MultiDOF Systems

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Introductory Remarks

The Homogeneous Problem

Modal Analysis

Examples

Multi Degrees of Freedom Systems

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Outline

Introductory Remarks
  An Example
  The Equation of Motion, a System of Linear Differential Equations
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem
  The Homogeneous Equation of Motion
  Eigenvalues and Eigenvectors
  Eigenvectors are Orthogonal

Modal Analysis
  Eigenvectors are a base
  EoM in Modal Coordinates
  Initial Conditions

Examples
  2 DOF System

MultiDOF Systems
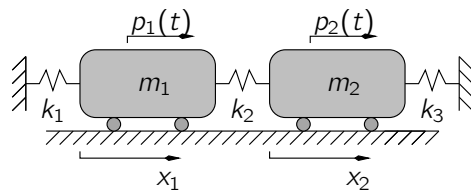
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Introductory Remarks
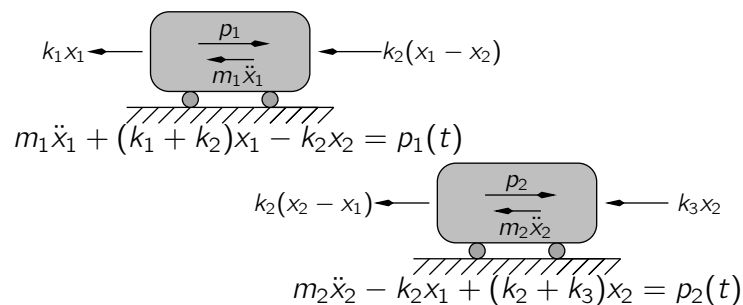
The Homogeneous Problem

Modal Analysis

Examples

Introductory Remarks

Consider an undamped system with two masses and two degrees of freedom.

We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally, write an equation of dynamic equilibrium for each mass.

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t)$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t)$$

MultiDOF Systems

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

The equation of motion of a 2DOF system

With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

The equation of motion of a 2DOF system

MultiDOF Systems

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

Introducing the loading vector $\mathbf{p}$, the vector of inertial forces $\mathbf{f}_I$ and the vector of elastic forces $\mathbf{f}_S$,

$$\mathbf{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \mathbf{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\mathbf{f}_I + \mathbf{f}_S = \mathbf{p}(t).$$

$\mathbf{f}_S = \mathbf{K}\,\mathbf{x}$

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements $\mathbf{x} = \left\{ x_1\, x_2 \right\}^T$ in terms of a matrix product.
In our example it is

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{x} = \mathbf{K}\,\mathbf{x}$$

introducing the stiffness matrix $\mathbf{K}$.
The stiffness matrix $\mathbf{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\mathbf{K}$ is hence a *square matrix* $\underset{\mathrm{ndof}\times\mathrm{ndof}}{\mathbf{K}}$

$\mathbf{f}_I = \mathbf{M}\,\ddot{\mathbf{x}}$

MultiDOF Systems

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

Analogously, introducing the mass matrix $\mathbf{M}$ that, for our example, is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\mathbf{f}_I = \mathbf{M}\,\ddot{\mathbf{x}}.$$

Also the mass matrix $\mathbf{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

Matrix Equation

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Introductory Remarks
  An Example
  The Equation of Motion
  Matrices are Linear Operators
  Properties of Structural Matrices
  An example

The Homogeneous Problem

Modal Analysis

Examples

Finally it is possible to write the equation of motion in matrix format:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t).$$

Of course, we can consider the damping forces too, taking into account the velocity vector $\dot{\mathbf{x}}$, introducing a *damping matrix* $\mathbf{C}$ and writing

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t),$$

however it is now more productive to keep our attention on undamped systems.

Properties of **K**

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Introductory Remarks
An Example
The Equation of Motion
Matrices are Linear Operators
Properties of Structural Matrices
An example

The Homogeneous Problem

Modal Analysis

Examples

- **K** is symmetrical, because the elastic force that acts on mass i due to an unit displacement of mass j, $f_{S,i} = k_{ij}$ is equal to the force on mass j due to unit diplacement of mass i, $f_{S,j} = k_{ji}$ in virtue of *Betti's theorem*.

- The strain energy V for a discrete system can be written

$$V = \frac{1}{2}\mathbf{x}^T \mathbf{f}_S = \frac{1}{2}\mathbf{x}^T \mathbf{K}\mathbf{x},$$

  because the strain energy is positive it follows that **K** is a positive definite matrix.

Properties of **M**

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Introductory Remarks
An Example
The Equation of Motion
Matrices are Linear Operators
Properties of Structural Matrices
An example

The Homogeneous Problem

Modal Analysis

Examples

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive, as well as the stiffness matrix is symmetrical and definite positive.

En passant, take note that the kinetic energy for a discrete system is

$$T = \frac{1}{2}\dot{\mathbf{x}}^T \mathbf{M}\dot{\mathbf{x}}.$$

Generalisation of previous results

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Introductory Remarks
An Example
The Equation of Motion
Matrices are Linear Operators
Properties of Structural Matrices
An example

The Homogeneous Problem

Modal Analysis

Examples

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with one exception.

For a general structural system, **M** could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy could be zero.

The problem

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Introductory Remarks
An Example
The Equation of Motion
Matrices are Linear Operators
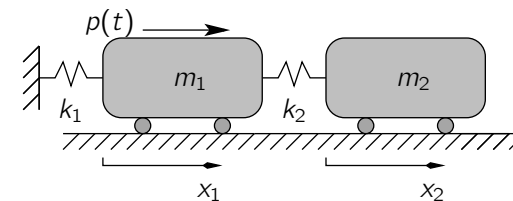Properties of Structural Matrices
An example

The Homogeneous Problem

Modal Analysis

Examples

Graphical statement of the problem

$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

The equations of motion

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = p_0 \sin \omega t,$$
$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0.$$

… but we prefer the matrix notation …

The steady state solution

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Introductory
Remarks
  An Example
  The Equation of Motion
  Matrices are Linear
  Operators
  Properties of Structural
  Matrices
  An example

The Homogeneous
Problem

Modal Analysis

Examples

because using the matrix notation we can follow the same steps we used to find the steady-state response of a *SDOF* system. First, the equation of motion

$$m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \sin \omega t$$

substituting $\mathbf{x}(t) = \boldsymbol{\xi} \sin \omega t$ and simplifying $\sin \omega t$, dividing by k, with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

$$\left(\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{\xi} = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \boldsymbol{\xi} = \Delta_{st} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

solving for $\boldsymbol{\xi}/\Delta_{st}$ gives

$$\frac{\boldsymbol{\xi}}{\Delta_{st}} = \frac{\begin{bmatrix} 1 - \beta^2 & 1 \\ 1 & 3 - 2\beta^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}{2\beta^4 - 5\beta^2 + 2} = \frac{\begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix}}{(2\beta^2 - 1)(\beta^2 - 2)}.$$

The solution, graphically

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Introductory
Remarks
  An Example
  The Equation of Motion
  Matrices are Linear
  Operators
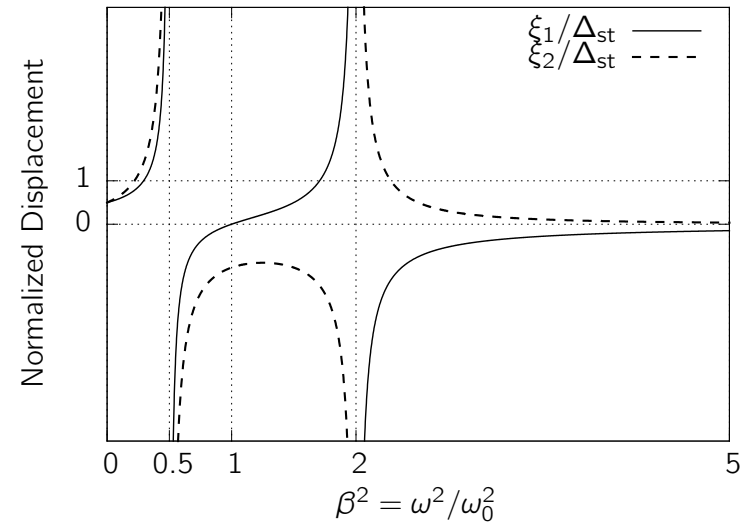 Properties of Structural
  Matrices
  An example

The Homogeneous
Problem

Modal Analysis

Examples

Steady-State Response for a 2 DOF System, Harmonic Load

Homogeneous equation of motion

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Systems

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Introductory
Remarks

The Homogeneous
Problem
  The Homogeneous
  Equation of Motion
  Eigenvalues and
  Eigenvectors
  Eigenvectors are
  Orthogonal

Modal Analysis

Examples

To understand the behaviour of a *MDOF* system, we start writing *the homogeneous equation of motion*,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* $\boldsymbol{\psi}$:

$$\mathbf{x}(t) = \boldsymbol{\psi}(A \sin \omega t + B \cos \omega t).$$

Substituting in *the equation of free vibrations*, we have

$$\left(\mathbf{K} - \omega^2 \mathbf{M} \right) \boldsymbol{\psi}(A \sin \omega t + B \cos \omega t) = \mathbf{0}$$

Eigenvalues

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Systems

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Introductory
Remarks

The Homogeneous
Problem
  The Homogeneous
  Equation of Motion
  Eigenvalues and
  Eigenvectors
  Eigenvectors are
  Orthogonal

Modal Analysis

Examples

The previous equation must hold for every value of t, so it can be simplified removing the time dependency:

$$\left(\mathbf{K} - \omega^2 \mathbf{M} \right) \boldsymbol{\psi} = \mathbf{0}.$$

This is a homogeneous linear equation, with unknowns ψ_i and the coefficients that depends on the parameter ω^2.

Speaking of homogeneous systems, we know that there is always a trivial solution, $\boldsymbol{\psi} = \mathbf{0}$, and that different non-zero solutions are available when the determinant of the matrix of coefficients is equal to zero,

$$\det \left(\mathbf{K} - \omega^2 \mathbf{M} \right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of ω^2 for which the above equation (the *equation of frequencies*) is verified.

Eigenvalues, cont.

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Introductory
Remarks

The Homogeneous
Problem
The Homogeneous
Equation of Motion
Eigenvalues and
Eigenvectors
Eigenvectors are
Orthogonal

Modal Analysis

Examples

For a system with N degrees of freedom the expansion of $\det\left(\mathbf{K} - \omega^2\mathbf{M}\right)$ is an algebraic polynomial of degree N in ω^2.

A polynomial of degree N has in general N real or complex conjugate roots.

When the matrices involved are symmetric, as it is always the case for structural matrices, the roots, ω_i^2, $i = 1, \ldots, N$ are all real.

If both $\mathbf{K}$ and $\mathbf{M}$ are positive definite matrices, condition that is always satisfied by stable structural systems, the roots (the eigenvalues) are also positive,

$$\omega_i^2 > 0 \text{ for } i = 1, \ldots, N.$$

Eigenvectors

MultiDOF
Systems

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Introductory
Remarks

The Homogeneous
Problem
The Homogeneous
Equation of Motion
Eigenvalues and
Eigenvectors
Eigenvectors are
Orthogonal

Modal Analysis

Examples

Substituting one of the N roots ω_i^2 in the characteristic equation,

$$\left(\mathbf{K} - \omega_i^2\mathbf{M}\right)\boldsymbol{\psi}_i = \mathbf{0}$$

the resulting system of $N - 1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue ω_i^2.

A common choice for the normalisation of the eigenvectors is *normalisation with respect to the mass matrix*, $\boldsymbol{\psi}_i^T\mathbf{M}\boldsymbol{\psi}_i = 1$

Please consider that substituting different values for ω_i^2 in the characteristic equation you obtain different linear systems, with different $\boldsymbol{\psi}$ solutions.

Initial Conditions

MultiDOF
Systems

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Introductory
Remarks

The Homogeneous
Problem
The Homogeneous
Equation of Motion
Eigenvalues and
Eigenvectors
Eigenvectors are
Orthogonal

Modal Analysis

Examples

The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\mathbf{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i(A_i \sin\omega_i t + B_i \cos\omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

MultiDOF
Systems

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Introductory
Remarks

The Homogeneous
Problem
The Homogeneous
Equation of Motion
Eigenvalues and
Eigenvectors
Eigenvectors are
Orthogonal

Modal Analysis

Examples

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i\omega_i(A_i \cos\omega_i t - B_i \sin\omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\mathbf{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \mathbf{x}_0, \qquad \dot{\mathbf{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i\omega_i A_i = \dot{\mathbf{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji}\omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \ldots, N.$$

Orthogonality - 1

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Introductory Remarks

The Homogeneous Problem
The Homogeneous Equation of Motion
Eigenvalues and Eigenvectors
Eigenvectors are Orthogonal

Modal Analysis

Examples

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\mathbf{K}\,\boldsymbol{\psi}_r = \omega_r^2 \mathbf{M}\,\boldsymbol{\psi}_r$$
$$\mathbf{K}\,\boldsymbol{\psi}_s = \omega_s^2 \mathbf{M}\,\boldsymbol{\psi}_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M}\,\boldsymbol{\psi}_r$$
$$\boldsymbol{\psi}_r^T \mathbf{K}\,\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s$$

Orthogonality - 2

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Introductory Remarks

The Homogeneous Problem
The Homogeneous Equation of Motion
Eigenvalues and Eigenvectors
Eigenvectors are Orthogonal

Modal Analysis

Examples

The term $\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r\right)^T = \boldsymbol{\psi}_r^T \mathbf{K}^T\,\boldsymbol{\psi}_s$$

but $\mathbf{K}$ is symmetrical, $\mathbf{K}^T = \mathbf{K}$ and we have

$$\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \mathbf{K}\,\boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^T \mathbf{M}\,\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s.$$

Orthogonality - 3

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Introductory Remarks

The Homogeneous Problem
The Homogeneous Equation of Motion
Eigenvalues and Eigenvectors
Eigenvectors are Orthogonal

Modal Analysis

Examples

Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^T \mathbf{K}\,\boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^T \mathbf{K}\,\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2)\,\boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\boldsymbol{\psi}_r^T \mathbf{M}\,\boldsymbol{\psi}_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 4

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Introductory Remarks

The Homogeneous Problem
The Homogeneous Equation of Motion
Eigenvalues and Eigenvectors
Eigenvectors are Orthogonal

Modal Analysis

Examples

The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \mathbf{K}\,\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M}\,\boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \mathbf{M}\,\boldsymbol{\psi}_i$$

and

$$\boldsymbol{\psi}_i^T \mathbf{K}\,\boldsymbol{\psi}_i = \omega_i^2 M_i.$$

Eigenvectors are a base

MultiDOF Systems

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Introductory Remarks

The Homogeneous Problem

Modal Analysis
Eigenvectors are a base
EoM in Modal Coordinates
Initial Conditions

Examples

The eigenvectors are linearly independent, so for every vector $\mathbf{x}$ we can write

$$\mathbf{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j.$$

The coefficients are readily given by premultiplication of $\mathbf{x}$ by $\boldsymbol{\psi}_i^T \mathbf{M}$, because

$$\boldsymbol{\psi}_i^T \mathbf{M} \mathbf{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_j q_j = \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\boldsymbol{\psi}_j^T \mathbf{M} \mathbf{x}}{M_j}.$$

Eigenvectors are a base

MultiDOF Systems

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Introductory Remarks

The Homogeneous Problem

Modal Analysis
Eigenvectors are a base
EoM in Modal Coordinates
Initial Conditions

Examples

Generalising our results for the displacement vector to the acceleration vector, we can write

$$\mathbf{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\mathbf{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \psi_{ij} \ddot{q}_j(t).$$

Introducing $\mathbf{q}(t)$, the vector of *modal coordinates* and $\boldsymbol{\Psi}$, the *eigenvector matrix*, whose columns are the eigenvectors,

$$\mathbf{x}(t) = \boldsymbol{\Psi} \mathbf{q}(t), \qquad \ddot{\mathbf{x}}(t) = \boldsymbol{\Psi} \ddot{\mathbf{q}}(t).$$

EoM in Modal Coordinates...

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Introductory Remarks

The Homogeneous Problem

Modal Analysis
Eigenvectors are a base
EoM in Modal Coordinates
Initial Conditions

Examples

Substituting the last two equations in the equation of motion,

$$\mathbf{M} \boldsymbol{\Psi} \ddot{\mathbf{q}} + \mathbf{K} \boldsymbol{\Psi} \mathbf{q} = \mathbf{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} \ddot{\mathbf{q}} + \boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi} \mathbf{q} = \boldsymbol{\Psi}^T \mathbf{p}(t)$$

introducing the so called *starred* matrices we can finally write

$$\mathbf{M}^\star \ddot{\mathbf{q}} + \mathbf{K}^\star \mathbf{q} = \mathbf{p}^\star(t)$$

where $\mathbf{p}^\star(t) = \boldsymbol{\Psi}^T \mathbf{p}(t)$, and the scalar equation are

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j.$$

... are N independent equations!

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Introductory Remarks

The Homogeneous Problem

Modal Analysis
Eigenvectors are a base
EoM in Modal Coordinates
Initial Conditions

Examples

We must examine the structure of the starred symbols. The generic element, with indexes i and j, of the *starred* matrices can be expressed in terms of single eigenvectors,

$$m_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_j \qquad = \qquad \delta_{ij} M_i,$$
$$k_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{K} \boldsymbol{\psi}_j \qquad = \omega_i^2 \delta_{ij} M_i.$$

where δ_{ij} is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p_i^\star = \boldsymbol{\psi}_i^T \mathbf{p}(t)$ we have a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \ldots, N$$

Initial Conditions Revisited

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Systems

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Introductory
Remarks
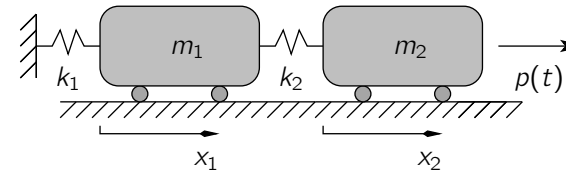
The Homogeneous
Problem

Modal Analysis
Eigenvectors are a base
EoM in Modal Coordinates
Initial Conditions

Examples

The initial displacements can be written in modal coordinates,

$$\mathbf{x}_0 = \boldsymbol{\Psi}\,\mathbf{q}_0$$

and premultiplying both members by $\boldsymbol{\Psi}^T\mathbf{M}$ we have the following relationship:

$$\boldsymbol{\Psi}^T\mathbf{M}\,\mathbf{x}_0 = \boldsymbol{\Psi}^T\mathbf{M}\,\boldsymbol{\Psi}\,\mathbf{q}_0 = \mathbf{M}^\star\mathbf{q}_0.$$

Premultiplying by the inverse of $\mathbf{M}^\star$ and taking into account that $\mathbf{M}^\star$ is diagonal,

$$\mathbf{q}_0 = (\mathbf{M}^\star)^{-1}\,\boldsymbol{\Psi}^T\mathbf{M}\,\mathbf{x}_0 \quad \Rightarrow \quad q_{i0} = \frac{\boldsymbol{\psi}_i^T\mathbf{M}\,\mathbf{x}_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\boldsymbol{\psi}_i^T\mathbf{M}\,\dot{\mathbf{x}}_0}{M_i}$$

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2 DOF System

MultiDOF
Systems

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Introductory
Remarks

The Homogeneous
Problem

Modal Analysis

Examples
2 DOF System

$$k_1 = k, \quad k_2 = 2k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin\omega t.$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix}\sin\omega t,$$

$$\mathbf{M} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

Equation of frequencies

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Introductory
Remarks

The Homogeneous
Problem

Modal Analysis

Examples
2 DOF System

The equation of frequencies is

$$\|\mathbf{K} - \omega^2\mathbf{M}\| = \begin{Vmatrix} 3k - 2\omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{Vmatrix} = 0.$$

Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0, \qquad (1)$$

with $\omega_0^2 = k/m$ and $\omega^2 = \Lambda\omega_0^2$,

$$2\Lambda^2 - 7\Lambda + 2 = 0 \quad \Rightarrow \quad \Lambda = \frac{7 \mp \sqrt{33}}{4}. \qquad (2)$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m}\frac{7-\sqrt{33}}{4} \qquad \omega_2^2 = \frac{k}{m}\frac{7+\sqrt{33}}{4}$$
$$\omega_1^2 = 0.31386\frac{k}{m} \qquad \omega_2^2 = 3.18614\frac{k}{m}$$

Eigenvectors

MultiDOF
Systems

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Introductory
Remarks

The Homogeneous
Problem

Modal Analysis

Examples
2 DOF System

Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\,(3 - 2 \times 0.31386)\psi_{11} - 2k\psi_{21} = 0$$

while substituting ω_2^2 gives

$$k\,(3 - 2 \times 3.18614)\psi_{12} - 2k\psi_{22} = 0.$$

Solving with the arbitrary assignment $\psi_{21} = \psi_{22} = 1$ gives the *unnormalised* eigenvectors,

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} +0.84307 \\ +1.00000 \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -0.59307 \\ +1.00000 \end{Bmatrix}.$$

Normalisation

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2 DOF System

We compute first M_1 and M_2,

$$M_1 = \boldsymbol{\psi}_1^T \mathbf{M}\, \boldsymbol{\psi}_1$$

$$= \{0.84307, \quad 1\} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix}$$

$$= \{1.68614m, \quad m\} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} = 2.42153m$$

$$M_2 = 1.70346m$$

the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{2.42153}, \qquad \alpha_2 = \sqrt{1.70346}.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalised eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.54177 & -0.45440 \\ +0.64262 & +0.76618 \end{bmatrix}$$

Modal Loadings

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Modal Analysis

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2 DOF System

The modal loading is

$$\mathbf{p}^{\star}(t) = \boldsymbol{\Psi}^T\, \mathbf{p}(t)$$

$$= p_0 \begin{bmatrix} +0.54177 & +0.64262 \\ -0.45440 & +0.76618 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \sin \omega t$$

$$= p_0 \begin{Bmatrix} +0.64262 \\ +0.76618 \end{Bmatrix} \sin \omega t$$

Modal EoM

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The Homogeneous Problem

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2 DOF System

Substituting its modal expansion for $\mathbf{x}$ into the equation of motion and premultiplying by $\boldsymbol{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} m\ddot{q}_1 + 0.31386k\, q_1 = +0.64262\, p_0 \sin \omega t \\ m\ddot{q}_2 + 3.18614k\, q_2 = +0.76618\, p_0 \sin \omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by m both equations, we have

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = +0.64262\, \dfrac{p_0}{m} \sin \omega t \\ \ddot{q}_2 + \omega_2^2 q_2 = +0.76618\, \dfrac{p_0}{m} \sin \omega t \end{cases}$$

Particular Integral

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The Homogeneous Problem

Modal Analysis

Examples

2 DOF System

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 \left(\omega_1^2 - \omega^2 \right) \sin \omega t = \frac{p_1^{\star}}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{p_1^{\star}}{m} \frac{1}{\omega_1^2 - \omega^2}$$

with $\omega_1^2 = K_1/m \;\Rightarrow\; m = K_1/\omega_1^2$:

$$C_1 = \frac{p_1^{\star}}{K_1} \frac{\omega_1^2}{\omega_1^2 - \omega^2} = \Delta_{\text{st}}^{(1)} \frac{1}{1 - \beta_1^2} \quad \text{with } \Delta_{\text{st}}^{(1)} = \frac{p_1^{\star}}{K_1} = 2.047 \frac{p_0}{k} \text{ and } \beta_1 = \frac{\omega}{\omega_1}$$

of course

$$C_2 = \Delta_{\text{st}}^{(2)} \frac{1}{1 - \beta_2^2} \quad \text{with } \Delta_{\text{st}}^{(2)} = \frac{p_2^{\star}}{K_2} = 0.2404 \frac{p_0}{k} \text{ and } \beta_2 = \frac{\omega}{\omega_2}$$

Integrals

The integrals, for our loading, are thus

$$
\begin{cases}
q_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + \Delta_{\text{st}}^{(1)} \dfrac{\sin \omega t}{1 - \beta_1^2} \\[2mm]
q_2(t) = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t + \Delta_{\text{st}}^{(2)} \dfrac{\sin \omega t}{1 - \beta_2^2}
\end{cases}
$$

for a system initially at rest

$$
\begin{cases}
q_1(t) = \Delta_{\text{st}}^{(1)} \dfrac{1}{1 - \beta_1^2} \left(\sin \omega t - \beta_1 \sin \omega_1 t \right) \\[2mm]
q_2(t) = \Delta_{\text{st}}^{(2)} \dfrac{1}{1 - \beta_2^2} \left(\sin \omega t - \beta_2 \sin \omega_2 t \right)
\end{cases}
$$

we are interested in structural degrees of freedom, too... disregarding transient

$$
\begin{cases}
x_1(t) = \left(\psi_{11} \dfrac{\Delta_{\text{st}}^{(1)}}{1 - \beta_1^2} + \psi_{12} \dfrac{\Delta_{\text{st}}^{(2)}}{1 - \beta_2^2} \right) \sin \omega t = \left(\dfrac{1.10926}{1 - \beta_1^2} - \dfrac{0.109271}{1 - \beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t \\[3mm]
x_2(t) = \left(\psi_{21} \dfrac{\Delta_{\text{st}}^{(1)}}{1 - \beta_1^2} + \psi_{22} \dfrac{\Delta_{\text{st}}^{(2)}}{1 - \beta_2^2} \right) \sin \omega t = \left(\dfrac{1.31575}{1 - \beta_1^2} + \dfrac{0.184245}{1 - \beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t
\end{cases}
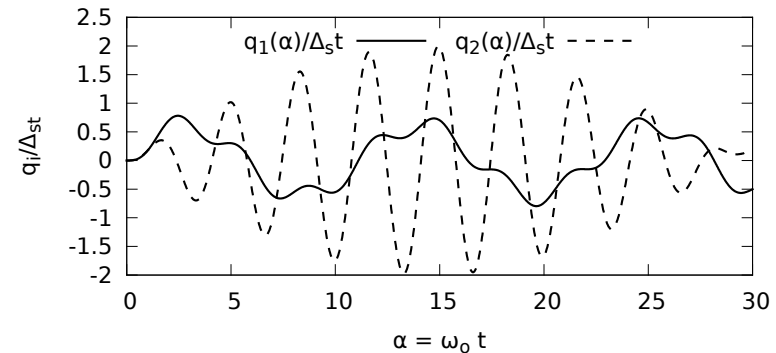$$

The response in modal coordinates

To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$.
This implies that $\beta_1 = \dfrac{\omega}{\omega_1} = \dfrac{2.0}{\sqrt{0.31386}} = 6.37226$ and $\beta_2 = \dfrac{\omega}{\omega_2} = \dfrac{2.0}{\sqrt{3.18614}} = 0.62771$.

In the graph above, the responses are plotted against an adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{\text{st}} = \dfrac{p_0}{k}$

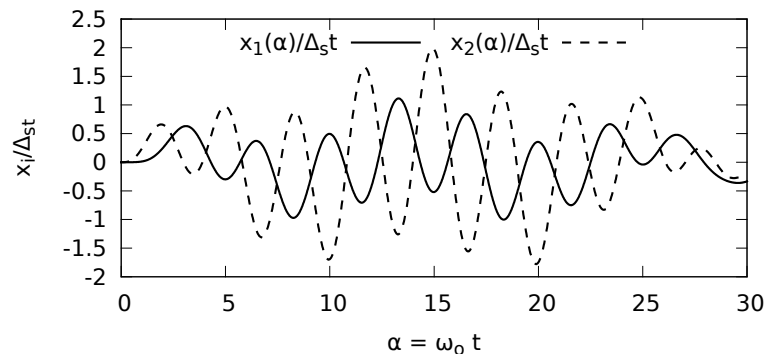## The response in structural coordinates

Using the same normalisation factors, here are the response functions in terms of $x_1 = \psi_{11} q_1 + \psi_{12} q_2$ and $x_2 = \psi_{21} q_1 + \psi_{22} q_2$: