1 Collision

The velocity of a body falling in a constant gravitational field, disregarding air drag, is $v(t) = gt$ while the distance ran is $s(t) = 1/2gt^2 = d \quad \Rightarrow \quad t = \sqrt{2d/g}$, substituting in v(t) we have that the velocity of the second body at impact is $v_2 = \sqrt{2gh}$.

The initial conditions of the new system with mass $m = m_1 + m_2$ are

$$\begin{aligned} x_0 &= 0 \\ \dot{x}_0 &= \frac{v_2 m_2}{m} = \frac{m_2}{m}\sqrt{2gh} \end{aligned}$$

by the conservation of momentum and the integral of motion, taking into account the particular integral $\xi(t) = m_2 g/k$, due to the static displacement associated with the weight of the second body, is

$$x(t) = A\cos\omega t + B\sin\omega t + m_2 g/k = A\cos\omega t + B\sin\omega t + \Delta.$$

Imposing the initial conditions imply

$$\begin{aligned} x(0) &= A + m_2 g/k = 0 & \Rightarrow \qquad A &= -m_2 g/k = -\Delta \\ \dot{x}(0) &= \omega B = \frac{m_2}{m}\sqrt{2gh} & \Rightarrow \qquad B &= \frac{m_2}{\omega m}\sqrt{2gh} \end{aligned}$$

and the maximum displacement is

$$x_{\mathrm{max}} = \sqrt{A^2 + B^2} + \Delta = \sqrt{\Delta^2 + B^2} + \Delta.$$

Substituting our previous results, with $\omega^2 = k/m$, we have

$$x_{\mathrm{max}}(g) = \sqrt{\left(\frac{g m_2}{k}\right)^2 + 2gh\frac{m_2^2}{km}} + \frac{g m_2}{k}.$$

The largest planet is Jupiter and it is $g_{\mathrm{Jup}} = 24.79\,\mathrm{m\,s^{-2}}$, so we can plot $x_{\mathrm{max}}(g)$ for $0 \le g \le 25\,\mathrm{m\,s^{-2}}$ (upon substitution of all the numerical constants) in figure 1: examination of the figure shows that we have a maximum deflection of 60 mm for a value of g slightly less than $10\,\mathrm{m\,s^{-2}}$...

But we can do better, from $x_{\mathrm{max}} - \Delta = \sqrt{\Delta^2 + B^2}$ we have, squaring both members, $x_{\mathrm{max}}^2 - 2x_{\mathrm{max}}\Delta + \Delta^2 = \Delta^2 + B^2$ and, simplifying,

$$x_{\mathrm{max}}^2 - 2x_{\mathrm{max}}\Delta = B^2$$

that gives an expression that is linear in g, so that we can easily solve and have

$$g = \frac{1}{2}\frac{m}{m_2}\frac{k x_{\mathrm{max}}^2}{m x_{\mathrm{max}} + m_2 h}.$$

When we substitute all the known numerical values in the right member, we find

$$g = 9.78260869565\,\mathrm{m\,s^{-2}}$$

and we can conclude that the misterious planet is the Earth.

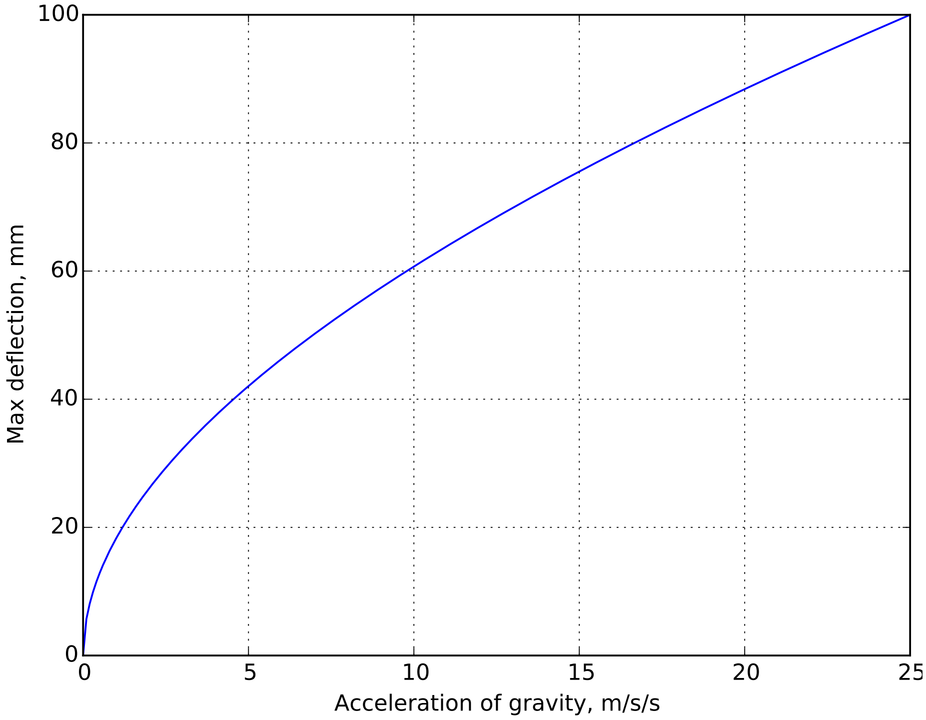

Figure 1: max displacement as a function of g.

2 Vibration Isolation

The requested trasmissibility ratio is $\mathrm{TR} = {}^{400}\!/_{1200} = {}^{1}\!/_{3}$ and for $\zeta = 0$ it is

$$\mathrm{TR} = \frac{1}{\beta^2 - 1} \le \frac{1}{3} \Rightarrow \beta^2 \ge 1 + \frac{1}{\mathrm{TR}} \Rightarrow \beta^2 \ge 4$$

substituting $\omega_n^2 = k/m$ in $\beta^2 = \omega^2/\omega_n^2$ it is

$$m\omega^2 \ge 4k \Rightarrow k \le \frac{m\omega^2}{4}$$

substituting the numerical values we have

$$k\Big|_{\zeta=0} = 1\,776\,528\,792.2\,\mathrm{N\,m^{-1}}.$$

For $\zeta > 0$ it is

$$\frac{\sqrt{1+(2\zeta\beta)^2}}{\sqrt{(\beta^2-1)^2+(2\zeta\beta)^2}} \le \mathrm{TR} \Rightarrow \mathrm{TR}^2(\beta^2-1)^2 + (\mathrm{TR}^2-1)(2\zeta\beta)^2 - 1 \ge 0$$

expanding we have a disequation of 2nd degree in β^2,

$$\mathrm{TR}^2{\beta^2}^2 + (4\zeta^2\mathrm{TR}^2 - 2\mathrm{TR}^2 - 4\zeta^2)\beta^2 + \mathrm{TR}^2 - 1 \ge 0$$

that it is verified for values of β^2 external to the roots.

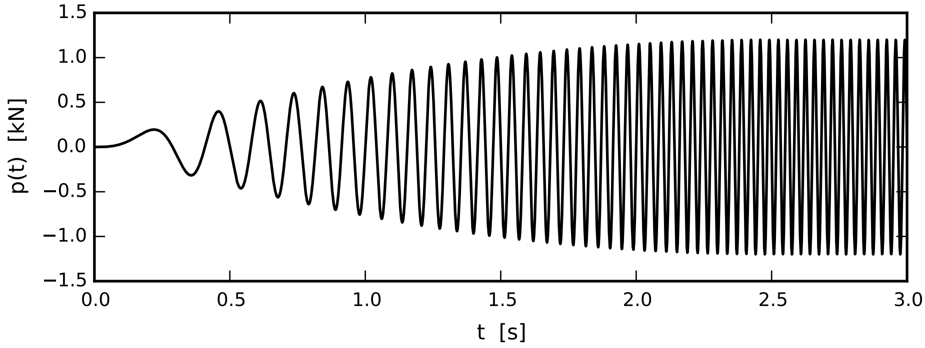

Figure 2: the load during the transient.

In our case we exclude the negative root and have

$$\beta^2 > \frac{4\zeta^2 + 2\mathrm{TR}^2 - 4\mathrm{TR}^2\zeta^2 + \sqrt{(4\mathrm{TR}^2\zeta^2 - 2\mathrm{TR}^2 - 4\zeta^2)^2 + 4(1-\mathrm{TR}^2)\mathrm{TR}^2}}{2\mathrm{TR}^2} = 4.07728835478$$

and it has to be, as before,

$$k \leq \frac{m\,\omega^2}{\beta^2} = 1\,742\,853\,227.55\,\mathrm{N\,m^{-1}}$$

and eventually

$$c = 2\zeta\sqrt{k\,m} = 2\,240\,405.609\,56\,\mathrm{N\,s\,m^{-1}}.$$

3 Numerical Integration

The plot of the load during the transient is shown on figure 2, note that the number of peaks in a unit of time varies during the transient, meaning that during the transient the system exhibits a resonant response.

The response can be computed numerically, in terms of $x(t)$ and $\dot{x}(t)$, and the transmitted force can subsequently be computed as

$$f_{\mathrm{tr}}(t) = k\,x(t) + c\,\dot{x}(t).$$

In figures 3 and 4 you can see, respectively, the transmitted force in the case of an undamped suspension and of a slightly damped suspension ($\zeta = 6\%$). As you can see, there is a significant amplification of the transmitted force with respect to the unbalanced load, even for the damped system.

As a non requested addendum, in the next figure, figure 5, the transmitted force for a damped system with $\zeta = 15\%$, the transmitted force is lower, approximately 50% of the $\zeta = 6\%$ force, but we have more than doubled the damping ratio.

Eventually, in figure 6 we have a comparison of the responses for different values of the damping, it's worth noting that, in the beginning, the responses are very close to each other... that's because the damping forces are proportional to the velocity and the velocities are lower during the first phase of the transient.

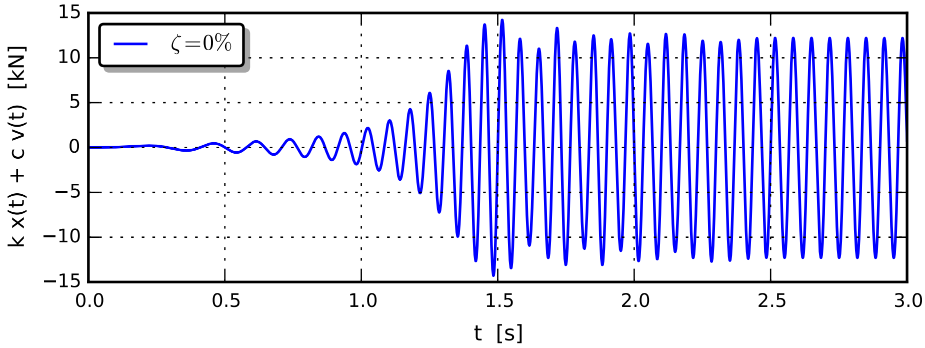

Figure 3: transmitted force, $\zeta = 0$..

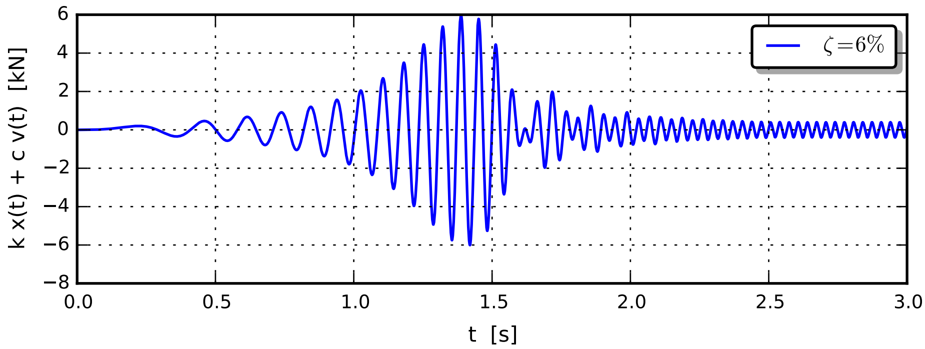

Figure 4: transmitted force, $\zeta = 6\%$..

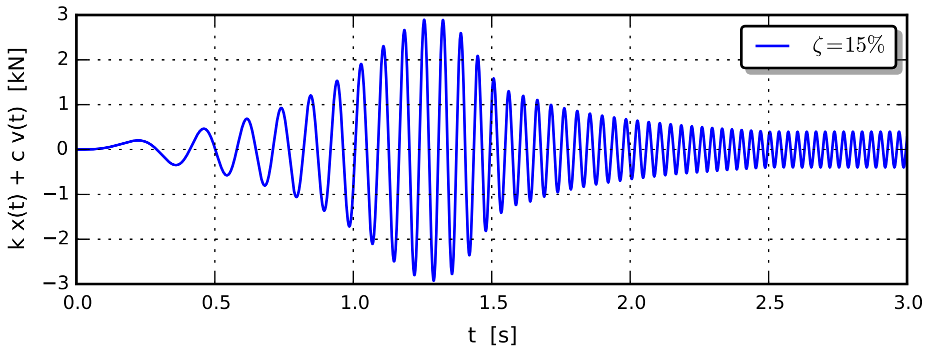

Figure 5: transmitted force, $\zeta = 15\%$..

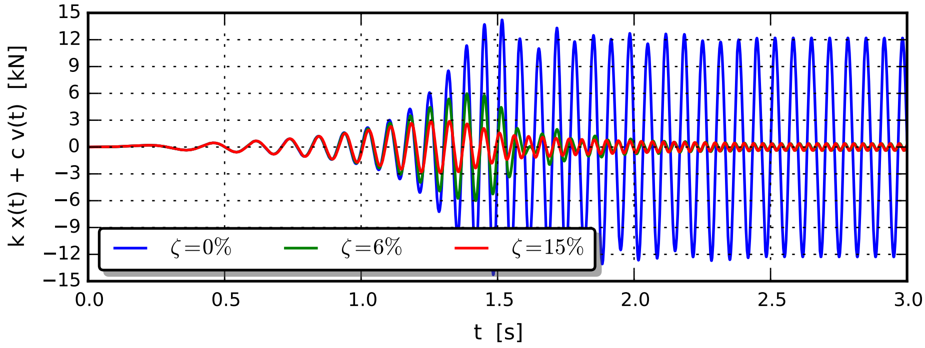

Figure 6: comparison of the transmitted forces.

4 SDOF System

The problem was solved using the following Python program

```
from fractions import Fraction as fr

# these functions are just placeholders to express the intent of the code
def d(v):    return v
def ddot(v): return v

# the units used later
k, m, p, L, v = 1, 1, 1, 1, 1

# the springs' stiffnesses
k1, k3 = k, k

# length, mass and rotatory inertia of central body
L2 = 2*L
m2 = m*L2
j2 = fr(m2*L2**2, 12)

# the rotations of the beams, in terms of the free displacement
r2 = fr(-v,L)
r1, r3 = -r2, -r2

# the coordinates of the _interesting_ points
# the origin is in the respective CIR
xk1, xg2, yg2, yk3 = 2*L, 0, L, -2*L

# the displacements of the _interesting_ points
vk1 = +r1*xk1
vg2 = +r2*xg2
ug2 = -r2*yg2
uk3 = -r3*yk3

# the virtual works of (i)nertial, (s)pring and (e)xternal forces
```

```
vwi = -m2*ddot(ug2)*d(ug2) -m2*ddot(vg2)*d(vg2) -j2*ddot(r2)*d(r2)
vws = -k1*vk1*d(vk1) - k3*uk3*d(uk3)
vwe = +p*d(v)

print "Here␣it␣is␣the␣equation␣of␣dynamic␣equilibrium:"
print "␣␣␣%s*m*ddot(v)␣%s*k*v␣+%s*p␣=␣0"%(vwi, vws, vwe)
```

Executing the program above produces the following output

```
Here it is the equation of dynamic equilibrium:
   -8/3*m*ddot(v) -8*k*v +1*p = 0
```

5 Rayleigh Quotient Method

Most of the problem boils down to writing the structural matrices... The mass it's easy

$$\boldsymbol{M} = m \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and the stiffness is only a bit more difficult

$$\boldsymbol{K} = k \begin{bmatrix} +2 & -2 & 00 \\ -2 & +5 & -3 \\ 00 & -3 & +7 \end{bmatrix}.$$

With $\boldsymbol{x} = Z\boldsymbol{\phi}_o \sin\omega t$ the maximum strain energy is

$$V = \frac{1}{2} Z^2 \boldsymbol{\phi}_o^T \boldsymbol{K} \boldsymbol{\phi}_o$$

and the max kinetic energy is

$$T = \frac{1}{2} \omega^2 Z^2 \boldsymbol{\phi}_o^T \boldsymbol{M} \boldsymbol{\phi}_o,$$

equating and solving for ω^2 gives

$$\omega^2 = \frac{\boldsymbol{\phi}_o^T \boldsymbol{K} \boldsymbol{\phi}_o}{\boldsymbol{\phi}_o^T \boldsymbol{M} \boldsymbol{\phi}_o} = 0.1875 \frac{k}{m}$$

We refine this result writing the max strain energy in terms of the work of the inertial forces,

$$V = \frac{1}{2}(-\omega^2 Z \boldsymbol{M} \boldsymbol{\phi}_o)^T (-\omega^2 Z \boldsymbol{K}^{-1} \boldsymbol{M} \boldsymbol{\phi}_o) = \frac{1}{2} Z^2 \omega^4 \phi_o^T \boldsymbol{M} \boldsymbol{K}^{-1} \boldsymbol{M} \boldsymbol{\phi}_o,$$

so that equating to T once more and solving for ω^2 gives

$$\omega^2 = \frac{\boldsymbol{\phi}_o^T \boldsymbol{M} \boldsymbol{\phi}_o}{\phi_o^T \boldsymbol{M} \boldsymbol{K}^{-1} \boldsymbol{M} \boldsymbol{\phi}_o} = 0.18616677 \frac{k}{m}.$$

Our last approximation is obtained computing the max kinetic energy starting from the max velocity associated to the deflections due to the inertial forces,

$v_{\max} = Z\omega^3 \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}_o$, and equating to the previous approximation to the max strain energy we find

$$\omega^2 = \frac{\phi_o^T \boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}_o}{\phi_o^T \boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\phi}_o} = 0.18597848\frac{k}{m}.$$

The characteristic polinomial, with $\omega^2 = \Lambda\frac{k}{m}$, is

$$60\Lambda^3 - 199\Lambda^2 + 164\Lambda - 2 = 0$$

whose roots are $\Lambda_1 = 0.185942964024$, $\Lambda_2 = 1.01840624343$ and $\Lambda_3 = 2.11231745924$.

6 MDOF System

$$\boldsymbol{M} = m\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{F} = \frac{1}{12}\frac{L^3}{EJ}\begin{bmatrix}11 & -6 & -9\\ -6 & 9 & 11\\ -9 & 11 & 16\end{bmatrix}$$

$$\boldsymbol{K} = \frac{3}{34}\frac{EJ}{L^3}\begin{bmatrix}23 & -3 & 15\\ -3 & 95 & -67\\ 15 & -67 & 63\end{bmatrix}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix}0.31295657 & & \\ & 1.5504954 & \\ & & 13.09243038\end{bmatrix}$$

$$\boldsymbol{\Psi} = \begin{bmatrix}-0.48309956 & 0.51458873 & -0.04258233\\ 0.42117533 & 0.46008195 & 0.78162391\\ 0.59652407 & 0.50864672 & -0.62083617\end{bmatrix}$$

$$\boldsymbol{q}_o = \boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x}_o = \begin{bmatrix}-0.96619913\\ 1.02917745\\ -0.08516467\end{bmatrix}$$

$$\begin{aligned}\ddot{q}_1 + 0.312957\,q_1 &= 0\\ \ddot{q}_2 + 1.5505\,q_2 &= 0\\ \ddot{q}_3 + 13.0924\,q_3 &= 0\end{aligned}$$

$$\begin{aligned}q_1(t) &= -0.966199 \times \cos(0.559425t)\\ q_2(t) &= +1.029177 \times \cos(1.245189t)\\ q_3(t) &= -0.085165 \times \cos(3.618346t)\end{aligned}$$

$$x_1(t) = +0.466770\cos(0.559425t) + 0.529603\cos(1.245189t) + 0.003627\cos(3.618346t)$$

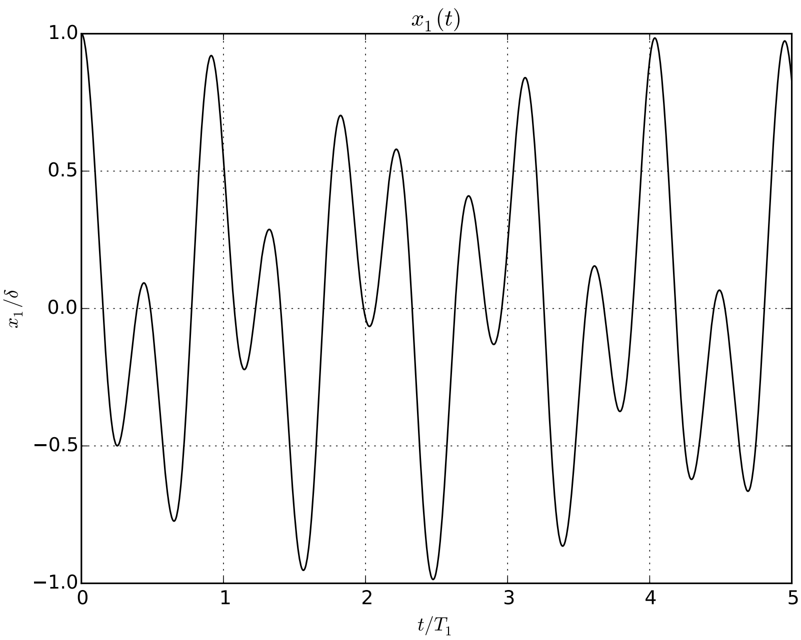

Figure 7: normalized horizontal displacement