SDOF linear oscillator

Response to Harmonic Loading

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Outline of parts 1 and 2

Outline of parts 3 and 4

Part I

Response of an Undamped Oscillator to Harmonic Load

The Equation of Motion

For an undamped *SDOF* system subjected to an harmonic excitation, characterized by the amplitude p_0 and frequency ω, we can write this equation of dynamic equilibrium:

$$m\,\ddot{x} + k\,x = p_0\,\sin\omega t.$$

The solution to the above differential equation is the homogeneous solution plus a particular integral $\xi(t)$,

$$x(t) = A\sin\omega_\mathrm{n} t + B\cos\omega_\mathrm{n} t + \xi(t)$$

with

$$m\ddot{\xi}(t) + k\xi(t) = p_0\sin\omega t.$$

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The ratio of the excitation frequency to the system natural frequency is the frequency ratio $\beta = {}^{\omega}/_{\omega_n}$.

The Particular Integral

The particular integral, as it happens, can be written in terms of an undetermined coefficient C multiplying a sine, with the same frequency ω as the excitation:

$$\begin{aligned}
\xi(t) &= C\,\sin\omega t,\\
\dot\xi(t) &= \omega\,C\,\cos\omega t,\\
\ddot\xi(t) &= -\omega^2\,C\,\sin\omega t.
\end{aligned}$$

The Particular Integral

1. Substituting $\xi(t)$ for $x(t)$ in $m\ddot{x}(t) + kx(t) = p_0 \sin\omega t$ we have

$$C\,(k - \omega^2 m) \sin\omega t = p_0 \sin\omega t$$

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5. with $\beta = \omega/\omega_\text{n}$, we get: $C = \frac{p_0}{k} \frac{1}{1 - \beta^2}$.

The Particular Integral

We can now write the particular solution, with the dependencies on β singled out in the second term:

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The general integral for $p(t) = p_0 \sin\omega t$ is hence

$$x(t) = A\sin\omega_\mathrm{n} t + B\cos\omega_\mathrm{n} t + \frac{p_0}{k}\,\frac{1}{1-\beta^2}\,\sin\omega t.$$

Response Ratio and Dynamic Amplification Factor

$\Delta_{\mathrm{st}} = p_0/k$ being the *static deformation*, defining the *Response Ratio*, $R(t;\beta) = \frac{1}{1-\beta^2}\sin\omega t$, we can write

$$\xi(t) = \Delta_{\mathrm{st}}\, R(t;\beta).$$

Introducing the *dynamic amplification factor* $D(\beta) = \frac{1}{1-\beta^2}$

$$\xi(t) = \Delta_{\mathrm{st}}\, D(\beta)\sin\omega t.$$

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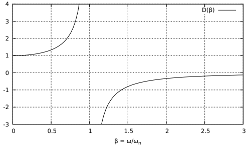
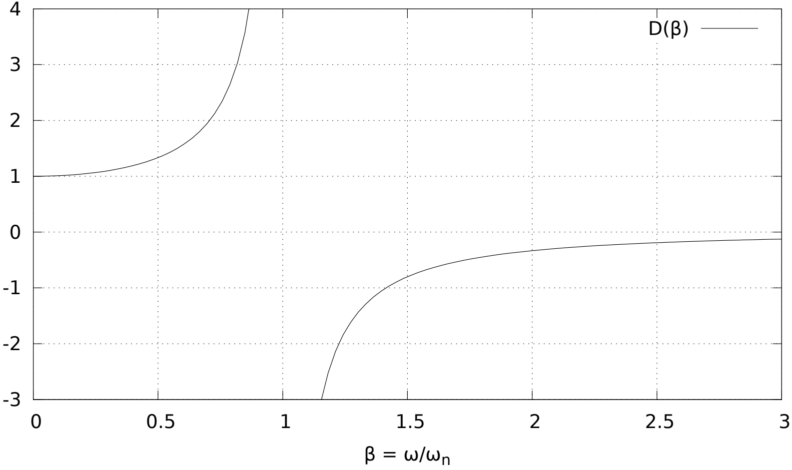
$D(\beta)$ is stationary and almost equal to 1 when $\omega \ll \omega_{\mathrm{n}}$ (this is a *quasi*-static behaviour), it grows out of bound when $\beta \Rightarrow 1$ (resonance), it is negative for $\beta > 1$ and goes to 0 when $\omega \gg \omega_{\mathrm{n}}$ (high-frequency loading).



Dynamic Amplification Factor, the plot



Response from Rest Conditions

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Let's start with $x(t)$, then evaluate $x(0)$ and finally equate this last expression to 0:

$$x(t) = A \sin \omega_\text{n} t + B \cos \omega_\text{n} t + \Delta_\text{st} D(\beta) \sin \omega t,$$

$$x(0) = B = 0.$$

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$$x(0) = B = 0.$$

We do as above for the velocity:

$$\dot{x}(t) = \omega_\mathrm{n}\,(A\,\cos\omega_\mathrm{n} t - B\,\sin\omega_\mathrm{n} t) + \Delta_\mathrm{st}\,D(\beta)\,\omega\cos\omega t,$$
$$\dot{x}(0) = \omega_\mathrm{n}\,A + \omega\,\Delta_\mathrm{st}\,D(\beta) = 0 \Rightarrow$$
$$\Rightarrow A = -\Delta_\mathrm{st}\,\frac{\omega}{\omega_\mathrm{n}} D(\beta) = -\Delta_\mathrm{st}\,\beta D(\beta)$$

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$$\dot{x}(t) = \omega_n \left(A \cos\omega_n t - B \sin\omega_n t\right) + \Delta_{st} D(\beta)\,\omega \cos\omega t,$$
$$\dot{x}(0) = \omega_n A + \omega\,\Delta_{st} D(\beta) = 0 \Rightarrow$$
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Substituting, A and B in $x(t)$ above, collecting Δ_{st} and $D(\beta)$ we have, for $p(t) = p_0 \sin\omega t$, the response from rest:

$$x(t) = \Delta_{st}\ D(\beta) \left(\sin\omega t - \beta \sin\omega_n t\right).$$

Response from Rest Conditions, cont.

For $p(t) = p_0 \cos \omega t$ you can show that $\xi(t) = \Delta_{\text{st}}\, D(\beta)\, \cos \omega t$ and the general integral is

$$x(t) = A \sin \omega_{\text{n}} t + B \cos \omega_{\text{n}} t + \Delta_{\text{st}}\, D(\beta)\, \cos \omega t.$$

Response from Rest Conditions, cont.

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$$x(t) = A \sin \omega_{\mathrm{n}} t + B \cos \omega_{\mathrm{n}} t + \Delta_{\mathrm{st}}\, D(\beta)\, \cos \omega t.$$

For a system starting from rest, with $p(t) = p_0 \cos \omega t$ it is

$$\begin{aligned} x(0) &= B + \Delta_{\mathrm{st}}\, D(\beta) = 0, \\ \dot{x}(0) &= A = 0, \end{aligned}$$

so that, solving for A and B and substituting in the general integral we have, collecting Δ_{st} and $D(\beta)$

$$x(t) = \Delta_{\mathrm{st}}\; D(\beta) \left(\cos \omega t - \cos \omega_{\mathrm{n}} t\right).$$

Beating

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$$\omega = \bar{\omega} - \Delta\omega, \quad \omega_\mathrm{n} = \bar{\omega} + \Delta\omega,$$

and substituting in the time dependency of the response to a cosine excitation,

$$s(t) = \cos \omega t - \cos \omega_\mathrm{n} t = \cos(\bar{\omega} t - \Delta\omega t) - \cos(\bar{\omega} t + \Delta\omega t).$$

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Next step is application of the *sum and difference formulas*, namely

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

that leads to

$$\begin{aligned} s(t) &= \cos(\bar{\omega} t)\cos(\Delta\omega t) + \sin(\bar{\omega} t)\sin(\Delta\omega t) \\ &\quad - \cos(\bar{\omega} t)\cos(\Delta\omega t) + \sin(\bar{\omega} t)\sin(\Delta\omega t) \\ &= 2\sin(\bar{\omega} t)\sin(\Delta\omega t) \end{aligned}$$

Beating, cont.

From $\omega = \bar\omega - \Delta\omega$, $\omega_\text{n} = \bar\omega + \Delta\omega$ we have

$$\bar\omega = \frac{\omega_\text{n} + \omega}{2}, \quad \Delta\omega = \frac{\omega_\text{n} - \omega}{2}$$

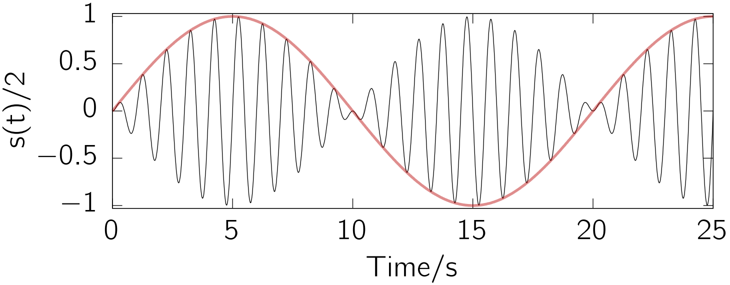
and we recognize that $\bar\omega$ is just the mean frequency and that $\Delta\omega \ll \omega, \omega_\text{n}, \bar\omega$ when the excitation and the natural frequency are close to each other.
Hence our final representation of the time dependency,

$$s(t) = 2\sin(\bar\omega t)\sin(\Delta\omega t),$$

can be interpreted as a sine of (relatively) high frequency modulated by a sine of (relatively) low frequency.

Beating, cont.

To give an example of the previous derivation, and to help visualing the beating phenomenon, let's consider the following parameters, $\bar{\omega} = 2\pi$, $\Delta\omega = 0.05\bar{\omega}$ or, equivalently, $T_n \approx 1\,\mathrm{s}$ and $T_\Delta = 20\,\mathrm{s}$.
A graphical representation of the time dependency of the response is



In red, the enveloping function $\sin(\Delta\omega t)$.

Resonant Response from Rest Conditions

The dynamic amplification factor, $D(\beta)$, is infinite for $\beta = 1$, but (of course) this doesn't imply that exciting a system with a harmonic force with frequency equal to its natural frequency suddenly produces an infinite response...

Resonant Response from Rest Conditions

We have seen that the response to a *resonant* sine loading with zero initial conditions is (expliciting the dependency on β)

$$x(t;\beta) = \Delta_{\mathrm{st}} \left. \frac{(\sin \beta\omega_n t - \beta \sin \omega_{\mathrm{n}} t)}{1-\beta^2} \right|_{\beta=1}$$

and while the denominator is equal to zero also the numerator is equal to zero... so we have an indeterminate form.

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To determine the resonant response, we have to compute the limit

$$x(t) = \lim_{\beta\to 1} \Delta_{\text{st}} \frac{(\sin\beta\omega_n t - \beta\,\sin\omega_\text{n} t)}{1-\beta^2}$$

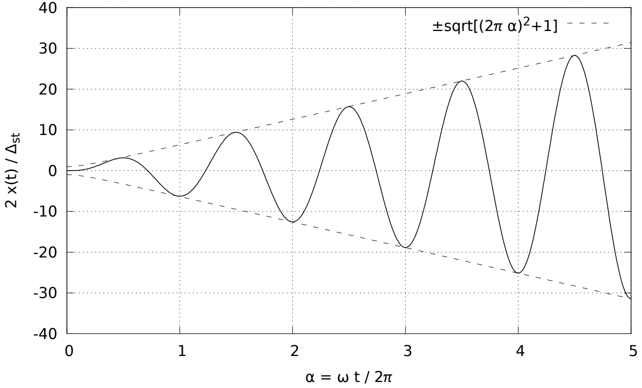
using, e.g., the *de l'Hôpital* rule.

Resonant Response from Rest Conditions

$$
\begin{aligned}
x(t) &= \lim_{\beta\to 1} \Delta_{\text{st}} \frac{(\sin\beta\omega_n t - \beta\ \sin\omega_\text{n} t)}{1-\beta^2} \\
&= \Delta_{\text{st}} \left.\frac{\partial(\sin\beta\omega_\text{n} t - \beta\sin\omega_\text{n} t)/\partial\beta}{\partial(1-\beta^2)/\partial\beta}\right|_{\beta=1} \\
&= \Delta_{\text{st}} \left.\frac{\omega_\text{n} t\cos\beta\omega_\text{n} t - \sin\omega_\text{n} t}{-2\beta}\right|_{\beta=1} \\
&= \frac{\Delta_{\text{st}}}{2}\left(\sin\omega t - \boldsymbol{\omega t \cos\omega t}\right)
\end{aligned}
$$

As you can see, there is a term in quadrature with the loading whose amplitude grows linearly and without bounds.

Resonant Response, the plot



$$\frac{2}{\Delta_{\text{st}}}x(t) = \sin\omega t - \omega t\cos\omega t = \sin 2\pi\alpha - 2\pi\alpha\ \cos 2\pi\alpha.$$

note that the amplitude $\mathcal{A}$ of the *normalized* envelope, with respect to the normalized abscissa $\alpha = \omega t/2\pi$, is $\mathcal{A} = \sqrt{1+(2\pi\alpha)^2} \xrightarrow{\text{for large } \alpha} 2\pi\alpha$, as the two components of the response are in *quadrature*.

home work

Derive the expression for the resonant response with $p(t) = p_0 \cos \omega t$, $\omega = \omega_{\mathrm{n}}$.

Part II

Response of the Damped Oscillator to Harmonic Loading

The EoM for a Damped Oscillator

For a *SDOF* damped system, the equation of motion for a harmonic loading is:

$$m\,\ddot{x} + c\,\dot{x} + k\,x = p_0\,\sin\omega t.$$

Its particular solution is a harmonic function not in phase with the input: $\xi(t) = G\sin(\omega t - \theta)$;

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Being $\sin(a - b) = \sin a \cos b - \cos a \sin b$, with

$$G_1 = G\cos\theta, \qquad G_2 = -G\sin\theta,$$

we can write the more convenient representation:

$$\xi(t) = G_1\,\sin\omega t + G_2\,\cos\omega t,$$

where we have two harmonic components in quadrature.
It's easy to derive the parameters G, θ in terms of G_1, G_2:

$$G = \sqrt{G_1^2 + G_2^2}, \qquad \theta = -\arctan\frac{G_2}{G_1}.$$

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The Particular Integral

Substituting $x(t)$ with $\xi(t)$, dividing by m it is

$$\ddot{\xi}(t) + 2\zeta\omega_\mathrm{n}\dot{\xi}(t) + \omega_\mathrm{n}^2\xi(t) = \frac{p_0}{k}\frac{k}{m}\sin\omega t,$$

Substituting the most general expressions for the particular integral and its time derivatives

$$\begin{aligned}
\xi(t) &= G_1 \sin\omega t + G_2 \cos\omega t,\\
\dot{\xi}(t) &= \omega\,(G_1\, cos\omega t - G_2 \sin\omega t),\\
\ddot{\xi}(t) &= -\omega^2\,(G_1 \sin\omega t + G_2 \cos\omega t).
\end{aligned}$$

in the above equation it is

$$-\omega^2\,(G_1 \sin\omega t + G_2 \cos\omega t) + 2\zeta\omega_\mathrm{n}\omega\,(G_1\, cos\omega t - G_2 \sin\omega t) +$$
$$+\omega_\mathrm{n}^2(G_1 \sin\omega t + G_2 \cos\omega t) = \Delta_\mathrm{st}\omega_\mathrm{n}^2 \sin\omega t$$

The particular integral, 2

Dividing our last equation by ω_n^2 and collecting $\sin\omega t$ and $\cos\omega t$ we obtain

$$\big(G_1(1-\beta^2) - G_2 2\beta\zeta\big)\sin\omega t + \\ + \big(G_1 2\beta\zeta + G_2(1-\beta^2)\big)\cos\omega t = \Delta_\mathrm{st}\,\sin\omega t.$$

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Evaluating the eq. above for $t = \pi/2\omega$ and $t = 0$ we obtain a linear system of two equations in G_1 and G_2:

$$G_1(1-\beta^2) - G_2 2\zeta\beta = \Delta_{st}.$$
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$$G_1(1-\beta^2)-G_2 2\zeta\beta=\Delta_\mathrm{st}.$$
$$G_1 2\zeta\beta+G_2(1-\beta^2)=0.$$

The determinant of the linear system is

$$\det=(1-\beta^2)^2+(2\zeta\beta)^2$$

and its solution is

$$G_1=\Delta_\mathrm{st}\frac{(1-\beta^2)}{\det},\qquad G_2=\Delta_\mathrm{st}\frac{-2\zeta\beta}{\det}.$$

The Particular Integral, 3

Substituting G_1 and G_2 in our expression of the particular integral it is

$$\xi(t) = \frac{\Delta_{\text{st}}}{\det}\left((1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t\right).$$

The general integral for $p(t) = p_0 \sin\omega t$ is hence

$$x(t) = \exp(-\zeta\omega_{\text{n}} t)\left(A\, sin\omega_{\text{D}} t + B\, \cos\omega_{\text{D}} t\right) + \\ + \Delta_{\text{st}}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

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For $p(t) = p_{\sin}\sin\omega t + p_{\cos}\cos\omega t$, $\Delta_{\sin} = p_{\sin}/k$, $\Delta_{\cos} = p_{\cos}/k$ it is

$$x(t) = \exp(-\zeta\omega_{\text{n}}t)\left(A\, sin\omega_{\text{D}}t + B\, \cos\omega_{\text{D}}t\right) + \\ + \Delta_{\sin}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det} + \\ + \Delta_{\cos}\frac{(1-\beta^2)\cos\omega t + 2\beta\zeta\sin\omega t}{\det}.$$

Stationary Response

Examination of the general integral

$$x(t) = \exp(-\zeta\omega_\mathrm{n} t)\,(A\,sin\omega_\mathrm{D} t + B\ \cos\omega_\mathrm{D} t) + \\ + \Delta_\mathrm{st} \frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

shows that we have a *transient response*, that depends on the initial conditions and damps out for large values of the argument of the real exponential, and a so called *steady-state response*, corresponding to the particular integral, $x_\text{s-s}(t) \equiv \xi(t)$, that remains constant in amplitude and phase as long as the external loading is being applied.

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shows that we have a *transient response*, that depends on the initial conditions and damps out for large values of the argument of the real exponential, and a so called *steady-state response*, corresponding to the particular integral, $x_\text{s-s}(t) \equiv \xi(t)$, that remains constant in amplitude and phase as long as the external loading is being applied. From an engineering point of view, we have a specific interest in the steady-state response, as it is the long term component of the response.

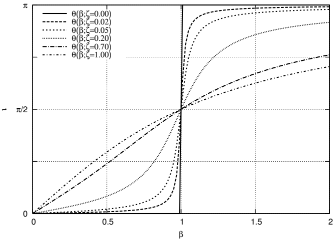
The Angle of Phase

To write the *stationary response* in terms of a *dynamic amplification factor*, it is convenient to reintroduce the amplitude and the phase difference θ and write:

$$\xi(t) = \Delta_{\text{st}}\, R(t; \beta, \zeta), \quad R = D(\beta,\zeta)\, \sin\left(\omega t - \theta\right).$$

Let's start analyzing the phase difference $\theta(\beta,\zeta)$. Its expression is:

$$\theta(\beta,\zeta) = \arctan \frac{2\zeta\beta}{1-\beta^2}.$$

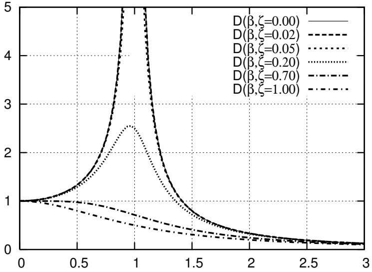


$\theta(\beta,\zeta)$ has a sharper variation around $\beta = 1$ for decreasing values of ζ, but it is apparent that, in the case of slightly damped structures, the response is approximately in phase for low frequencies of excitation, and in opposition for high frequencies. It is worth mentioning that for $\beta = 1$ we have that the response is in perfect quadrature with the load: this is very important to detect resonant response in dynamic tests of structures.

Dynamic Magnification Ratio

The dynamic magnification factor, $D = D(\beta, \zeta)$, is the amplitude of the stationary response normalized with respect to Δ_{st}:

$$D(\beta, \zeta) = \frac{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}{(1-\beta^2)^2 + (2\beta\zeta)^2} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}$$



- $D(\beta, \zeta)$ has larger peak values for decreasing values of ζ,
- the approximate value of the peak, very good for a slightly damped structure, is $1/2\zeta$,
- for larger damping, peaks move toward the origin, until for $\zeta = \frac{1}{\sqrt{2}}$ the peak is in the origin,
- for dampings $\zeta > \frac{1}{\sqrt{2}}$ we have no peaks.

Dynamic Magnification Ratio (2)

The location of the response peak is given by the equation

$$\frac{d\,D(\beta,\zeta)}{d\,\beta} = 0, \quad \Rightarrow \quad \beta^3 + (2\zeta^2 - 1)\beta = 0$$

the 3 roots are

$$\beta_i = 0, \pm\sqrt{1 - 2\zeta^2}.$$

We are interested in a real, positive root, so we are restricted to $0 < \zeta \le \frac{1}{\sqrt{2}}$. In this interval, substituting $\beta = \sqrt{1 - 2\zeta^2}$ in the expression of the response ratio, we have

$$D_{\max} = \frac{1}{2\zeta} \frac{1}{\sqrt{1 - \zeta^2}}.$$

For $\zeta = \frac{1}{\sqrt{2}}$ there is a maximum corresponding to $\beta = 0$. Note that, for a relatively large damping ratio, $\zeta = 20\%$, the error of $1/2\zeta$ with respect to $D_{\max}$ is in the order of 2%.

Harmonic Exponential Load

Consider the *EOM* for a load modulated by an exponential of imaginary argument:

$$\ddot{x} + 2\zeta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = \Delta_\mathrm{st}\omega_\mathrm{n}^2 \exp(i(\omega t - \phi)).$$

Note that the phase can be disregarded as we can represent its effects with a constant factor, as it is $\exp(i(\omega t - \phi)) = \exp(i\omega t)/\exp(i\phi)$.

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The particular solution and its derivatives are

$$\xi = G\exp(i\omega t), \quad \dot{\xi} = i\omega G\exp(i\omega t), \quad \ddot{\xi} = -\omega^2 G\exp(i\omega t),$$

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$$G = \Delta_\mathrm{st}\left[\frac{1}{(1-\beta^2) + i(2\zeta\beta)}\right] = \Delta_\mathrm{st}\left[\frac{(1-\beta^2) - i(2\zeta\beta)}{(1-\beta^2)^2 + (2\zeta\beta)^2}\right].$$

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Notice how simpler it is to represent the stationary response of a damped oscillator using exponential harmonics.

Measuring Support Accelerations

We have seen that in seismic analysis the loading is proportional to the ground acceleration.
A simple oscillator, when properly damped, may serve the scope of measuring support accelerations.

Measuring Support Accelerations, 2

With the equation of motion valid for a harmonic support acceleration:

$$\ddot{x} + 2\zeta\beta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = -a_g \sin\omega t,$$

the stationary response is $\xi = \frac{m\,a_g}{k}\,D(\beta,\zeta)\,\sin(\omega t - \theta)$.
If the damping ratio of the oscillator is $\zeta \approxeq 0.7$, then the

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Books have been written on the *correction* of accelerometers' records to most accurately recover the real ground acceleration.

Measuring Displacements

Consider now a harmonic *displacement* of the support, $u_g(t) = u_g \sin \omega t$.
The support acceleration (disregarding the sign) is $a_g(t) = \omega^2 u_g \sin \omega t$.

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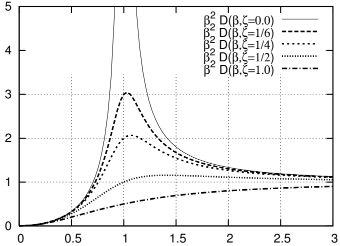
We see that the displacement of the instrument is approximately equal to the support displacement for all the excitation frequencies greater than the natural frequency of the instrument, for a damping ratio $\zeta \approxeq .5$.



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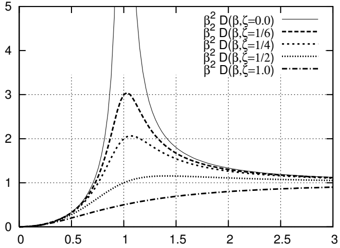


It is possible to measure the support displacement measuring the deflection of the oscillator, within an almost constant scale factor, for excitation frequencies larger than ω_n.

Part III

Vibration Isolation

Vibration Isolation

Vibration isolation is a subject too broad to be treated in detail, we'll present the basic principles involved in two problems,

1. prevention of harmful vibrations in supporting structures due to oscillatory forces produced by operating equipment,
2. prevention of harmful vibrations in sensitive instruments due to vibrations of their supporting structures.

Force Isolation

Consider a rotating machine that produces an oscillatory force $p_0 \sin \omega t$ due to unbalance in its rotating part, that has a total mass m and is mounted on a spring-damper support.

Its steady-state relative displacement is given by

$$x_{\text{s-s}} = \frac{p_0}{k}\, D \sin(\omega t - \theta).$$

This result depend on the assumption that the supporting structure deflections are negligible respect to the relative system motion.

The steady-state spring and damper forces are

$$f_S = k\, x_{\text{ss}} = p_0\, D \sin(\omega t - \theta),$$

$$f_D = c\, \dot{x}_{\text{ss}} = \frac{c p_0\, D\, \omega}{k} \cos(\omega t - \theta) = 2\, \zeta\, \beta\, p_0\, D \cos(\omega t - \theta).$$

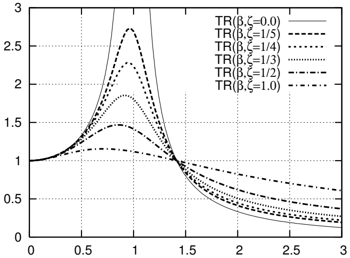
Transmitted force

The spring and damper forces are in quadrature, so the amplitude of the steady-state reaction force is given by

$$f_{\text{max}} = p_0\, D\, \sqrt{1 + (2\zeta\beta)^2}$$

The ratio of the maximum transmitted force to the amplitude of the applied force is the *transmissibility ratio* (TR),

$$\text{TR} = \frac{f_{\text{max}}}{p_0} = D\, \sqrt{1 + (2\zeta\beta)^2}.$$



1. For $\beta < \sqrt{2}$, TR is always greater than 1: the transmitted force is amplified. 2. For $\beta > \sqrt{2}$, TR is always smaller than 1 and for the same β TR decreases with ζ.

Displacement Isolation

Another problem concerns the harmonic support motion $u_g(t) = u_{g_0} \exp i\omega t$ forcing a steady-state relative displacement of some supported (spring+damper) equipment of mass m (using exp notation) $x_{ss} = u_{g_0}\beta^2 D \exp i\omega t$, and the mass total displacement is given by

$$x_{\text{tot}} = x_{\text{s-s}} + u_g(t) = u_{g_0}\left(\frac{\beta^2}{(1-\beta^2) + 2\,i\,\zeta\beta} + 1\right) \exp i\omega t$$

$$= u_{g_0}\,(1 + 2i\zeta\beta)\frac{1}{(1-\beta^2) + 2\,i\,\zeta\beta} \exp i\omega t$$

but $1 + 2i\zeta\beta = \text{abs}(1 + 2i\zeta\beta) \exp i\varphi$ so

$$x_{\text{tot}} = u_{g_0}\sqrt{1 + (2\zeta\beta)^2}\, D \exp\, i\,(\omega t + \varphi).$$

If we define the transmissibility ratio TR as the ratio of the maximum total response to the support displacement amplitude, we find that, as in the previous case,

$$\text{TR} = D\sqrt{1 + (2\zeta\beta)^2}.$$

Isolation Effectiveness

Define the isolation effectiveness,

$$\mathrm{IE} = 1 - \mathrm{TR},$$

IE=1 means complete isolation, i.e., $\beta = \infty$, while IE=0 is no isolation, and takes place for $\beta = \sqrt{2}$.
As effective isolation requires low damping, we can approximate $\mathrm{TR} \approxeq 1/(\beta^2 - 1)$, in which case we have $\mathrm{IE} = (\beta^2 - 2)/(\beta^2 - 1)$.
Solving for β^2, we have $\beta^2 = (2 - \mathrm{IE})/(1 - \mathrm{IE})$, but

$$\beta^2 = \omega^2/\omega_\mathrm{n}^2 = \omega^2\,(m/k) = \omega^2\,(W/gk) = \omega^2\,(\Delta_\mathrm{st}/g)$$

where W is the weight of the mass and Δ_st is the static deflection under self weight. Finally, from $\omega = 2\pi f$ we have

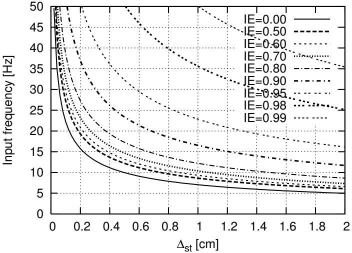
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_\mathrm{st}}\,\frac{2 - \mathrm{IE}}{1 - \mathrm{IE}}}$$

Isolation Effectiveness (2)

The strange looking

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{\mathrm{st}}}\frac{2-\mathrm{IE}}{1-\mathrm{IE}}}$$

can be plotted f vs Δ_{st} for different values of IE, obtaining a design chart.



Knowing the frequency of excitation and the required level of vibration isolation efficiency (IE), one can determine the minimum static deflection (proportional to the spring flexibility) required to achieve the required IE. It is apparent that any isolation system must be very flexible to be effective.

Part IV

Evaluation of Viscous Damping Ratio

Evaluation of damping

The mass and stiffness of phisycal systems of interest are usually evaluated easily, but this is not feasible for damping, as the energy is dissipated by different mechanisms, some one not fully understood... it is even possible that dissipation cannot be described in term of viscous-damping, But it generally is possible to measure an equivalent viscous-damping ratio by experimental methods:

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- free-vibration decay method,
- resonant amplification method,
- half-power (bandwidth) method,
- resonance cyclic energy loss method.

Free vibration decay

We already have discussed the free-vibration decay method,

$$\zeta = \frac{\delta_m}{2\pi\, m\, (\omega_\mathrm{n}/\omega_D)}$$

with $\delta_m = \ln \frac{x_n}{x_{n+m}}$, *logarithmic decrement*. The method is simple and its requirements are minimal, but some care must be taken in the interpretation of free-vibration tests, because the damping ratio decreases with decreasing amplitudes of the response, meaning that for a very small amplitude of the motion the effective values of the damping can be underestimated.

Resonant amplification

This method assumes that it is possible to measure the stiffness of the structure, and that damping is small. The experimenter (*a*) measures the steady-state response x_{ss} of a SDOF system under a harmonic loading for a number of different excitation frequencies (eventually using a smaller frequency step when close to the resonance), (*b*) finds the maximum value $D_{max} = \frac{\max\{x_{ss}\}}{\Delta_{st}}$ of the dynamic magnification factor, (*c*) uses the approximate expression (good for small ζ) $D_{max} = \frac{1}{2\zeta}$ to write

$$D_{max} = \frac{1}{2\zeta} = \frac{\max\{x_{ss}\}}{\Delta_{st}}$$

and finally (*d*) has

$$\zeta = \frac{\Delta_{st}}{2\max\{x_{ss}\}}.$$

The most problematic aspect here is getting a good estimate of Δ_{st}, if the results of a static test aren't available.

Half Power

The adimensional frequencies where the response is $1/\sqrt{2}$ times the peak value can be computed from the equation

$$\frac{1}{\sqrt{(1-\beta^2)^2+(2\beta\zeta)^2}}=\frac{1}{\sqrt{2}}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

squaring both sides and solving for β^2 gives

$$\beta_{1,2}^2=1-2\zeta^2\mp 2\zeta\sqrt{1-\zeta^2}$$

For small ζ we can use the binomial approximation and write

$$\beta_{1,2}=\left(1-2\zeta^2\mp 2\zeta\sqrt{1-\zeta^2}\right)^{\frac{1}{2}}\approxeq 1-\zeta^2\mp\zeta\sqrt{1-\zeta^2}$$

Half power (2)

From the approximate expressions for the difference of the half power frequency ratios,

$$\beta_2 - \beta_1 = 2\zeta\sqrt{1-\zeta^2} \approxeq 2\zeta$$

and their sum

$$\beta_2 + \beta_1 = 2(1-\zeta^2) \approxeq 2$$

we can deduce that

$$\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1} = \frac{f_2 - f_1}{f_2 + f_1} \approxeq \frac{2\zeta\sqrt{1-\zeta^2}}{2(1-\zeta^2)} \approxeq \zeta, \text{ or } \zeta \approxeq \frac{f_2 - f_1}{f_2 + f_1}$$

where f_1, f_2 are the frequencies at which the steady state amplitudes equal $1/\sqrt{2}$ times the peak value, frequencies that can be determined from a dynamic test where detailed test data is available.

Resonance Cyclic Energy Loss

If it is possible to determine the phase of the s-s response, it is possible to measure ζ from the amplitude ρ of the resonant response.
At resonance, the deflections and accelerations are in quadrature with the excitation, so that the external force is equilibrated *only* by the viscous force, as both elastic and inertial forces are also in quadrature with the excitation.
The equation of dynamic equilibrium is hence:

$$p_0 = c\,\dot{x} = 2\zeta\omega_n m\,(\omega_n\rho).$$

Solving for ζ we obtain:

$$\zeta = \frac{p_0}{2m\omega_n^2\rho}.$$