SDOF linear oscillator

Response to Periodic and Non-periodic Loadings

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Outline

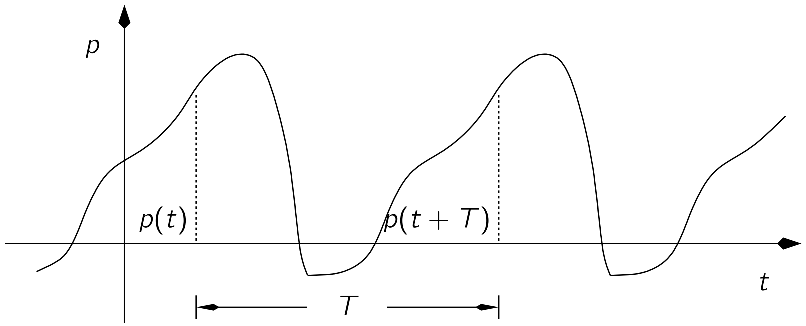
Response to Periodic Loading

Introduction

A periodic loading is characterized by the identity

$$p(t) = p(t + T)$$

where T is the *period* of the loading, and $\omega_1 = \frac{2\pi}{T}$ is its *principal frequency*.



Note that a function with period T/n is also periodic with period T.

Introduction

Periodic loadings can be expressed as an infinite series of harmonic functions using the Fourier theorem, e.g., for an antisymmetric loading you can write

$$p(t) = -p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t.$$

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_n$ it is:

$$x_{j,\text{s-s}} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)).$$

In general, it is possible to sum all steady-state responses, the infinite series giving the *SDOF* response to $p(t)$.
Due to the asymptotic behaviour of $D(\beta; \zeta)$ (D goes to zero for large, increasing β) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Fourier Series Coefficients

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.
Consider a loading of period T_p, its Fourier series is given by

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \omega_j t + \sum_{j=1}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j\,\omega_1 = j\frac{2\pi}{T_\text{p}},$$

where the harmonic amplitude coefficients have expressions:

$$a_0 = \frac{1}{T_\text{p}} \int_0^{T_\text{p}} p(t)\,\mathrm{d}t, \quad a_j = \frac{2}{T_\text{p}} \int_0^{T_\text{p}} p(t) \cos \omega_j t\,\mathrm{d}t,$$
$$b_j = \frac{2}{T_\text{p}} \int_0^{T_\text{p}} p(t) \sin \omega_j t\,\mathrm{d}t,$$

as, by orthogonality,
$\int_o^{T_\text{p}} p(t) cos\omega_j\,\mathrm{d}t = \int_o^{T_\text{p}} a_j \cos^2 \omega_j t\,\mathrm{d}t = \frac{T_\text{p}}{2} a_j$, etc etc.

Sampled Periodic Functions

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

(a) to divide the period in N equal parts $\Delta t = T_\text{p}/N$,
(b) measure or compute $p(t)$ at a discrete set of instants $t_1, t_2, \ldots, t_N$, with $t_m = m\Delta t$,

obtaining a discrete set of values $p_m,\ m = 1, \ldots, N$ (note that $p_0 = p_N$ by periodicity).
Using the trapezoidal rule of integration, with $p_0 = p_N$ we can write, for example, the cosine-wave amplitude coefficients,

$$a_j \cong \frac{2\Delta t}{T_\text{p}} \sum_{m=1}^{N} p_m \cos \omega_j t_m$$
$$= \frac{2}{N} \sum_{m=1}^{N} p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^{N} p_m \cos \frac{jm\,2\pi}{N}.$$

It's worth to note that the discrete function $\cos \frac{jm\,2\pi}{N}$ is periodic with period N.

Periodicity

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

$$\cos \frac{(n+N)\,2\pi}{N} = \cos\left(\frac{n\,2\pi}{N} + 2\pi\right) = \cos \frac{n\,2\pi}{N}$$

$$a_{j+N} = \frac{2}{N} \sum p_m \cos \frac{(j+N)\,m\,2\pi}{N}$$
$$= \frac{2}{N} \sum p_m \cos \frac{(jm + Nm)\,2\pi}{N}$$
$$= \frac{2}{N} \sum p_m \cos \left(\frac{jm\,2\pi}{N} + m\,2\pi\right)$$

$$a_{j+N} = \frac{2}{N} \sum p_m \cos \frac{jm\,2\pi}{N} = a_j.$$

Exponential Form

The Fourier series can be written in terms of the exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_\mathrm{p}} \int_0^{T_\mathrm{p}} p(t) \exp i\omega_j t \,\mathrm{d}t, \qquad j = -\infty, \ldots, +\infty.$$

For a sampled p_m we can write, using the trapezoidal integration rule and substituting $t_m = m\Delta t = m\,T_\mathrm{p}/N$, $\omega_j = j\,2\pi/T_\mathrm{p}$:

$$P_j \approxeq \frac{1}{N} \sum_{m=1}^{N} p_m \exp(-i\frac{2\pi\, j\, m}{N}),$$

Undamped Response

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

We have seen that the steady-state response to the jth sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\sin\omega_j t, \qquad \beta_j = \omega_j/\omega_\mathrm{n},$$

analogously, for the jth cosine-wave harmonic,

$$x_j = \frac{a_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\cos\omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k}\left\{a_0 + \sum_{j=1}^{\infty}\left[\frac{1}{1-\beta_j^2}\right](a_j\cos\omega_j t + b_j \sin\omega_j t)\right\}.$$

Damped Response

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

In the case of a damped oscillator, we must substitute the steady state response for both the jth sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k}\sum_{j=1}^{\infty}\frac{+(1-\beta_j^2)\,a_j - 2\zeta\beta_j\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\cos\omega_j t + \\ + \frac{1}{k}\sum_{j=1}^{\infty}\frac{+2\zeta\beta_j\,a_j + (1-\beta_j^2)\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\sin\omega_j t.$$

As usual, the exponential notation is neater,

$$x(t) = \sum_{j=-\infty}^{\infty}\frac{P_j}{k}\frac{\exp i\omega_j t}{(1-\beta_j^2)+i\,(2\zeta\beta_j)}.$$

Example

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

As an example, consider the loading $p(t) = \max\{p_0 \sin\frac{2\pi t}{T_\mathrm{p}},\ 0\}$

$$a_0 = \frac{1}{T_\mathrm{p}}\int_0^{T_\mathrm{p}/2} p_o \sin\frac{2\pi t}{T_\mathrm{p}}\,\mathrm{d}t = \frac{p_0}{\pi},$$

$$a_j = \frac{2}{T_\mathrm{p}}\int_0^{T_\mathrm{p}/2} p_o \sin\frac{2\pi t}{T_\mathrm{p}}\cos\frac{2\pi j t}{T_\mathrm{p}}\,\mathrm{d}t = \begin{cases} 0 & \text{for } j \text{ odd} \\ \frac{p_0}{\pi}\left[\frac{2}{1-j^2}\right] & \text{for } j \text{ even,}\end{cases}$$

$$b_j = \frac{2}{T_\mathrm{p}}\int_0^{T_\mathrm{p}/2} p_o \sin\frac{2\pi t}{T_\mathrm{p}}\sin\frac{2\pi j t}{T_\mathrm{p}}\,\mathrm{d}t = \begin{cases} \frac{p_0}{2} & \text{for } j = 1 \\ 0 & \text{for } n > 1.\end{cases}$$

Example cont.

Assuming $\beta_1 = 3/4$, from $p = \frac{p_0}{\pi}\left(1 + \frac{\pi}{2}\sin\omega_1 t - \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_2 t - \ldots\right)$ with the dynamic amplifiction factors

$$D_1 = \frac{1}{1-(1\frac{3}{4})^2} = \frac{16}{7},$$

$$D_2 = \frac{1}{1-(2\frac{3}{4})^2} = -\frac{4}{5},$$

$$D_4 = \frac{1}{1-(4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \ldots$$

etc, we have

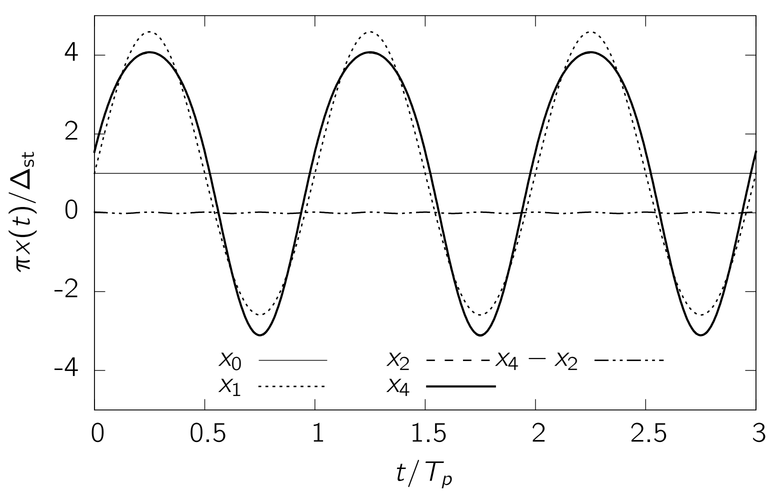
$$x(t) = \frac{p_0}{k\pi}\left(1 + \frac{8\pi}{7}\sin\omega_1 t + \frac{8}{15}\cos 2\omega_1 t + \frac{1}{60}\cos 4\omega_1 t + \ldots\right)$$

Take note, these solutions are particular solutions! If your solution has to respect given initial conditions, you must consider also the homogeneous solution.

Example cont.

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Introduction
Fourier Series Representation
Fourier Series of the Response
An example
An example
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings

$x_i = \sum_i a_i \cos\omega_i t + b + i \sin\omega_i t$



Outline of Fourier transform

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Fourier Transform
Extension of Fourier Series to non periodic functions
Response in the Frequency Domain
The Discrete Fourier Transform
Response to General Dynamic Loadings

Non periodic loadings

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Fourier Transform
Extension of Fourier Series to non periodic functions
Response in the Frequency Domain
The Discrete Fourier Transform
Response to General Dynamic Loadings

It is possible to extend the Fourier analysis to non periodic loading. Let's start from the Fourier series representation of the load $p(t)$,

$$p(t) = \sum_{-\infty}^{+\infty} P_r \exp(i\omega_r t), \quad \omega_r = r\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T_p},$$

introducing $P(i\omega_r) = P_r T_p$ and substituting,

$$p(t) = \frac{1}{T_p}\sum_{-\infty}^{+\infty} P(i\omega_r)\exp(i\omega_r t) = \frac{\Delta\omega}{2\pi}\sum_{-\infty}^{+\infty} P(i\omega_r)\exp(i\omega_r t).$$

Due to periodicity, we can modify the extremes of integration in the expression for the complex amplitudes,

$$P(i\omega_r) = \int_{-T_p/2}^{+T_p/2} p(t)\exp(-i\omega_r t)\, dt.$$

Non periodic loadings (2)

If the loading period is extended to infinity to represent the non-periodicity of the loading ($T_p \to \infty$) then (a) the frequency increment becomes infinitesimal ($\Delta\omega = \frac{2\pi}{T_p} \to d\omega$) and ($b$) the discrete frequency ω_r becomes a continuous variable, ω.
In the limit, for $T_p \to \infty$ we can then write

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(i\omega)\exp(i\omega t)\,d\omega$$
$$P(i\omega) = \int_{-\infty}^{+\infty} p(t)\exp(-i\omega t)\,dt,$$

which are known as the inverse and the direct Fourier Transforms, respectively, and are collectively known as the Fourier transform pair.

SDOF Response

In analogy to what we have seen for periodic loads, the response of a damped SDOF system can be written in terms of $H(i\omega)$, the complex frequency response function,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(i\omega)\,P(i\omega)\exp i\omega t\,dt, \quad \text{where}$$

$$H(i\omega) = \frac{1}{k}\left[\frac{1}{(1-\beta^2)+i(2\zeta\beta)}\right] = \frac{1}{k}\left[\frac{(1-\beta^2)-i(2\zeta\beta)}{(1-\beta^2)^2+(2\zeta\beta)^2}\right], \quad \beta = \frac{\omega}{\omega_n}.$$

To obtain the response *through frequency domain*, you should evaluate the above integral, but analytical integration is not always possible and also when it is possible it is usually very difficult, implying contour integration in the complex plane (e.g., the Example **E6-3** in Clough Penzien presents a detailed derivation).

Outline of the Discrete Fourier Transform

Discrete Fourier Transform

To overcome the analytical difficulties associated with the inverse Fourier transform, one can use appropriate numerical methods, leading to good approximations.
Consider a loading of finite period T_p, divided into N equal intervals $\Delta t = T_p/N$, and the set of values $p_s = p(t_s) = p(s\Delta t)$. We can approximate the complex amplitude coefficients with a sum,

$$P_r = \frac{1}{T_p}\int_0^{T_p} p(t)\exp(-i\omega_r t)\,dt, \quad \text{that, by trapezoidal rule, is}$$
$$\approx \frac{1}{N\Delta t}\left(\Delta t\sum_{s=0}^{N-1} p_s\exp(-i\omega_r t_s)\right) = \frac{1}{N}\sum_{s=0}^{N-1} p_s\exp(-i\frac{2\pi rs}{N}).$$

Discrete Fourier Transform (2)

In the last two passages we have used the relations

$$p_N = p_0, \quad \exp(i\omega_r t_N) = \exp(ir\Delta\omega T_p) = \exp(ir2\pi) = \exp(i0)$$

$$\omega_r\, t_s = r\Delta\omega\, s\Delta t = rs\,\frac{2\pi}{T_p}\,\frac{T_p}{N} = \frac{2\pi\, rs}{N}.$$

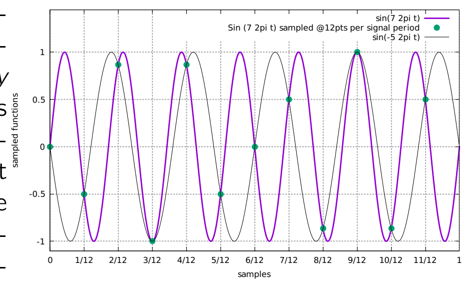
Take note that the discrete function $\exp(-i\frac{2\pi rs}{N})$, defined for integer r, s is periodic with period N, implying that the complex amplitude coefficients are themselves periodic with period N.

$$P_{r+N} = P_r$$

Starting in the time domain with N distinct complex numbers, p_s, we have found that in the frequency domain our load is described by N distinct complex numbers, P_r, so that we can say that our function is described by the same amount of information in both domains.

Aliasing

Only $N/2$ distinct frequencies ($\sum_0^{N-1} = \sum_{-N/2}^{+N/2}$) contribute to the load representation, what if the *frequency content* of the loading has contributions from frequencies higher than $\omega_{N/2}$? What happens is *aliasing*, i.e., the upper frequencies contributions are mapped to contributions of lesser frequency.



See the plot above: the contributions from the high frequency sines, *when sampled*, are indistinguishable from the contributions from lower frequency components, i.e., are *aliased* to lower frequencies!

Aliasing (2)

- The maximum frequency that can be described in the DFT is called the Nyquist frequency, $\omega_{\text{Ny}} = \frac{1}{2}\frac{2\pi}{\Delta t}$.
- It is usual in signal analysis to remove the signal's higher frequency components preprocessing the signal with a *filter* or a *digital filter*.
- It is worth noting that the *resolution* of the DFT in the frequency domain for a given sampling rate is proportional to the number of samples, i.e., to the duration of the sample.

The Fast Fourier Transform

The operation count in a DFT is in the order of N^2
A Fast Fourier Transform is an algorithm that reduces the operation count.
The first and simpler FFT algorithm is the *Decimation in Time* algorithm by Tukey and Cooley (1965).

Tukey and Cooley, 1965.

Assume N is even, and divide the DFT summation to consider even and odd indices s

$$X_r = \sum_{s=0}^{N-1} x_s e^{-\frac{2\pi i}{N} sr}, \qquad r = 0, \ldots, N-1$$

$$= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N}(2q)r} + \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N}(2q+1)r}$$

collecting $e^{-\frac{2\pi i}{N} r}$ in the second term and letting $\frac{2q}{N} = \frac{q}{N/2}$

$$= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N/2} qr} + e^{-\frac{2\pi i}{N} r} \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N/2} qr}$$

We have two DFT's of length $N/2$, the operations count is hence $2(N/2)^2 = N^2/2$, but we have to combine these two halves in the full DFT.

Tukey and Cooley, 1965.

Say that

$$X_r = E_r + e^{-\frac{2\pi i}{N} r} O_r$$

where E_r and O_r are the even and odd half-DFT's, of which we computed only coefficients from 0 to $N/2-1$.
To get the full sequence we have to note that

1. the E and O DFT's are periodic with period $N/2$, and
2. $\exp(-2\pi i(r+N/2)/N) = e^{-\pi i} \exp(-2\pi i r/N) = -\exp(-2\pi i r/N)$,

so that we can write

$$X_r = \begin{cases} E_r + \exp(-2\pi i r/N) O_r & \text{if } r < N/2, \\ E_{r-N/2} - \exp(-2\pi i r/N) O_{r-N/2} & \text{if } r \geq N/2. \end{cases}$$

The algorithm that was outlined can be applied to the computation of each of the half-DFT's when $N/2$ were even, so that the operation count goes to $N^2/4$. If $N/4$ were even …

Pseudocode for CT algorithm


```
def fft2(X, N):
  if N = 1 then
     Y = X
  else
     Y0 = fft2(X0, N/2)
     Y1 = fft2(X1, N/2)
     for k = 0 to N/2-1
       Y_k        = Y0_k + exp(2 pi i k/N) Y1_k
       Y_(k+N/2)  = Y0_k - exp(2 pi i k/N) Y1_k
     endfor
  endif
return Y
```



```
from cmath import exp, pi

def d_fft(x,n):
    """Direct fft of x, a list of n=2**m complex values"""
    return _fft(x,n,[exp(-2*pi*1j*k/n) for k in range(n/2)])

def i_fft(x,n):
    """Inverse fft of x, a list of n=2**m complex values"""
    transform = _fft(x,n,[exp(+2*pi*1j*k/n) for k in range(n/2)])
    return [x/n for x in transform]
def _fft(x, n, twiddle):
    """Decimation in Time FFT, to be called by d_fft and i_fft.
    x   is the signal to transform, a list of complex values
    n   is its length, results are undefined if n is not a power of 2
    tw  is a list of twiddle factors, precomputed by the caller

    returns a list of complex values, to be normalized in case of an
    inverse transform"""

    if n == 1: return x # bottom reached, DFT of a length 1 vec x is x

    # call fft with the even and the odd coefficients in x
    # the results are the so called even and odd DFT's
    y_0 = _fft(x[0::2], n/2, tw[::2])
    y_1 = _fft(x[1::2], n/2, tw[::2])

    # assemble the partial results "in_place":
    # 1st half of full DFT is put in even DFT, 2nd half in odd DFT
    for k in range(n/2):
        y_0[k], y_1[k] = y_0[k]+tw[k]*y_1[k], y_0 [k]-tw[k]*y_1[k]

    # concatenate the two halves of the DFT and return to caller
    return y_0+y_1
```

```
def main():
    """Run some test cases"""
    from cmath import cos, sin, pi

    def testit(title, seq):
        """utility to format and print a vector and the ifft of its fft"""
        l_seq = len(seq)
        print "-"*5, title, "-"*5
        print "\n".join([
          "%10.6f :: %10.6f, %10.6fj" % (a.real, t.real, t.imag)
          for (a, t) in zip(seq, i_fft(d_fft(seq, l_seq), l_seq))
          ])

    length = 32

    testit("Square wave", [+1.0+0.0j]*(length/2) + [-1.0+0.0j]*(length/2))
    testit("Sine wave",   [sin((2*pi*k)/length) for k in range(length)])
    testit("Cosine wave", [cos((2*pi*k)/length) for k in range(length)])

if __name__ == "__main__":
    main()
```


Dynamic Response (1)

To evaluate the dynamic response of a linear SDOF system in the frequency domain, use the inverse DFT,

$$x_s = \sum_{r=0}^{N-1} V_r \exp(i\frac{2\pi\, rs}{N}), \quad s = 0, 1, \ldots, N-1$$

where $V_r = H_r\, P_r$. P_r are the discrete complex amplitude coefficients computed using the direct DFT, and H_r is the discretization of the complex frequency response function, that for viscous damping is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2)+i(2\zeta\beta_r)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2)-i(2\zeta\beta_r)}{(1-\beta_r^2)^2+(2\zeta\beta_r)^2}\right], \quad \beta_r = \frac{\omega_r}{\omega_n}.$$

while for hysteretic damping is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2)+i(2\zeta)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2)-i(2\zeta)}{(1-\beta_r^2)^2+(2\zeta)^2}\right].$$

Some words of caution

If you're going to approach the application of the complex frequency response function without proper concern, you're likely to be hurt.
Let's say $\Delta\omega = 1.0$, $N = 32$, $\omega_\text{n} = 3.5$ and $r = 30$, what do you think it is the value of β_{30}? If you are thinking $\beta_{30} = 30\,\Delta\omega/\omega_\text{n} = 30/3.5 \approx 8.57$ you're wrong!
Due to aliasing, $\omega_r = \begin{cases} r\Delta\omega & r \le N/2 \\ (r-N)\Delta\omega & r > N/2 \end{cases}$,
note that in the upper part of the DFT the coefficients correspond to negative frequencies and, staying within our example, it is $\beta_{30} = (30-32)\times 1/3.5 \approx -0.571$.
If N is even, $P_{N/2}$ is the coefficient corresponding to the Nyquist frequency, if N is odd $P_{\frac{N-1}{2}}$ corresponds to the largest positive frequency, while $P_{\frac{N+1}{2}}$ corresponds to the largest negative frequency.

Response to General Dynamic Loading

Response to a short duration load

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)]\,\mathrm{d}t.$$

When one notes that, for small t_0, the displacement is of the order of t_0^2 while the velocity is in the order of t_0, it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approxeq \int_0^{t_0} p(t)\,\mathrm{d}t.$$

Response to a short duration load

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Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m}\int_0^{t_0} p(t)\,\mathrm{d}t,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{1}{m\omega_n}\int_0^{t_0} p(t)\,\mathrm{d}t\,\ \sin\omega_n(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading.

Undamped SDOF

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For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau)\,d\tau$ and the response is

$$dx(t-\tau) = \frac{p(\tau)\,d\tau}{m\omega_n}\sin\omega_n(t-\tau), \qquad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\,\sin\omega_n(t-\tau)\,d\tau, \qquad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Damped SDOF

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The derivation of the equation of motion for a generic load is analogous to what we have seen for undamped SDOF, the infinitesimal contribution to the response at time t of the load at time τ is

$$dx(t) = \frac{p(\tau)}{m\omega_D}\,d\tau\,\sin\omega_D(t-\tau)\exp(-\zeta\omega_n(t-\tau)) \qquad t \geq \tau$$

and integrating all infinitesimal contributions one has

$$x(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\,\sin\omega_D(t-\tau)\exp(-\zeta\omega_n(t-\tau))\,d\tau, \qquad t \geq 0.$$

Evaluation of Duhamel integral, undamped

Using the trig identity

$$\sin(\omega_n t - \omega_n \tau) = \sin\omega_n t \cos\omega_n \tau - \cos\omega_n t \sin\omega_n \tau$$

the Duhamel integral is rewritten as

$$\begin{aligned} x(t) &= \frac{\int_0^t p(\tau)\cos\omega_n\tau\,d\tau}{m\omega_n}\sin\omega_n t - \frac{\int_0^t p(\tau)\sin\omega_n\tau\,d\tau}{m\omega_n}\cos\omega_n t \\ &= \mathcal{A}(t)\sin\omega_n t - \mathcal{B}(t)\cos\omega_n t \end{aligned}$$

where

$$\begin{cases} \mathcal{A}(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau\,d\tau \\ \mathcal{B}(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n\tau\,d\tau \end{cases}$$

Numerical evaluation of Duhamel integral, undamped

Usual numerical procedures can be applied to the evaluation of $\mathcal{A}$ and $\mathcal{B}$, e.g., using the trapezoidal rule, one can have, with $\mathcal{A}_N = \mathcal{A}(N\Delta\tau)$ and $y_N = p(N\Delta\tau)\cos(N\Delta\tau)$

$$\mathcal{A}_{N+1} = \mathcal{A}_N + \frac{\Delta\tau}{2m\omega_n}(y_N + y_{N+1})\,.$$

Evaluation of Duhamel integral, damped

For a damped system, it can be shown that

$$x(t) = \mathcal{A}(t)\sin\omega_D t - \mathcal{B}(t)\cos\omega_D t$$

with

$$\mathcal{A}(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\frac{\exp\zeta\omega_n\tau}{\exp\zeta\omega_n t}\cos\omega_D\tau\,d\tau,$$

$$\mathcal{B}(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\frac{\exp\zeta\omega_n\tau}{\exp\zeta\omega_n t}\sin\omega_D\tau\,d\tau.$$

Numerical evaluation of Duhamel integral, damped

Numerically, using e.g. Simpson integration rule and $y_N = p(N\Delta\tau)\cos\omega_D\tau$,

$$\begin{aligned} &\mathcal{A}_{N+2} = \mathcal{A}_N \exp(-2\zeta\omega_n\Delta\tau) + \\ &\frac{\Delta\tau}{3m\omega_D}\left[y_N\exp(-2\zeta\omega_n\Delta\tau) + 4y_{N+1}\exp(-\zeta\omega_n\Delta\tau) + y_{N+2}\right] \\ &\qquad\qquad N = 0, 2, 4, \cdots \end{aligned}$$

Transfer Functions

The response of a linear SDOF system to arbitrary loading can be evaluated by a convolution integral in the time domain,

$$x(t) = \int_0^t p(\tau)\, h(t-\tau)\, d\tau,$$

with the unit impulse response function $h(t) = \frac{1}{m\omega_D} \exp(-\zeta\omega_n t)\, \sin(\omega_D t)$, or through the frequency domain using the Fourier integral

$$x(t) = \int_{-\infty}^{+\infty} H(\omega) P(\omega) \exp(i\omega t)\, d\omega,$$

where $H(\omega)$ is the complex frequency response function.

Transfer Functions

SDOF linear oscillator
Giacomo Boffi
Response to Periodic Loading
Fourier Transform
The Discrete Fourier Transform
Response to General Dynamic Loadings
Response to infinitesimal impulse
Numerical integration of Duhamel integral
Undamped SDOF systems
Damped SDOF systems
Relationship between time and frequency domain

These response functions, or *transfer* functions, are connected by the direct and inverse Fourier transforms:

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\, \exp(-i\omega t)\, dt,$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega)\, \exp(i\omega t)\, d\omega.$$

Relationship of transfer functions

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Giacomo Boffi
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Fourier Transform
The Discrete Fourier Transform
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We write the response and its Fourier transform:

$$x(t) = \int_0^t p(\tau) h(t-\tau)\, d\tau = \int_{-\infty}^t p(\tau) h(t-\tau)\, d\tau$$

$$X(\omega) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^t p(\tau) h(t-\tau)\, d\tau \right] \exp(-i\omega t)\, dt$$

the lower limit of integration in the first equation was changed from 0 to $-\infty$ because $p(\tau) = 0$ for $\tau < 0$, and since $h(t-\tau) = 0$ for $\tau > t$, the upper limit of the second integral in the second equation can be changed from t to $+\infty$,

$$X(\omega) = \lim_{s\to\infty} \int_{-s}^{+s} \int_{-s}^{+s} p(\tau) h(t-\tau) \exp(-i\omega t)\, dt\, d\tau$$

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Introducing a new variable $\theta = t - \tau$ we have

$$X(\omega) = \lim_{s\to\infty} \int_{-s}^{+s} p(\tau) \exp(-i\omega\tau)\, d\tau \int_{-s-\tau}^{+s-\tau} h(\theta) \exp(-i\omega\theta)\, d\theta$$

with $\lim_{s\to\infty} s - \tau = \infty$, we finally have

$$X(\omega) = \int_{-\infty}^{+\infty} p(\tau) \exp(-i\omega\tau)\, d\tau \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta$$

$$= P(\omega) \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta$$

where we have recognized that the first integral is the Fourier transform of $p(t)$.

Relationship of transfer functions

Our last relation was

$$X(\omega) = P(\omega) \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta$$

but $X(\omega) = H(\omega)P(\omega)$, so that, noting that in the above equation the last integral is just the Fourier transform of $h(\theta)$, we may conclude that, effectively, $H(\omega)$ and $h(t)$ form a Fourier transform pair.