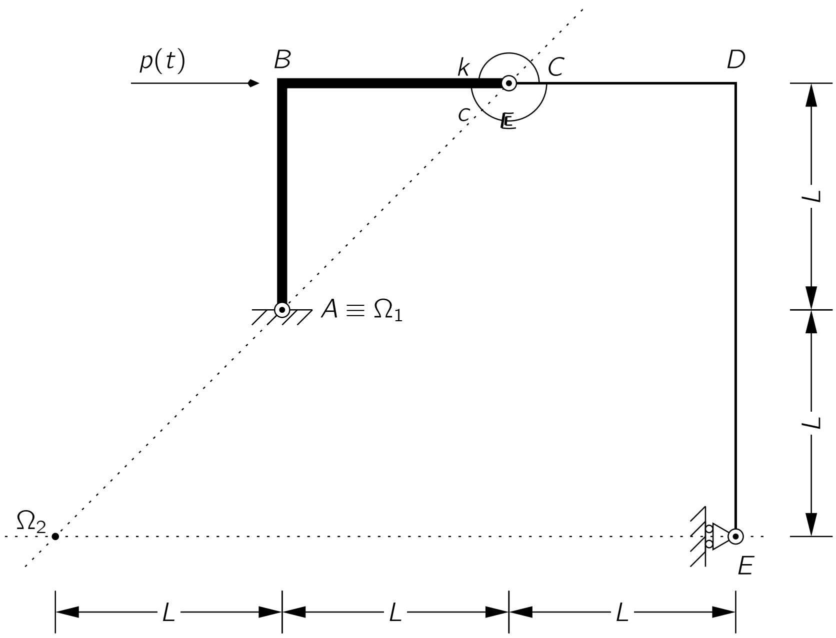
Plots

In the following, a plot of the response in the interval $0 \leq t \leq 10t_0$ and a zoom of the same plot in a neighborhood of the peak value of the response





4 Generalized Coordinates (rigid bodies)



The articulated system in figure, composed by

- two rigid bars, (1) ABC and (2) CDE,
- three kinematical constraints, (1) a hinge in A, (2) an internal hinge in C and (3) a vertical roller in E,
- two deformable constraints, (1) a rotational spring[1] in C, its stiffness k, (2) a rotational dashpot[2] in E, its damping coefficient c,

is excited by a time varying horizontal force applied in D, $p(t)$

Considering that the bar ABC has a constant unit mass $\overline{m}$, with $\overline{m}\,L = m$ while all the other parts of the system are massless, and using v_E (the vertical displacement of the roller initially in E) as the generalized coordinate

1. compute the generalized parameters m^*, c^* and k^*,
2. compute the generalized loading $p^*(t)$ and
3. write the equation of dynamic equilibrium.

[1]A rotational spring gives, on each connected beam, a couple proportional and opposite to the *relative* rotation between the beams. Note that indicating with ϕ_{AB} the rotation of beam A relative to beam B, it is $\phi_{AB} = -\phi_{BA}$.

[2]Same as a rotational spring, but the couple is proportional to the time derivative of the relative rotation.

Solution

Rotations Assuming an upwards $v \equiv v_E$, the rotation of the second rod is $\theta_2 = v/3L$ while, by congruence in C, the rotation of the first rod is $\theta_1 = 2\theta_2 = 3v/3L$.

Relative Rotation The relative rotation between the second and the first rod is $\theta_{21} = \theta_2 - \theta_1 = -v/3L$ and the variation of the relative rotation is $\delta\theta_{21} = -\delta v/3L$.

Spring and Damper Moments The spring moment is $W_{2,\mathrm{S}} = -k\theta_{21} = kv/3L$ and by the same reasoning the damping moment is $W_{2,\mathrm{D}} = c\dot{v}/3L$.

Accelerations The rotational acceleration of the first rod is $\ddot{\theta}_1 = 2\ddot{v}/3L$, the acceleration of the centre of mass G_V of the vertical part of the first rod has components $\ddot{u}_{G_V} = -\ddot{v}/3$, $\ddot{v}_{G_V} = 0$ while G_H the centre of mass of the horizontal part has acceleration components $\ddot{u}_{G_H} = -2\ddot{v}/3$, $\ddot{v}_{G_H} = \ddot{v}/3$. The

Inertial Forces Given that each part has mass m and ratational inertia $mL^2/12$

Virtual Works Collecting the virtual work of the inertial forces, of the inertial moments, of the damping moments and of the spring moments, collecting the virtual displacements, it is

$$\left[-\left(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}\right)m\ddot{v}-\left(\frac{4}{9L^2}+\frac{4}{9L^2}\right)\frac{mL^2}{12}\ddot{v}-\frac{1}{9L^2}c\dot{v}-\frac{1}{9L^2}kv-\frac{2}{3}P(t)\right]\delta v=0$$

Simplifying, negating and taking the external force on the other side

$$\frac{20}{27}m\ddot{v}+\frac{1}{9}\frac{c}{L^2}\dot{v}+\frac{1}{9}\frac{k}{L^2}=-\frac{2}{3}P(t).$$

5 Rayleigh Quotient

A straight, uniform beam of length L is clamped at $x = 0$ and is simply supported at $x = L$. It is possible to approximate its free vibrations using a shape function $\psi(x)$,

$$v(x,t) = Z_0\,\psi(x)\,\sin(\omega t), \qquad 0 \le x \le L.$$

Using a polynomial shape function $\psi(x) = \sum_{i=0}^{n} a_i (x/L)^i$ and imposing the kinematical conditions at $x = 0$ ($\psi(0) = 0$, $\psi'(0) = 0$), you'll find that the constant term and the linear term in the polynomial must be equal to zero.

1. Using a shape function $\psi(x) = \xi^2 + a\xi^3$, $\xi = {}^{x}\!/_{L}$, such that $\psi(L) = 0$ estimate, by the means of Rayleigh quotient, ω^2.

2. Using a shape function $\psi(x) = \xi^2 + a\xi^3 + b\xi^4$ such that $\psi(L) = 0$ and $M(L) = 0$, compute a new estimate of ω^2.
3. [Optional] Using a shape function $\psi(x) = \xi^2 + a\xi^3 + b\xi^4 + c\xi^5$ you can obey all the constraints and still have at your disposal a free parameter. Find the minimum value of the Rayleigh quotient computed as a function of the free parameter.

Solution

Cubic Shape Function Using $\psi = \xi^2 - \xi^3$, $0 \le \xi \le 1$ as our shape function, we respect all the kinematical boundary conditions, in particular it is $\psi(1) = 0$.

The Rayleigh quotient formula gives

$$\omega^2 = \frac{EJ}{m} \frac{\int_0^L \psi''^2(x)\,\mathrm{d}x}{\int_0^L \psi^2(x)\,\mathrm{d}x}$$

and using $\psi'' = \frac{2-6\xi}{L^2}$ we'll find

$$\omega^2 = \frac{\frac{4EJ}{L^3}}{\frac{mL}{105}} = 420\frac{EJ}{L^4 m}$$

Given that a very good approximation to the first eigenvalue of a uniform, clamped-simply supported beam is

$$\omega_1^2 \approx \frac{EJ}{mL^4}\left(\frac{5}{4}\pi\right)^4 \approx 237.8\frac{EJ}{mL^4}$$

our first trial, with a cubic shape function, is not very satisfactory.

Quartic Shape Function We are going to determine a quartic polynomial

$$\psi = \xi^2 + a_3\xi^3 + a_4\xi^4, \qquad L^2\psi'' = 2 + 6a_3\xi + 12a_4\xi^2,$$

such that $\psi(1) = 0$ and $\psi''(1) = 0$. Substituting and simplifying, we have

$$1 + 1a_3 + 1a_4 = 0,$$
$$1 + 3a_3 + 6a_4 = 0.$$

Solving

$$a_3 = -\frac{5}{3}, \qquad a_4 = +\frac{2}{3},$$

substituting in ψ and ψ''

$$\psi = \xi^2 - \frac{5}{3}\xi^3 + \frac{2}{3}\xi^4, \qquad L^2\psi'' = 2 - 10\xi + 8\xi^2.$$

Computing the integrals and taking the Rayleigh quotient, gives

$$\begin{aligned} EJ\int_0^L \psi''^2(x)\,\mathrm{d}x &= \frac{4EJ}{5L^3}, \\ m\int_0^L \psi^2(x)\,\mathrm{d}x &= mL\frac{19}{5670}, \\ \omega^2 &= \frac{4536}{19}\frac{EJ}{mL^4} \approx 238.737\frac{EJ}{mL^4}. \end{aligned}$$

Comparing this result with the (almost) exact soltion, it is apparent that a shape function that respects *also* the mechanical boundary conditions can give very good results.

Quintic Shape Function We write our shape function as

$$\psi = \xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5,$$

we evaluate the function and its second derivative at $x = L$ and by imposing that they are equal to zero we have

$$a_3 = (4a_5 - 5)/3, \qquad a_4 = (2 - 7a_5)/3$$

and substituting we have

$$\begin{aligned} \psi &= \xi^2 + \frac{4a_5 - 5}{3}\xi^3 + \frac{2 - 7a_5}{3}\xi^4 + a_5\xi^5, \\ L^2\psi'' &= 2 + (8a_5 - 10)\xi + (8 - 28a_5)\xi^2 + 20a_5\xi^3. \end{aligned}$$

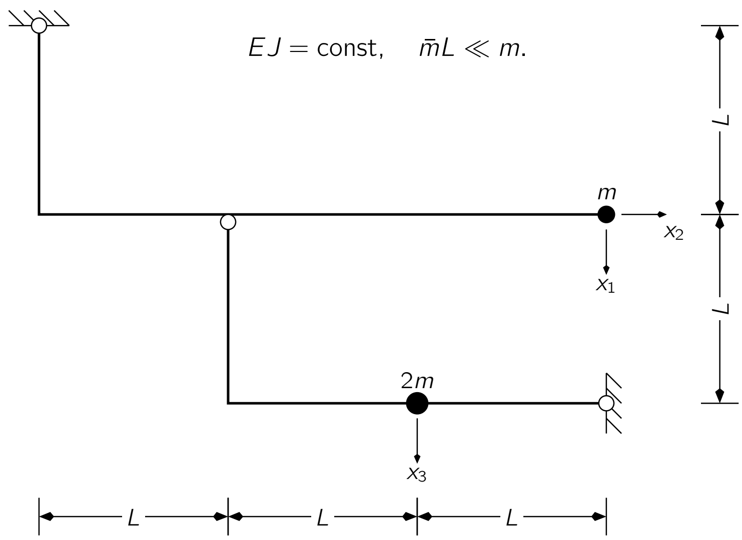
Computing the integrals and taking the Rayleigh quotient, gives

$$\begin{aligned} EJ\int_0^L \psi''^2(x)\,\mathrm{d}x &= (16a_5^2 + 28a_5 + 21)\frac{4EJ}{105L^3} \\ m\int_0^L \psi^2(x)\,\mathrm{d}x &= (104a_4^2 + 28a_5 + 209)\frac{mL}{62370}, \\ \omega^2 &= 2376\frac{16a_5^2 + 28a_5 + 21}{104a_4^2 + 28a_5 + 209}\frac{EJ}{mL^4}. \end{aligned}$$

Plotting $\omega^2(a_5)$ and using successive zooms, it is possible to say that the best estimate of ω^2 is found for $a_5 = 0.0634883$ and $\omega^2 = 238.4820EJ/(mL^4)$.

The improvement is not negligible but, all in all, it doesn't seem worth the extra effort needed for its computation.

6 3 DOF System



A three hinged arch supports two different bodies of negligible dimensions, whose total mass is much greater than the mass of the structure. Axial and shear deformations can be neglected.

1. Discuss the choice of the dynamical degrees of freedom given in figure.

With the positions $\omega_0^2 = {}^k\!/_m$ and $k = {}^{EJ}\!/_{L^3}$,

2. compute the three eigenvalues of the system and the corresponding eigenvectors, normalizing the eigenvectors with respect to the mass matrix $\boldsymbol{M}$ ($\boldsymbol{\Psi}$ being the eigenvectors' matrix, it must be $\boldsymbol{\Psi}^T \boldsymbol{M} \boldsymbol{\Psi} = m\boldsymbol{I}$).

Considering that the system is at rest when $t = 0$ and is then loaded by

$$\boldsymbol{p}(t) = \frac{kL}{2000} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \sin(\frac{\omega_0}{6} t),$$

(3) write the three *modal* equations of motion, (4) integrate the modal equations of motion and write the three equations of modal displacement, $q_i = q_i(t)$ (you should be able to write your results in terms of the unit length L), (5) find the analytical expression of $u_3 = u_3(t)$, showing your intermediate results and (6) plot u_3 in the interval $0 \le \omega_0\, t \le 10$.

Hint

$$\boldsymbol{F} = \boldsymbol{K}^{-1} = \frac{L^3}{EJ}\frac{1}{6}\begin{bmatrix} 408 & -98 & 53 \\ -98 & \cdots & -13 \\ 53 & -13 & \cdots \end{bmatrix}, \qquad \boldsymbol{K} = \frac{EJ}{L^3}\frac{3}{200}\begin{bmatrix} 11 & 19 & -42 \\ 19 & \cdots & 22 \\ -42 & 22 & \cdots \end{bmatrix}.$$

Solution

The structural matrices, with $k = {}^{EJ}/_{L^3}$, are

$$\boldsymbol{M} = m\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}, \quad \boldsymbol{F} = \frac{1}{6k}\begin{bmatrix}408 & -98 & 53\\ -98 & 28 & -13\\ 53 & -13 & 8\end{bmatrix}, \quad \boldsymbol{K} = k\frac{3}{200}\begin{bmatrix}11 & 19 & -42\\ 19 & 91 & 22\\ -42 & 22 & 364\end{bmatrix}.$$

The equation of free vibrations can be written, using the position $\omega^2 = \omega_0^1\Lambda^2$

$$\left(\frac{3}{200}\begin{bmatrix}11 & 19 & -42\\ 19 & 91 & 22\\ -42 & 22 & 364\end{bmatrix} - \Lambda^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\right)\boldsymbol{\psi} = \boldsymbol{0}$$

and has non trivial solutions when the determinant of the coefficient matrix is equal to zero. Expanding the determinant and simplifying gives

$$500\Lambda^6 - 2130\Lambda^4 + 2034\Lambda^2 - 27 = 0,$$

whose roots are

$$\Lambda_1^2 = 0.013463559176, \quad \Lambda_2^2 = 1.41797294149, \quad \Lambda_3^2 = 2.82856349934.$$

The associated eigenvectors can be collected in the eigenvector matrix,

$$\boldsymbol{\Psi} = \begin{bmatrix}+0.95646241 & +0.25012888 & -0.15038354\\ -0.23221417 & +0.96433364 & +0.12703235\\ +0.12501249 & -0.06122164 & +0.69327036\end{bmatrix}.$$

The eigenvector are normalized, such that the modal mass and the modal stiffness are

$$M_i = m, \quad K_i = M_i\omega_i^2 = m\Lambda_i^2\omega_0^2.$$

Given the fixed load vector shape,

$$p_i(t) = \boldsymbol{\psi}_i^T\begin{Bmatrix}0\\ 1\\ 0\end{Bmatrix}\ldots = \psi_{2i}\frac{kL}{2000}\sin(\frac{1}{6}\omega_0 t) = \psi_{2i}k\delta\sin(\Lambda_0\omega_0 t),$$

where $\delta = {}^{L}/_{2000}$ and $\Lambda_0 = {}^{1}/_{6}$.

The generic modal equation of motion, after division of all then terms by m, is

$$\ddot{q}_i + \omega_0^2\Lambda_i^2 q_i = \psi_{2i}\omega_0^2\delta\ \sin(\Lambda_0\omega_0 t)$$

and, substituting the values of Λ's and ψ's, it is

$$\ddot{q}_1 + 0.013464\ \omega_0^2 q_1 = -0.232214\ \omega_0^2\delta\sin(\frac{1}{6}\omega_0 t),$$

$$\ddot{q}_2 + 1.417973\ \omega_0^2 q_2 = +0.964334\ \omega_0^2\delta\sin(\frac{1}{6}\omega_0 t),$$

$$\ddot{q}_3 + 2.828563\ \omega_0^2 q_3 = +0.127032\ \omega_0^2\delta\sin(\frac{1}{6}\omega_0 t).$$

A particular integral is

$$\xi_i = C_i \sin(\Lambda_0\omega_0 t)$$

and substituting in the equation of motion it is

$$(\Lambda_i^2 - \Lambda_0^2)\omega_0^2 C_i \sin(\Lambda_0\omega_0 t) = \psi_{2i}\omega_0^2\delta \sin(\Lambda_0\omega_0 t) \quad \Rightarrow \quad C_i = \frac{\psi_{2i}}{\Lambda_i^2 - \Lambda_0^2}\,\delta.$$

The system starting from rest condition, it is

$$q_i(0) = 0, \quad \dot{q}_i(0) = 0$$

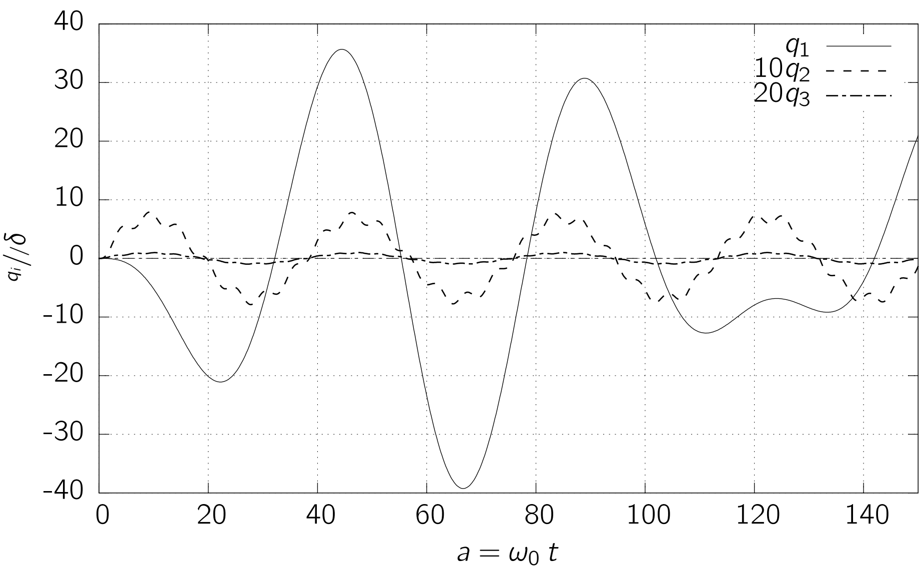
and the modal response functions can be written

$$q_i(t) = \frac{\psi_{2i}}{\Lambda_i^2 - \Lambda_0^2}\,\delta \left(\sin(\Lambda_0\omega_0 t) - \frac{\Lambda_0}{\Lambda_i}\sin(\Lambda_i\omega_0 t)\right)$$

or, substituting the numerical values and using an adimensional time $a = \omega_0 t$,

$$\begin{aligned}
\frac{q_1}{\delta} &= 16.222623 \sin(\frac{1}{6}\,a) - 23.301822 \sin(0.116033\,a),\\
\frac{q_2}{\delta} &= 0.693668 \sin(\frac{1}{6}\,a) - 0.097088 \sin(1.190787\,a),\\
\frac{q_3}{\delta} &= 0.045356 \sin(\frac{1}{6}\,a) - 0.004495 \sin(1.681833\,a).
\end{aligned}$$

Here it is a plot of the modal responses, note that I extended a bit (15 times!) the time range for the plot. Note also that q_2 and q_3 were scaled to make them visible on the plot!



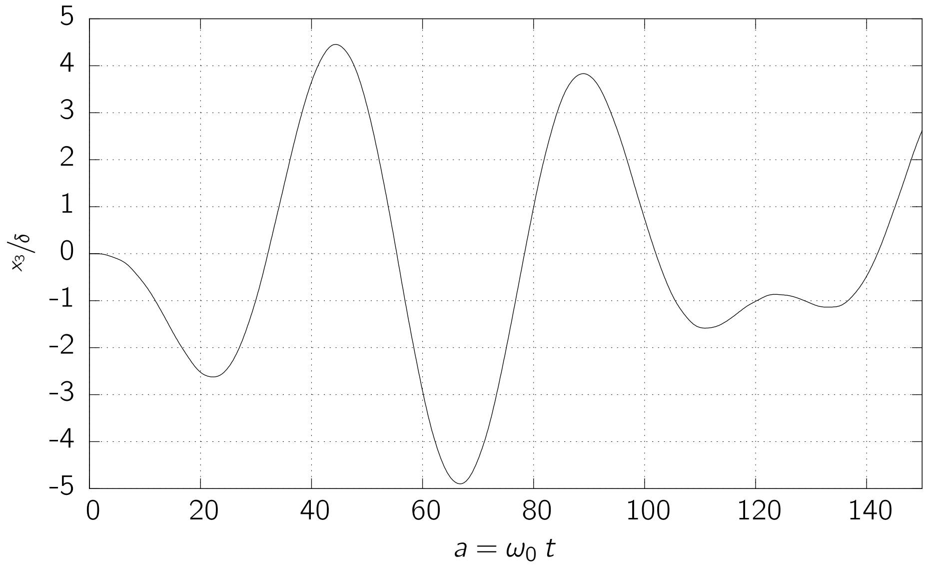
The displacement component x_3 is given by

$$x_3(t) = \sum \psi_{3i}\, q_i(t),$$

that substituting the numerical values and simplifying gives

$$\frac{x_5}{\delta} = 2.017\ \sin(\frac{a}{6}) - 2.913\ \sin(0.116\, a) + \frac{5.944}{1000}\ \sin(1.191\, a) - \frac{3.116}{1000}\ \sin(1.682\, a)$$

and here it is the corresponding plot.



Just to be sure, I integrated numerically the equation of motion, and here it is the comparison