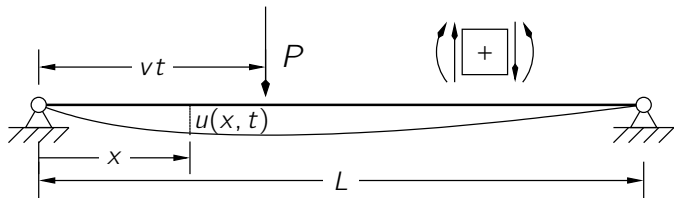


# Continuous Systems

## an example

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A uniform beam ( $m(x) = m$ ,  $EJ(x) = EJ$ ) of length  $L$  is loaded by a moving load  $P$ , moving with constant velocity,  $v(t) = v$ , in the interval  $0 \leq t \leq t_0 = L/v = t_0$ .

Using the sign conventions indicated above, compute and plot the midspan displacement  $u(L/2, t)$  and the midspan bending moment  $M_b(L/2, t)$  as functions of time in the interval  $0 \leq t \leq t_0$  for different values of the velocity.

NB: the beam is at rest for  $t = 0$ .

For an uniform beam, the equation of dynamic equilibrium is

$$m \frac{\partial^2 u(x, t)}{\partial t^2} + EJ \frac{\partial^4 u(x, t)}{\partial x^4} = p(x, t).$$

In our example, the loading function must be defined in terms of  $\delta(x)$ , the Dirac's delta distribution,

$$p(x, t) = P \delta(x - vt).$$

Problem  
statement

Solution

Equation of motion

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The Dirac's delta is a *generalized* function of one variable, defined by

$$\delta(x - x_0) \equiv 0 \quad \text{and} \quad \int f(x) \delta(x - x_0) dx = f(x_0).$$

Note that the Dirac distribution and the Kronecker's symbol  $\delta_{ij}$  are two different things.

The solution will be computed by separation of variables

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The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\begin{aligned}\phi_n(x) &= \sin \beta_n x, & \beta_n &= \frac{n\pi}{L}, \\ m_n &= \frac{mL}{2}, & \omega_n^2 &= \beta_n^4 \frac{EJ}{m} = n^4 \pi^4 \frac{EJ}{mL^4}.\end{aligned}$$

For an uniform beam, the orthogonality relationships are

$$m \int_0^L \phi_n(x) \phi_m(x) dx = m_n \delta_{nm},$$

$$EJ \int_0^L \phi_n(x) \phi_m^{IV}(x) dx = k_n \delta_{nm} = m_n \omega_n^2 \delta_{nm}.$$

in the equations above  $\delta$  is the Kroneker's  $\delta$  symbol, a completely different thing from Dirac's  $\delta$  distribution.

# Decoupling the EOM

Continuous  
Systems

Giacomo Boffi

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

Problem  
statement

Solution

Equation of motion



# Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

1. The equation of motion is written in terms of the modal series representation of  $u(x, t)$ :

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2. every term is multiplied by  $\phi_n$  and integrated over the length of the beam

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3. we use the orthogonality relationships and the definition of  $\delta$ ,

$$m_n \ddot{q}(t) + k_n q(t) = P \phi_n(vt) = P \sin \frac{n\pi vt}{L}, \quad n = 1, \dots, \infty.$$

Considering that the initial conditions are nil for all the modal equations, with  $\bar{\omega}_n = n\pi v/L$  and  $\beta_n = \bar{\omega}_n/\omega_n$  the individual solutions are given by

$$q_n(t) = \frac{P}{k_n} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}$$

Problem  
statement

Solution

Equation of motion

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With  $k_n = m_n \omega_n^2 = \frac{mL}{2} n^4 \pi^4 \frac{EJ}{mL^4} = n^4 \pi^4 \frac{EJ}{2L^3}$ , it is

$$q_n(t) = \frac{2PL^3}{n^4 \pi^4 EJ} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}.$$

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It is apparent that for  $\beta_n^2 = 1$  there is resonance.

The critical velocity  $v_{cr,n}$  for mode  $n$  is given by  $\beta_n = 1$ , substituting  $\omega_n = n^2\omega_1$  we have  $n\pi v_{cr,n}/L/n^2\omega_1 = 1$  that gives  $v_{cr,n} = n\omega_1 L/\pi = n v_{cr,1} = n v_{cr}$ , where  $v_{cr} = \omega_1 L/\pi$ .

With the position  $v = \kappa v_{cr}$  it is

$$\bar{\omega}_n = \kappa n \omega_1 \text{ and } \beta_n = n \kappa \omega_1 / n^2 \omega_1 = \kappa / n.$$

Problem  
statement

Solution

Equation of motion

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$$\bar{\omega}_n = \kappa n \omega_1 \text{ and } \beta_n = n \kappa \omega_1 / n^2 \omega_1 = \kappa / n.$$

The solution can be rewritten as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left( \sin\left(\frac{\kappa}{n}\omega_n t\right) - \frac{\kappa}{n} \sin \omega_n t \right),$$

for  $0 \leq t \leq \frac{L}{v}$ .



Introducing an adimensional time coordinate  $\xi$  with  $t = t_0 \xi$ , noting that  $\omega_n = n^2 \omega_1$  we can write the argument of the first sine as follows:

$$\frac{\kappa}{n} \omega_n t = \kappa n \omega_1 \xi t_0 = n \xi t_0 \kappa v_{cr} \pi / L = n \pi \xi \times (v t_0) / L = n \pi \xi.$$

In a similar way we have  $\omega_n t = n^2 \pi \xi / \kappa$ .

Substituting in the equation of the modal responses the new expressions for the sine arguments, it is

$$q_n(\xi) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right)$$

for  $0 \leq \xi \leq 1$ .

If we denote with  $\mathbb{X}(t)$  the position of the load at time  $t$ , it is  $\mathbb{X}(t) = vt = \xi L$ , or  $\xi = \mathbb{X}/L$  and the expression  $u(x, \xi) = \sum q_n(\xi)\phi_n(x)$  can be interpreted as the displacement in  $x$  when the load is positioned in  $\mathbb{X} = \xi L$ .

The displacement and the bending moment are given by

$$u(x, \xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{x}{L})}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right),$$

$$\begin{aligned} M_b(x, \xi) &= -EJ \frac{\partial^2 u(x, \xi)}{\partial x^2} = \\ &= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{x}{L})}{n^2 - \kappa^2} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right). \end{aligned}$$

The maximum values of the midspar deflection and bending moment are obtained when  $P$  is placed at midspar,

$$u_{\text{stat}} = \frac{PL^3}{48EJ}, \quad M_{\text{b stat}} = \frac{PL}{4}.$$

It is convenient to normalize the responses with respect to these maxima to have an appreciation of the dynamical effects.

The normalized midspan displacement

$\eta(\xi) = u(L/2, \xi)/u_{\text{stat}}$  has the expression

$$\eta(\xi) = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2(n^2 - \kappa^2)} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right),$$

where  $\sin(n\pi/2) = 1, 0, -1, 0, 1, \dots$  for  $n = 1, 2, 3, 4, 5, \dots$

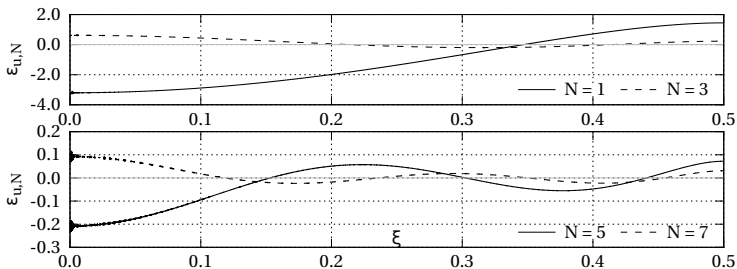
Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n^2 - \kappa^2} \left( \sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right).$$

Partial sums with  $N$  terms will be denoted in the following by  $\eta_N(\xi)$  and  $\mu_N(\xi)$ .

The normalized midspan statical displacement for a load  $P$  placed at  $\mathbb{X} = \xi L$  is  $\eta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$  for  $0 \leq \xi \leq 1/2$  and we can define a percent error function (using  $\kappa = 10^{-6}$  to obtain a good approximation to the static response)

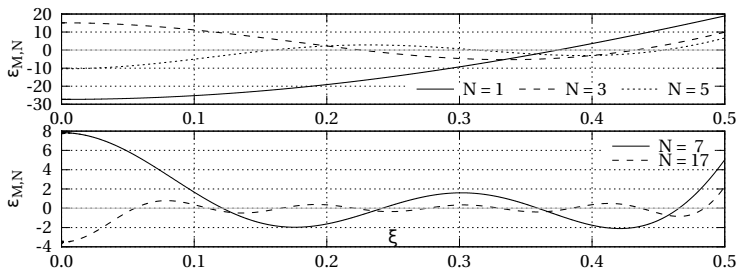
$$\epsilon_{u,N}(\xi) = 100 \left( 1 - \frac{\eta_N(\xi)|_{\kappa=10^{-6}}}{\eta_{\text{stat}}(\xi)} \right) \quad \text{for } 0 \leq \xi \leq 1/2,$$



With 5 terms the approximation is in the order of  $1/1000$ .

Analogously we can use the midspan bending moment, normalized with respect to  $PL/4$ ,  $\mu_{\text{stat}}(\xi) = 2\xi$  to define another percent error function

$$\epsilon_{M,N} = 100 \left( 1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)} \right)$$



With 17 terms the approximation is in the order of 4%. As usual, worse convergence for internal forces.

Finally, we plot the normalized displacement and the normalized bending moment different values of the velocity (i.e., for different values of  $\kappa$ ).

Note that for the displacement I used  $N = 11$  while for the bending moment I used  $N = 25$ .

Problem  
statement

Solution

Equation of motion





# Bending moments

