SDOF linear oscillator

Response to Impulsive Loads & Step by Step Methods

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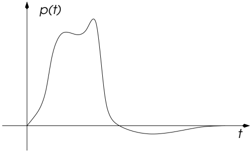
Outline

Response to Impulsive Loadings

Nature of Impulsive Loadings

An impulsive load is characterized

- by a single principal impulse, and
- by a relatively short duration.



- Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
- Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
- For this reason, in the following we'll consider only the undamped response to impulsive loads.

Definition of *Peak Response*

When dealing with the response to an impulsive loading of duration t_0 of a SDOF system, with natural period of vibration T_n we are mostly interested in the *peak response* of the system.

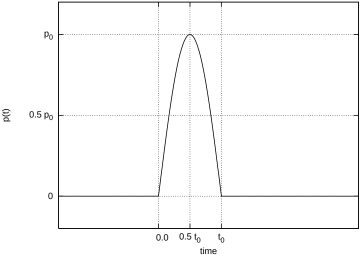
> *The* **peak response** *is the maximum of the absolute value of the response ratio,* $R_{max} = \max\{|R(t)|\}$.

- If $t_0 \ll T_\text{n}$ necessarily R_max happens after the end of the loading, and its value can be determined studying the free vibrations of the dynamic system.
- On the other hand, if the excitation lasts *enough* to have at least a local extreme (maximum or minimum) during the excitation we have to consider the more difficult problem of completely determining the response during the application of the impulsive loading.

Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \frac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load. Note that $\omega\, t_0 = \pi$.

Response to sine-wave impulse

Consider an undamped *SDOF* initially at rest, with natural period T_n, excited by a half-sine impulse of duration t_0.
The frequency ratio is $\beta = {}^{T_\text{n}}\!/_{2t_0}$ and the response ratio in the interval $0 < t < t_0$ is

$$R(t) = \frac{1}{1-\beta^2}\left(\sin\omega t - \beta\sin\frac{\omega t}{\beta}\right). \qquad [\text{NB: } \frac{\omega}{\beta} = \omega_\text{n}]$$

It is $(1-\beta^2)R(t_0) = -\beta\sin{}^{\pi}\!/_{\beta}$ and $(1-\beta^2)\dot{R}(t_0) = -\omega\,(1+\cos{}^{\pi}\!/_{\beta})$, consequently for $t_o \le t$ the response ratio is

$$R(t) = \frac{-\beta}{1-\beta^2}\left((1+\cos\frac{\pi}{\beta})\sin\omega_\text{n}(t-t_0) + \sin\frac{\pi}{\beta}\cos\omega_\text{n}(t-t_0)\right)$$

Maximum response to sine impulse

We have an extreme, and a possible peak value, for $0 \le t \le t_0$ if

$$\dot{R}(t) = \frac{\omega}{1-\beta^2}(\cos \omega t - \cos \frac{\omega t}{\beta}) = 0.$$

That implies that $\cos \omega t = \cos {}^{\omega t}/_{\beta} = \cos -{}^{\omega t}/_{\beta}$, whose roots are

$$\omega t = \mp\omega t/\beta + 2n\pi,\ n = 0, \mp1, \mp2, \mp3, \ldots.$$

It is convenient to substitute $\omega t = \pi\alpha$, where $\alpha = t/t_0$:

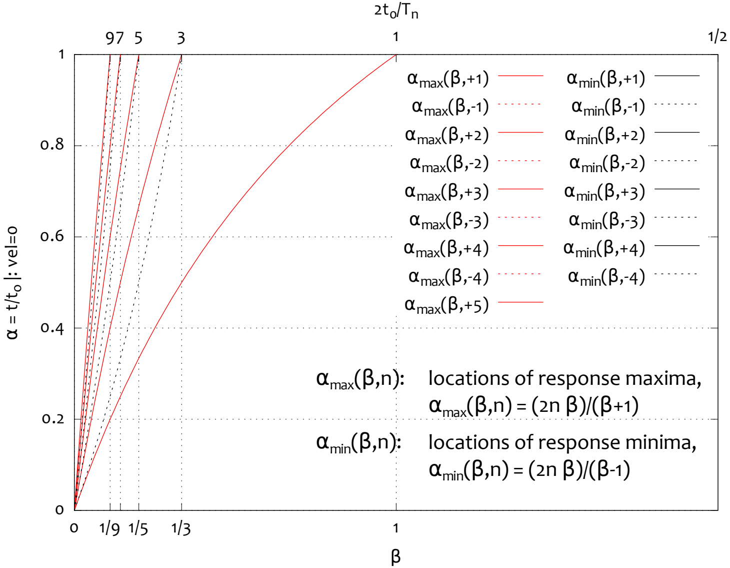
$$\pi a = \pi\left(\mp\frac{a}{\beta} + 2n\right), \quad n = 0, \mp1, \mp2, \ldots, \quad 0 \le a \le 1.$$

Eventually solving for α one has

$$\alpha = \frac{2n\beta}{\beta \mp 1}, \quad n = 0, \mp1, \mp2, \ldots, \quad 0 < \alpha < 1.$$

The next slide regards the characteristics of these roots.

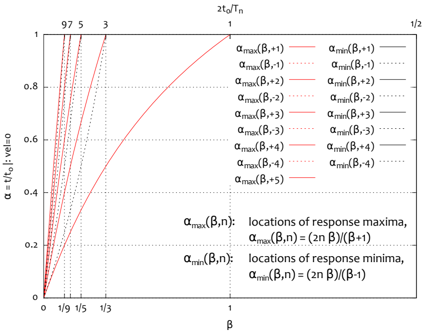
$\alpha(\beta, n)$



2t0/Tn
9 7 5 3 1 1/2
α = t/t0 |: vel=0
β
0 1/9 1/5 1/3 1
0 0.2 0.4 0.6 0.8 1
αmax(β,+1) αmin(β,+1)
αmax(β,-1) αmin(β,-1)
αmax(β,+2) αmin(β,+2)
αmax(β,-2) αmin(β,-2)
αmax(β,+3) αmin(β,+3)
αmax(β,-3) αmin(β,-3)
αmax(β,+4) αmin(β,+4)
αmax(β,-4) αmin(β,-4)
αmax(β,+5)
αmax(β,n): locations of response maxima,
αmax(β,n) = (2n β)/(β+1)
αmin(β,n): locations of response minima,
αmin(β,n) = (2n β)/(β-1)

$\alpha(\beta, n)$

In summary, to find the maximum of the response for an assigned $\beta < 1$, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k t_0$, (c) choose the maximum of the maxima.



- No roots of type α_{min} for $n > 0$;
- no roots of type α_{max} for $n < 0$;
- no roots for $\beta > 1$, i.e., no roots for $t_0 < \frac{T_n}{2}$;
- only one root of type α_{max} for $\frac{1}{3} < \beta < 1$, i.e., $\frac{T_n}{2} < t_0 < \frac{3T_n}{2}$;
- three roots, two maxima and one minimum, for $\frac{1}{5} < \beta < \frac{1}{3}$;
- five roots, three maxima and two minima, for $\frac{1}{7} < \beta < \frac{1}{5}$;
- etc etc.

Maximum response for $\beta > 1$

For $\beta > 1$, the maximum response takes place for $t > t_0$, and its absolute value (see slide *Response to sine-wave impulse*) is

$$R_{\max} = \frac{\beta}{1-\beta^2}\sqrt{(1+\cos\frac{\pi}{\beta})^2 + \sin^2\frac{\pi}{\beta}},$$

using a simple trigonometric identity we can write

$$R_{\max} = \frac{\beta}{1-\beta^2}\sqrt{2+2\cos\frac{\pi}{\beta}}$$

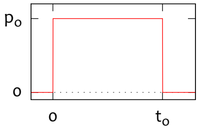
but $1+\cos 2\phi = (\cos^2\phi + \sin^2\phi) + (\cos^2\phi - \sin^2\phi) = 2\cos^2\phi$, so that

$$R_{\max} = \frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}.$$

Rectangular Impulse

Consider a rectangular impulse of duration t_0,

$$p(t) = p_0 \begin{cases} 1 & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



The response ratio and its time derivative are

$$R(t) = 1 - \cos \omega_n t, \qquad \dot{R}(t) = \omega_n \sin \omega_n t,$$

and we recognize that we have maxima $R_{max} = 2$ for $\omega_n t = n\pi$, with the condition $t \le t_0$. Hence we have no maximum during the loading phase for $t_0 < T_n/2$, and at least one maximum, of value $2\Delta_{st}$, if $t_0 \ge T_n/2$.

Rectangular Impulse (2)

For shorter impulses, the maximum response ratio is not attained during loading, so we have to compute the amplitude of the free vibrations after the end of loading (remember, as $t_0 \leq T_n/2$ the velocity is positive at $t = t_0$!).

$$R(t) = (1 - \cos\omega_n t_0)\cos\omega_n(t - t_0) + (\sin\omega_n t_0)\sin\omega_n(t - t_0).$$

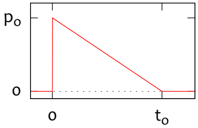
The amplitude of the response ratio is then

$$A = \sqrt{(1 - \cos\omega_n t_0)^2 + \sin^2\omega_n t_0} = \\ = \sqrt{2(1 - \cos\omega_n t_0)} = 2\sin\frac{\omega_n t_0}{2}.$$

Triangular Impulse

Let's consider the response of a *SDOF* to a triangular impulse,

$$p(t) = p_0\,(1 - t/t_0) \text{ for } 0 < t < t_0$$



As usual, we must start finding the minimum duration that gives place to a maximum of the response in the loading phase, that is

$$R(t) = \frac{1}{\omega_n t_0} \sin \omega_n \frac{t}{t_0} - \cos \omega_n \frac{t}{t_0} + 1 - \frac{t}{t_0}, \quad 0 < t < t_0.$$

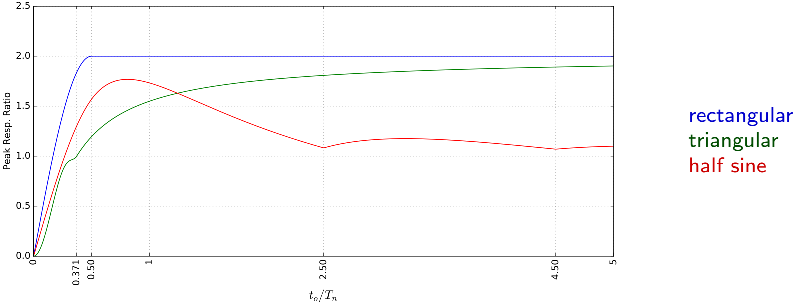
Taking the first derivative and setting it to zero, one can see that the first maximum occurs for $t = t_0$ for $t_0 = 0.37101\,T_n$, and substituting one can see that $R_{max} = 1$.

Triangular Impulse (2)

For load durations shorter than $0.37101\,T_\mathrm{n}$, the maximum occurs after loading and it's necessary to compute the displacement and velocity at the end of the load phase.
For longer loads, the maxima are in the load phase, so that one has to find the all the roots of $\dot{R}(t)$, compute all the extreme values and finally sort out the absolute value maximum.

Shock or response spectra

We have seen that the response ratio is determined by the ratio of the impulse duration to the natural period of the oscillator. One can plot the maximum displacement ratio $R_{\max}$ as a function of t_o/T_n for various forms of impulsive loads.



Such plots are commonly known as displacement-response spectra, or simply as response spectra.

Approximate Analysis

For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor $\approxeq 1$

On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the load.
The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Approximate Analysis (2)

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)]\ dt.$$

When one notes that, for small t_0, the displacement is of the order of t_0^2 while the velocity is in the order of t_0, it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approxeq \int_0^{t_0} p(t)\ dt.$$

Approximate Analysis (3)

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t)\, dt,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{1}{m\omega_\text{n}} \int_0^{t_0} p(t)\, dt\; \sin \omega_\text{n}(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading.

Review of Numerical Methods

Previous Methods

Both the Duhamel integral and the Fourier transform methods lie on on the principle of superposition, i.e., superposition of the responses

- to a succession of infinitesimal impulses, using a convolution (Duhamel) integral, when operating in time domain
- to an infinity of infinitesimal harmonic components, using the frequency response function, when operating in frequency domain.

The principle of superposition implies *linearity*, but this assumption is often invalid, e.g., a severe earthquake is expected to induce inelastic deformation in a code-designed structure.

State Vector, Linear and Non Linear Systems

The internal state of a linear dynamical system, considering that the mass, the damping and the stiffness do not vary during the excitation, is described in terms of its displacements and its velocity, i.e., the so called *state vector*

$$x = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}.$$

For a non linear system the state vector must include other information, e.g. the current tangent stiffness, the cumulated plastic deformations, the internal damage, ...

Step-By-Step Methods

Step-by-step Methods

The so-called step-by-step methods restrict the assumption of linearity to the duration of a (usually short) *time step* .

Given an initial system state, in step-by-step methods we divide the time in *steps* of known, short duration h_i (usually $h_i = h$, a constant) and from the initial system state at the beginning of each step we compute the final system state at the end of each step.

The final state vector in step i will be the initial state in the subsequent step, $i + 1$.

Step-by-step Methods, 2

Operating independently the analysis for each time step there are no requirements for superposition and non linear behaviour can be considered assuming that the structural properties remain constant during each time step.

In many cases, the non linear behaviour can be reasonably approximated by a *local* linear model, valid for the duration of the time step.

If the approximation is not good enough, usually a better approximation can be obtained reducing the time step.

Advantages of s-b-s methods

Generality step-by-step methods can deal with every kind of non-linearity, e.g., variation in mass or damping or variation in geometry and not only with mechanical non-linearities.

Efficiency step-by-step methods are very efficient and are usually preferred also for linear systems in place of Duhamel integral.

Extensibility step-by-step methods can be easily extended to systems with many degrees of freedom, simply using matrices and vectors in place of scalar quantities.

Disadvantages of s-b-s methods

The step-by-step methods are approximate numerical methods, that can give only an approximation of true response. The causes of error are

roundoff using too few digits in calculations.

truncation using too few terms in series expressions of quantities,

instability the amplification of errors deriving from roundoff, truncation or modeling in one time step in all following time steps, usually depending on the time step duration.

Errors may be classified as

- phase shifts or change in frequency of the response,
- artificial damping, the numerical procedure removes or adds energy to the dynamic system.

Examples Of SbS Methods

Piecewise exact method

- We use the exact solution of the equation of motion for a system excited by a linearly varying force, so the source of all errors lies in the piecewise linearisation of the force function and in the approximation due to a local linear model.
- We will see that an appropriate time step can be decided in terms of the number of points required to accurately describe either the force or the response function.

Piecewise exact method

For a generic time step of duration h, consider

- $\{x_0, \dot{x}_0\}$ the initial state vector,
- p_0 and p_1, the values of $p(t)$ at the start and the end of the integration step,
- the linearised force

$$p(\tau) = p_0 + \alpha\tau,\ 0 \le \tau \le h,\ \alpha = (p(h) - p(0))/h,$$

- the forced response

$$x = e^{-\zeta\omega\tau}(A\cos(\omega_D\tau) + B\sin(\omega_D\tau)) + (\alpha k\tau + kp_0 - \alpha c)/k^2,$$

where k and c are the stiffness and damping of the SDOF system.

Piecewise exact method

Evaluating the response x and the velocity $\dot{x}$ for $\tau = 0$ and equating to $\{x_0, \dot{x}_0\}$, writing $\Delta_{st} = p(0)/k$ and $\delta(\Delta_{st}) = (p(h) - p(0))/k$, one can find A and B

$$A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_\mathrm{D}}$$

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}$$

substituting and evaluating for $\tau = h$ one finds the state vector at the end of the step.

Piecewise exact method

With

$$\mathcal{S}_{\zeta,h} = \sin(\omega_\mathrm{D} h)\exp(-\zeta\omega h) \text{ and } \mathcal{C}_{\zeta,h} = \cos(\omega_\mathrm{D} h)\exp(-\zeta\omega h)$$

and the previous definitions of Δ_{st} and $\delta(\Delta_{st})$, finally we can write

$$x(h) = A\,\mathcal{S}_{\zeta,h} + B\,\mathcal{C}_{\zeta,h} + (\Delta_{st} + \delta(\Delta_{st})) - \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h}$$

$$\dot{x}(h) = A(\omega_\mathrm{D}\mathcal{C}_{\zeta,h} - \zeta\omega\mathcal{S}_{\zeta,h}) - B(\zeta\omega\mathcal{C}_{\zeta,h} + \omega_\mathrm{D}\mathcal{S}_{\zeta,h}) + \frac{\delta(\Delta_{st})}{h}$$

where

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}, \quad A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_\mathrm{D}}.$$

Example

We have a damped system that is excited by a load in resonance with the system, we know the exact response and we want to compute a step-by-step approximation using different step lengths.

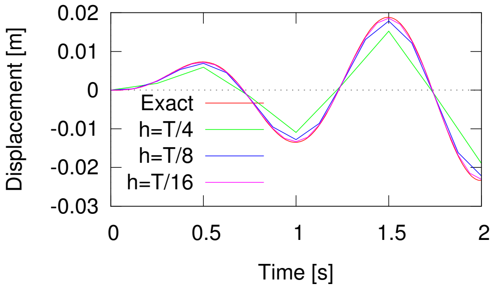
m=1000kg,

$k = 4\pi^2\,1000\mathrm{N/m}$,

$\omega = 2\pi$,

$\zeta = 0.05$,

$p(t) = 4\pi^2 5\,\mathrm{N}\,\sin(2\pi\,t)$



It is apparent that you have a very good approximation when the linearised loading is a very good approximation of the input function, let's say $h \le T/10$.