Generalized Single Degree of Freedom Systems

Giacomo Boffi

Dipartimento di Ingegneria Civile e Ambientale, Politecnico di Milano

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Outline

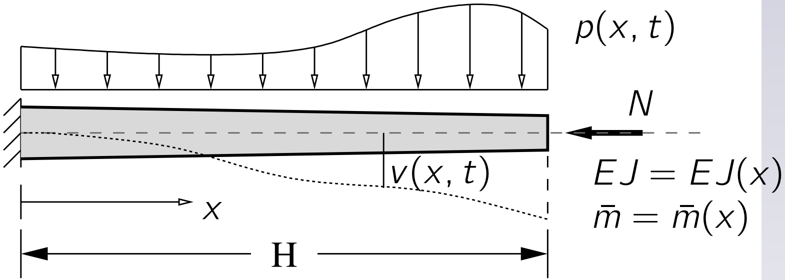
Let's start with an example...

Consider a cantilever, with varying properties $\bar{m}$ and EJ, subjected to a load that is function of both time t and position x,

$$p = p(x, t).$$

The transverse displacements v will be function of time and position,

$$v = v(x, t)$$



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Note that $\Psi(x)$, the *shape function*, is adimensional, while $Z(t)$ is dimensionally a generalised displacement, usually chosen to characterise the structural behaviour.
In our example we can use the displacement of the tip of the chimney, thus implying that $\Psi(H) = 1$ because

$$Z(t) = v(H, t) \quad \text{and}$$
$$v(H, t) = \Psi(H)\, Z(t)$$

Principle of Virtual Displacements

For a flexible system, the PoVD states that, at equilibrium,

$$\delta W_{\mathrm{E}} = \delta W_{\mathrm{I}}.$$

The virtual work of external forces can be easily computed, the virtual work of internal forces is usually approximated by the virtual work done by bending moments, that is

$$\delta W_{\mathrm{I}} \approx \int M \, \delta\chi$$

where χ is the curvature and $\delta\chi$ the virtual increment of curvature.

δW_{E}

The external forces are $p(x,t)$, N and the forces of inertia f_{I}; we have, by separation of variables, that $\delta v = \Psi(x)\delta Z$ and we can write

$$\delta W_{\text{p}} = \int_0^H p(x,t)\delta v \,\mathrm{d}x = \left[\int_0^H p(x,t)\Psi(x)\,\mathrm{d}x\right]\,\delta Z = p^\star(t)\,\delta Z$$

$$\begin{aligned}\delta W_{\text{Inertia}} &= \int_0^H -\bar{m}(x)\ddot{v}\delta v\,\mathrm{d}x = \int_0^H -\bar{m}(x)\Psi(x)\ddot{Z}\Psi(x)\,\mathrm{d}x\,\delta Z \\ &= \left[\int_0^H -\bar{m}(x)\Psi^2(x)\,\mathrm{d}x\right]\,\ddot{Z}(t)\,\delta Z = m^\star\ddot{Z}\,\delta Z.\end{aligned}$$

The virtual work done by the axial force deserves a separate treatment...

δW_{N}

The virtual work of N is $\delta W_{\mathrm{N}} = N\delta u$ where δu is the variation of the vertical displacement of the top of the chimney.

We start computing the vertical displacement of the top of the chimney in terms of the rotation of the axis line, $\phi \approx \Psi'(x)Z(t)$,

$$u(t) = H - \int_0^H \cos\phi \,\mathrm{d}x = \int_0^H (1-\cos\phi)\,\mathrm{d}x,$$

substituting the well known approximation $cos\phi \approx 1 - \frac{\phi^2}{2}$ in the above equation we have

$$u(t) = \int_0^H \frac{\phi^2}{2}\,\mathrm{d}x = \int_0^H \frac{\Psi'^2(x)Z^2(t)}{2}\,\mathrm{d}x$$

hence

$$\delta u = \int_0^H \Psi'^2(x)Z(t)\delta Z\,\mathrm{d}x = \int_0^H \Psi'^2(x)\,\mathrm{d}x\; Z\delta Z$$

δW_{Int}

Approximating the internal work with the work done by bending moments, for an infinitesimal slice of beam we write

$$\mathrm{d}W_{\text{Int}} = \frac{1}{2} M v''(x,t)\,\mathrm{d}x = \frac{1}{2} M \Psi''(x) Z(t)\,\mathrm{d}x$$

with $M = EJ(x)v''(x)$

$$\delta(\mathrm{d}W_{\text{Int}}) = EJ(x)\Psi''^2(x) Z(t) \delta Z\,\mathrm{d}x$$

integrating

$$\delta W_{\text{Int}} = \left[\int_0^H EJ(x)\Psi''^2(x)\,\mathrm{d}x\right] Z\delta Z = k^\star\, Z\,\delta Z$$

Remarks

- the shape function *must* respect the geometrical boundary conditions of the problem, i.e., both

$$\Psi_1 = x^2 \qquad \text{and} \qquad \Psi_2 = 1 - \cos\frac{\pi x}{2H}$$

are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$

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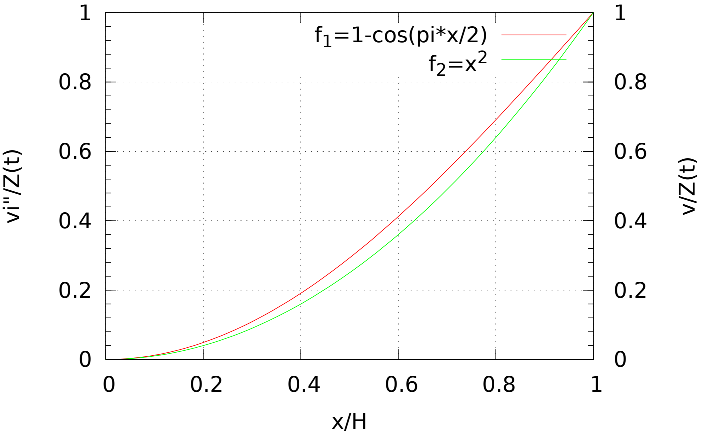

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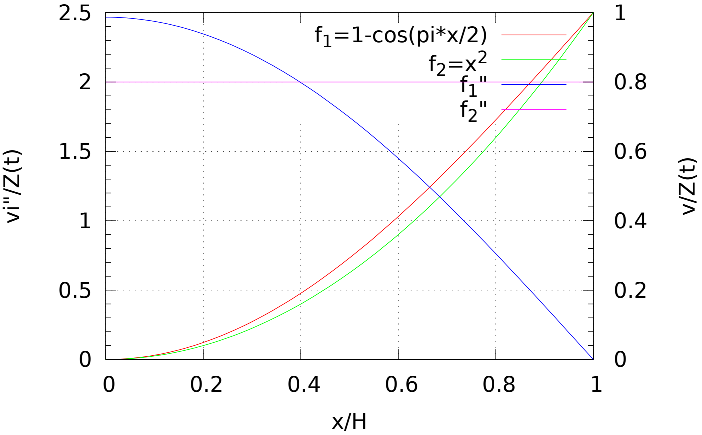

f1=1-cos(pi*x/2)
f2=x2
f1"
f2"
2.5
2
1.5
1
0.5
0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
vi"/Z(t)
v/Z(t)
x/H

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 are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$
- better results are obtained when the second derivative of the shape function at least *resembles* the typical distribution of bending moments in our problem, so that between

$$\Psi_1'' = \text{constant} \qquad \text{and} \qquad \Psi_2\text{''} = \frac{\pi^2}{4H^2}\cos\frac{\pi x}{2H}$$

 the second choice is preferable.

Example

Using $\Psi(x) = 1 - \cos\frac{\pi x}{2H}$, with $\bar{m} =$ constant and $EJ =$ constant, with a load characteristic of seismic excitation, $p(t) = -\bar{m}\ddot{v}_g(t)$,

$$m^\star = \bar{m}\int_0^H (1-\cos\frac{\pi x}{2H})^2 \,\mathrm{d}x = \bar{m}(\frac{3}{2}-\frac{4}{\pi})H$$

$$k^\star = EJ\frac{\pi^4}{16H^4}\int_0^H \cos^2\frac{\pi x}{2H}\,\mathrm{d}x = \frac{\pi^4}{32}\frac{EJ}{H^3}$$

$$k_G^\star = N\frac{\pi^2}{4H^2}\int_0^H \sin^2\frac{\pi x}{2H}\,\mathrm{d}x = \frac{\pi^2}{8H}N$$

$$p_g^\star = -\bar{m}\ddot{v}_g(t)\int_0^H 1-\cos\frac{\pi x}{2H}\,\mathrm{d}x = -\left(1-\frac{2}{\pi}\right)\bar{m}H\,\ddot{v}_g(t)$$

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- We can use our previous results for flexible systems, based on the *SDOF* model, to give an estimate of the natural frequency $\omega^2 = k^\star/m^\star$
- A different approach, proposed by Lord Rayleigh, starts from different premises to give the same results but the *Rayleigh's Quotient* method is important because it offers a better understanding of the vibrational behaviour, eventually leading to successive refinements of the first estimate of ω^2.

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- disregarding damping, the energy of the system is constant during free vibrations,

$$V_{\max} + 0 = 0 + T_{\max}$$

Rayleigh' s Quotient Method

Now we write the expressions for $V_{\max}$ and $T_{\max}$,

$$V_{\max} = \frac{1}{2} Z_0^2 \int_S EJ(x)\Psi''^2(x)\,\mathrm{d}x,$$
$$T_{\max} = \frac{1}{2} \omega^2 Z_0^2 \int_S \bar{m}(x)\Psi^2(x)\,\mathrm{d}x,$$

equating the two expressions and solving for ω^2 we have

$$\omega^2 = \frac{\int_S EJ(x)\Psi''^2(x)\,\mathrm{d}x}{\int_S \bar{m}(x)\Psi^2(x)\,\mathrm{d}x}.$$

Recognizing the expressions we found for $k^\star$ and $m^\star$ we could question the utility of Rayleigh's Quotient...

Rayleigh's Quotient Method

- in Rayleigh's method we know the specific time dependency of the inertial forces

$$f_\mathrm{I} = -\bar{m}(x)\ddot{v} = \bar{m}(x)\omega^2 Z_0 \Psi(x) \sin\omega t$$

  f_I has the same *shape* we use for displacements.
- if Ψ were the real shape assumed by the structure in free vibrations, the displacements v due to a loading $f_\mathrm{I} = \omega^2 \bar{m}(x)\Psi(x)Z_0$ should be proportional to $\Psi(x)$ through a constant factor, with equilibrium respected in every point of the structure during free vibrations.

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- starting from a shape function $\Psi_0(x)$, a new shape function Ψ_1 can be determined normalizing the displacements due to the inertial forces associated with $\Psi_0(x)$, $f_I = \bar{m}(x)\Psi_0(x)$,

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- we are going to demonstrate that the new shape function is a better approximation of the true mode shape

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To keep inertia induced deformation proportional to Ψ_i we must consider the presence of additional elastic constraints.
This leads to the following considerations

- the frequency of vibration of a structure with additional constraints is higher than the true natural frequency,
- the criterium to discriminate between different shape functions is: better shape functions give lower estimates of the natural frequency, the true natural frequency being a lower bound of all estimates.

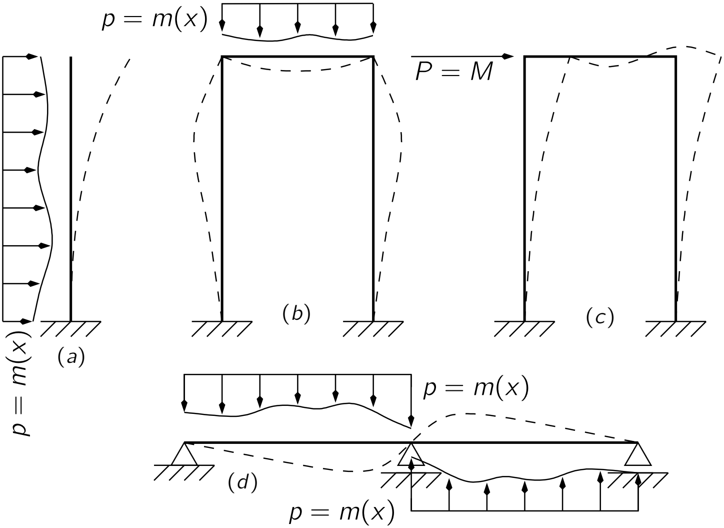
Selection of mode shapes 2

In general the selection of trial shapes goes through two steps,

1. the analyst considers the flexibilities of different parts of the structure and the presence of symmetries to devise an approximate shape,
2. the structure is loaded with constant loads directed as the assumed displacements, the displacements are computed and used as the shape function,

of course a little practice helps a lot in the the choice of a proper pattern of loading...

Selection of mode shapes 3



Refinement R_{00}

Choose a trial function $\Psi^{(0)}(x)$ and write

$$v^{(0)} = \Psi^{(0)}(x) Z^{(0)} \sin \omega t$$
$$V_{\max} = \frac{1}{2} Z^{(0)2} \int EJ\Psi^{(0)\prime\prime 2}\,\mathrm{d}x$$
$$T_{\max} = \frac{1}{2}\omega^2 Z^{(0)2} \int \bar{m}\Psi^{(0)2}\,\mathrm{d}x$$

our first estimate R_{00} of ω^2 is

$$\omega^2 = \frac{\int EJ\Psi^{(0)\prime\prime 2}\,\mathrm{d}x}{\int \bar{m}\Psi^{(0)2}\,\mathrm{d}x}.$$

Refinement R_{01}

We try to give a better estimate of $V_{\max}$ computing the external work done by the inertial forces,

$$p^{(0)} = \omega^2 \bar{m}(x) v^{(0)} = Z^{(0)} \omega^2 \Psi^{(0)}(x)$$

the deflections due to $p^{(0)}$ are

$$v^{(1)} = \omega^2 \frac{v^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \frac{Z^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \bar{Z}^{(1)},$$

where we write $\bar{Z}^{(1)}$ because we need to keep the unknown ω^2 in evidence. The maximum strain energy is

$$V_{\max} = \frac{1}{2} \int p^{(0)} v^{(1)} \, \mathrm{d}x = \frac{1}{2} \omega^4 Z^{(0)} \bar{Z}^{(1)} \int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, \mathrm{d}x$$

Equating to our previus estimate of $T_{\max}$ we find the R_{01} estimate

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x) \Psi^{(0)} \Psi^{(0)} \, \mathrm{d}x}{\int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, \mathrm{d}x}$$

Refinement R_{11}

With little additional effort it is possible to compute $T_{\max}$ from $v^{(1)}$:

$$T_{\max} = \frac{1}{2}\omega^2 \int \bar{m}(x) v^{(1)2}\,\mathrm{d}x = \frac{1}{2}\omega^6 \bar{\bar{Z}}^{(1)2} \int \bar{m}(x)\Psi^{(1)2}\,\mathrm{d}x$$

equating to our last approximation for $V_{\max}$ we have the R_{11} approximation to the frequency of vibration,

$$\omega^2 = \frac{Z^{(0)}}{\bar{\bar{Z}}^{(1)}} \frac{\int \bar{m}(x)\Psi^{(0)}\Psi^{(1)}\,\mathrm{d}x}{\int \bar{m}(x)\Psi^{(1)}\Psi^{(1)}\,\mathrm{d}x}.$$

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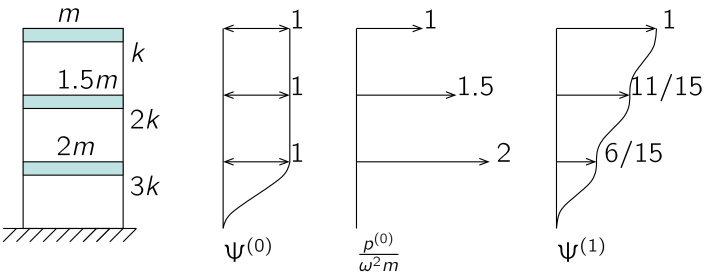
Of course the procedure can be extended to compute better and better estimates of ω^2 but usually the refinements are not extended beyond R_{11}, because it would be contradictory with the quick estimate nature of the Rayleigh's Quotient method and also because R_{11} estimates are usually very good ones.

Refinement Example



$$T = \frac{1}{2}\omega^2 \times 4.5 \times m\, Z_0^2$$

$$V = \frac{1}{2} \times 1 \times 3k\, Z_0^2$$

$$\omega^2 = \frac{3}{9/2}\frac{k}{m} = \frac{2}{3}\frac{k}{m}$$

$$v^{(1)} = \frac{15}{4}\frac{m}{k}\omega^2 \Psi^{(1)}$$

$$\bar{Z}^{(1)} = \frac{15}{4}\frac{m}{k}$$

$$V^{(1)} = \frac{1}{2}m\frac{15}{4}\frac{m}{k}\omega^4\,(1 + 33/30 + 4/5)$$

$$= \frac{1}{2}m\frac{15}{4}\frac{m}{k}\omega^4\,\frac{87}{30}$$

$$\omega^2 = \frac{\frac{9}{2}m}{m\frac{87}{8}\frac{m}{k}} = \frac{12}{29}\frac{k}{m} = 0.4138\frac{k}{m}$$