Response by Superposition

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Eigenvector Expansion

For a *N-DOF* system, it is possible and often advantageous to represent the displacements **x** in terms of a linear combination of the free vibration modal shapes, the eigenvectors, by the means of a set of modal coordinates,

$$\mathbf{x} = \sum \boldsymbol{\psi}_i q_i = \boldsymbol{\Psi}\mathbf{q}.$$

The eigenvectors play a role analogous to the role played by trigonometric functions in Fourier Analysis,

- they possess orthogonality properties,
- we will see that it is usually possible to approximate the response using only a few low frequency terms.

Inverting Eigenvector Expansion

The columns of the eigenmatrix $\boldsymbol{\Psi}$ are the N linearly indipendent eigenvectors $\boldsymbol{\psi}_i$, hence the eigenmatrix is non-singular and it is always correct to write $\mathbf{q} = \boldsymbol{\Psi}^{-1}\mathbf{x}$.
However, it is not necessary to invert the eigenmatrix:

Inverting Eigenvector Expansion

If we write, again,

$$\mathbf{x} = \sum \boldsymbol{\psi}_i q_i = \mathbf{\Psi q}.$$

and multiply both members by $\mathbf{\Psi}^T\mathbf{M}$, taking into account that $\mathbf{\Psi}^T\mathbf{M\Psi} = \mathbf{M}^\star$ we have

$$\mathbf{\Psi}^T\mathbf{Mx} = \mathbf{M}^\star\mathbf{q}$$

but $\mathbf{M}^\star$ is a diagonal matrix, hence $(\mathbf{M}^\star)^{-1} = \{\delta_{ij}/M_i\}$ and we can write

$$\mathbf{q} = \mathbf{M}^{\star -1}\mathbf{\Psi}^T\mathbf{Mx}, \qquad \text{or} \qquad q_i = \frac{\boldsymbol{\psi}_i^T\mathbf{Mx}}{M_i}$$

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Note: this formula works also when we don't know all the eigenvectors and the inversion of a partial, rectangular $\boldsymbol{\Psi}$ is not feasible.

Undamped System

The equation of motion is $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t)$.
Substituting in it $\mathbf{x} = \boldsymbol{\Psi}\mathbf{q}$, $\ddot{\mathbf{x}} = \boldsymbol{\Psi}\ddot{\mathbf{q}}$, pre multiplying both members by $\boldsymbol{\Psi}^T$ and exploiting the ortogonality rules, we have

$$M_i\,\ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \quad i = 1, \ldots, N.$$

with $p_i^\star(t) = \boldsymbol{\psi}_i^T \mathbf{p}(t)$.

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The equations of motion written in terms of nodal coordinates constitute a system of N interdipendent, *coupled* differential equations, written in terms of modal coordinates constitute a set of N indipendent, *uncoupled* differential equations.

Damped System

In general,

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t)$$

and with the usual stuff

$$M_i\,\ddot{q}_i + \boldsymbol{\psi}^T\mathbf{C}\,\boldsymbol{\Psi}\,\dot{\mathbf{q}} + \omega_i^2 M_i q_i = p_i^\star(t),$$

with $\boldsymbol{\psi}_i^T\mathbf{C}\,\boldsymbol{\psi}_j = c_{ij}$

$$M_i\,\ddot{q}_i + \sum_j c_{ij}\,\dot{q}_j + \omega_i^2 M_i q_i = p_i^\star(t),$$

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If we define the damping matrix as

$$\mathbf{C} = \sum_b \mathfrak{c}_b \mathbf{M}\left(\mathbf{M}^{-1}\mathbf{K}\right)^b,$$

we know that, as required,

$$c_{ij} = \delta_{ij}C_i \quad \text{with } C_i \left(= 2\zeta_i M_i\omega_i\right) = \sum_b \mathfrak{c}_b\left(\omega_i^2\right)^b.$$

Damped Systems, a Comment

If the response is computed by modal superposition, it is usually preferred a simpler but equivalent procedure: for each mode of interest the analyst imposes a given damping ratio and the integration of the modal equation of equilibrium is carried out as usual.

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The $\sum \mathfrak{c}_b \ldots$ procedure is useful when, e.g. for non-linear problems, the integration of the eq. of motion is carried out in nodal coordinates, because it is easier to specify damping properties globally as elastic modes properties (that can be measured or deduced from similar outsets) than to assign correct damping properties at the *FE* level and assembling $\mathbf{C}$ by the *FEM*.

Initial Conditions

For a set of generic initial conditions $\mathbf{x}_0$, $\dot{\mathbf{x}}_0$, we can easily have the initial conditions in modal coordinates:

$$\mathbf{q}_0 = \mathbf{M}^{\star -1} \boldsymbol{\Psi}^T \mathbf{M} \mathbf{x}_0$$

$$\dot{\mathbf{q}}_0 = \mathbf{M}^{\star -1} \boldsymbol{\Psi}^T \mathbf{M} \dot{\mathbf{x}}_0$$

and, for each mode, the total modal response can be obtained by superposition of a particular integral $\xi_i(t)$ and the general integral of the homogeneous associate,

$$\begin{aligned} q_i(t) = \xi_i(t) + & \\ & + e^{-\zeta_i \omega_i t} (q_{i,0} - \xi_i(0)) \cos \omega_{Di} t + \\ & + e^{-\zeta_i \omega_i t} \frac{(\dot{q}_{i,0} - \dot{\xi}_i(0)) + (q_{i,0} - \xi_i(0)) \zeta_i \omega_i}{\omega_{Di}} \sin \omega_{Di} t. \end{aligned}$$

Truncated sum

Having computed all $q_i(t)$, we can sum all the modal responses,

$$\mathbf{x}(t) = \boldsymbol{\psi}_1 q_1(t) + \boldsymbol{\psi}_2 q_2(t) + \cdots + \boldsymbol{\psi}_N q_N(t) = \sum^{N} \boldsymbol{\psi}_i q_i(t)$$

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A *truncated sum*, comprising only a few of the lower frequency modes,

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The importance of truncated sum approximation is twofold:

- less computational effort: less eigenpairs to calculate, less equation of motion to integrate etc
- in FEM models the higher modes are rough approximations to structural ones (mostly due to uncertainties in mass distribution details) and the truncated sum excludes potentially spurious contributions from the response.

Elastic Forces

Until now, we showed interest in displacements only, but we are interested in elastic forces too. We know that elastic forces can be expressed in terms of displacements and the stiffness matrix:

$$\mathbf{f}_S(t) = \mathbf{K}\,\mathbf{x}(t) = \mathbf{K}\boldsymbol{\psi}_1 q_1(t) + \mathbf{K}\boldsymbol{\psi}_2 q_2(t) + \cdots .$$

From the characteristic equation we know that

$$\mathbf{K}\boldsymbol{\psi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\psi}_i$$

substituting in the previous equation

$$\mathbf{f}_S(t) = \omega_1^2 \mathbf{M}\boldsymbol{\psi}_1 q_1(t) + \omega_2^2 \mathbf{M}\boldsymbol{\psi}_2 q_2(t) + \cdots .$$

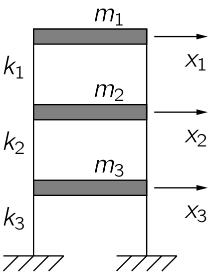
Elastic Forces, 2

Obviously the higher modes' force contributions, e.g.

$$\mathbf{f}_S(t) = \omega_1^2 \mathbf{M} \boldsymbol{\psi}_1 q_1(t) + \cdots + \omega_2^2 \mathbf{M} \boldsymbol{\psi}_2 q_2(t) + \cdots$$

in a truncated sum will be higher than displacement ones or, from a different point of view, to estimate internal forces within given accuracy, a greater number of modes must be considered in a truncated sum than the number required to estimate displacements within the same accuracy

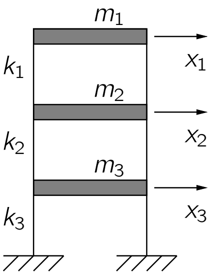
Example: problem statement



$$k_1 = 120\,\text{MN/m}, \quad m_1 = 200\,\text{t},$$
$$k_2 = 240\,\text{MN/m}, \quad m_2 = 300\,\text{t},$$
$$k_3 = 360\,\text{MN/m}, \quad m_3 = 400\,\text{t}.$$

1. The above structure is subjected to these initial conditions,
$$\mathbf{x}_0^T = \{5\,\text{mm} \quad 4\,\text{mm} \quad 3\,\text{mm}\},$$
$$\dot{\mathbf{x}}_0^T = \{0 \quad 9\,\text{mm/s} \quad 0\}.$$
Write the equation of motion using modal superposition.
2. The above structure is subjected to a half-sine impulse,
$$\mathbf{p}^T(t) = \{1 \quad 2 \quad 2\}\, 2.5\,\text{MN} \sin\frac{\pi t}{t_1}, \quad \text{with } t_1 = 0.02\,\text{s}.$$
Write the equation of motion using modal superposition.

Example: structural matrices



$$\begin{aligned} k_1 &= 120\,\text{MN/m}, & m_1 &= 200\,\text{t}, \\ k_2 &= 240\,\text{MN/m}, & m_2 &= 300\,\text{t}, \\ k_3 &= 360\,\text{MN/m}, & m_3 &= 400\,\text{t}. \end{aligned}$$

The structural matrices can be written

$$\mathbf{K} = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} = k\overline{\mathbf{K}}, \qquad \text{with } k = 120\,\frac{\text{MN}}{\text{m}},$$

$$\mathbf{M} = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = m\overline{\mathbf{M}}, \qquad \text{with } m = 100000\,kg.$$

Example: adimensional eigenvalues

We want the solutions of the characteristic equation, so we start writing that the determinant of the equation must be zero:

$$\left\|\overline{\mathbf{K}} - \frac{\omega^2}{k/m}\overline{\mathbf{M}}\right\| = \left\|\overline{\mathbf{K}} - \Omega^2\overline{\mathbf{M}}\right\| = 0,$$

with $\omega^2 = 1200\left(\frac{\text{rad}}{\text{s}}\right)^2\Omega^2$.
Expanding the determinant

$$\begin{Vmatrix} 1-2\Omega^2 & -1 & 0 \\ -1 & 3-3\Omega^2 & -2 \\ 0 & -2 & 5-4\Omega^2 \end{Vmatrix} = 0$$

we have the following algebraic equation of 3rd order in Ω^2

$$24\left(\Omega^6 - \frac{11}{4}\Omega^4 + \frac{15}{8}\Omega^2 - \frac{1}{4}\right) = 0.$$

Example: table of eigenvalues etc

Here are the adimensional roots Ω_i^2, $i = 1, 2, 3$, the dimensional eigenvalues $\omega_i^2 = 1200 \frac{\text{rad}^2}{\text{s}^2} \Omega_i^2$ and all the derived dimensional quantities:

n	1	2	3
Ω^2	0.175 73	0.8033	1.7710
$\omega^2/(\text{rad}^2\,\text{s}^{-2})$	210.88	963.96	2125.2
$\omega/(\text{rad}\,\text{s}^{-1})$	14.522	31.048	46.099
f/Hz	2.3112	4.9414	7.3370
T/s	0.432 68	0.202 37	0.1363

Example: eigenvectors and modal matrices

With $\psi_{1j} = 1$, using the 2nd and 3rd equations,

$$\begin{bmatrix} 3-3\Omega_j^2 & -2 \\ -2 & 5-4\Omega_j^2 \end{bmatrix} \begin{Bmatrix} \psi_{2j} \\ \psi_{3j} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The above equations must be solved for $j = 1, 2, 3$.
For $j = 1$, it is

$$\begin{cases} 2.47280290827\psi_{21} & -2\psi_{31} & = & 1 \\ -2\psi_{21} & +4.29707054436\psi_{31} & = & 0 \end{cases}$$

For $j = 2$,

$$\begin{cases} 0.5901013613\psi_{22} & -2\psi_{32} & = & 1 \\ -2\psi_{22} & +1.78680181507\psi_{32} & = & 0 \end{cases}$$

Finally, for $j = 3$,

$$\begin{cases} -2.31290426958\psi_{23} & -2\psi_{33} & = & 1 \\ -2\psi_{23} & -2.08387235944\psi_{33} & = & 0 \end{cases}$$

Example

The solutions are finally collected in the eigenmatrix

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 1 & 1 \\ +0.648535272183 & -0.606599092464 & -2.54193617967 \\ +0.301849953585 & -0.678977475113 & +2.43962752148 \end{bmatrix}.$$

The Modal Matrices are

$$\mathbf{M}^\star = \mathbf{\Psi}^T \mathbf{M}\,\mathbf{\Psi} = \begin{bmatrix} 362.6 & 0 & 0 \\ 0 & 494.7 & 0 \\ 0 & 0 & 4519.1 \end{bmatrix} \times 10^3\,\text{kg},$$

$$\mathbf{K}^\star = \mathbf{\Psi}^T \mathbf{K}\,\mathbf{\Psi} = \begin{bmatrix} 76.50 & 0 & 0 \\ 0 & 477.0 & 0 \\ 0 & 0 & 9603.9 \end{bmatrix} \times 10^6\,\frac{\text{N}}{\text{m}}$$

Example: initial conditions in modal coordinates

$$\mathbf{q}_0 = (\mathbf{M}^\star)^{-1}\boldsymbol{\psi}^T\mathbf{M}\begin{Bmatrix}5\\4\\3\end{Bmatrix}\,\text{mm} = \begin{Bmatrix}+5.9027\\-1.0968\\+0.1941\end{Bmatrix}\,\text{mm},$$

$$\dot{\mathbf{q}}_0 = (\mathbf{M}^\star)^{-1}\boldsymbol{\psi}^T\mathbf{M}\begin{Bmatrix}0\\9\\0\end{Bmatrix}\,\frac{\text{mm}}{\text{s}} = \begin{Bmatrix}+4.8288\\-3.3101\\-1.5187\end{Bmatrix}\,\frac{\text{mm}}{\text{s}}$$

Example: structural response

These are the displacements, in mm

$$x_1 = +5.91\cos(14.5t + .06) + 1.10\cos(31.0t - 3.04) + 0.20\cos(46.1t - 0.17)$$
$$x_2 = +3.83\cos(14.5t + .06) - 0.67\cos(31.0t - 3.04) - 0.50\cos(46.1t - 0.17)$$
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and these the elastic/inertial forces, in kN

$$f_1 = +249.\cos(14.5t + .06) + 212.\cos(31.0t - 3.04) + 084.\cos(46.1t - 0.17)$$
$$f_2 = +243.\cos(14.5t + .06) - 193.\cos(31.0t - 3.04) - 319.\cos(46.1t - 0.17)$$
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As expected, the contributions of the higher modes are more important for the forces, less important for the displacements.