PieceWise_Exact_Integration

April 4, 2016

```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        from numpy import pi, cos, sin, sqrt, exp
```

1 Piecewise Exact Integration

1.1 The Dynamical System

We want to study a damped SDOF system, so characterized

```
In [2]: T=1.0                   # Natural period of the oscillator
        w=2*pi                  # circular frequency of the oscillator

        m=1000.0                # oscillator's mass, in kg
        k=m*w*w                 # oscillator stifness, in N/m
        z=0.05                  # damping ratio over critical
        c=2*z*m*w               # damping

        wd=w*sqrt(1-z*z)        # damped circular frequency
        ratio=sqrt(1-z*z)       # ratio damped/undamped frequencies
```

The excitation is given by a force such that the static displacement is 5 mm, modulated by a sine in resonance with the dynamic sistem, i.e., $\omega = \omega_n$.

```
In [3]: D=0.005                 # static displacement, 5mm
        P=D*k                   # force amplitude
```

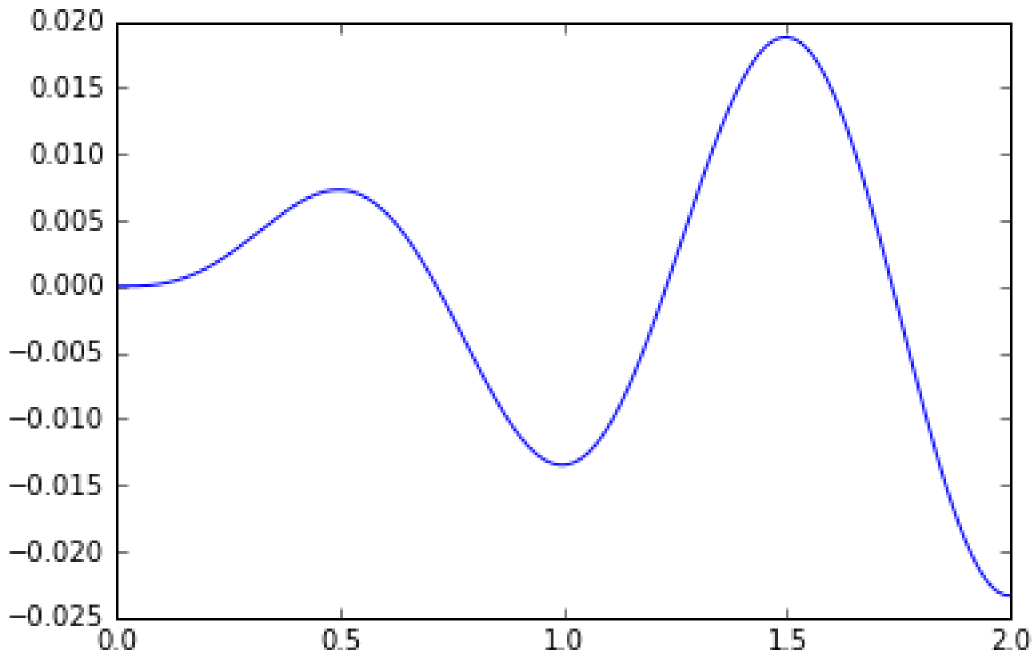
For such a system, we know exactly the response. The particular integral is

$$\xi(t) = -\frac{\cos\omega t}{2\zeta}$$

and imposing rest initial conditions it is

$$x(t) = \frac{\Delta_{st}}{2\zeta}((\frac{\zeta}{\sqrt{1=\zeta^2}}\sin\omega_D t + \cos\omega_D t)\exp(-\zeta\omega t) - \cos\omega t), \qquad \omega = \omega_n.$$

```
In [4]: def exact(t):
            return D*((z*sin(wd*t)/ratio+cos(wd*t))*exp(-z*w*t)-cos(w*t))/(2*z)
        t = np.linspace(0.0, 2.0, 1001)
        plt.plot(t, exact(t));
```



1.2 Numerical integration

Now we prepare for the numerical integration, first the constants that represent the homogeneous response at end of step

```
In [5]: def initstep(h):
            cdh=cos(wd*h)*exp(-z*w*h)
            sdh=sin(wd*h)*exp(-z*w*h)
            return cdh, sdh
```

then the actual step computations, where in terms of the initial state vector and the load variation the final state is derived.

```
In [6]: def step(x0,v0,p0,p1,h,cdh,sdh):
            dst=p0/k
            ddst=(p1-p0)/k
            B = x0 - dst + ((2*z)/w)*(ddst/h)
            A = (v0 + z*w*B - ddst/h)/wd
            x1 = A*sdh + B*cdh + dst + ddst - ddst/h * 2*z/w
            v1 = A*(wd*cdh-z*w*sdh) - B*(z*w*cdh+wd*sdh) + ddst/h
            return x1, v1
```

With those pieces in place, we can define a function that, for a given number of steps per period computes the response on the interval $0 \leq t \leq 2.0$.

```
In [7]: def resp(nstep):
            T = np.linspace(0.0, 2.0, 2*nstep + 1)
            X = np.zeros(2*nstep + 1)

            h=1./float(nstep)
```

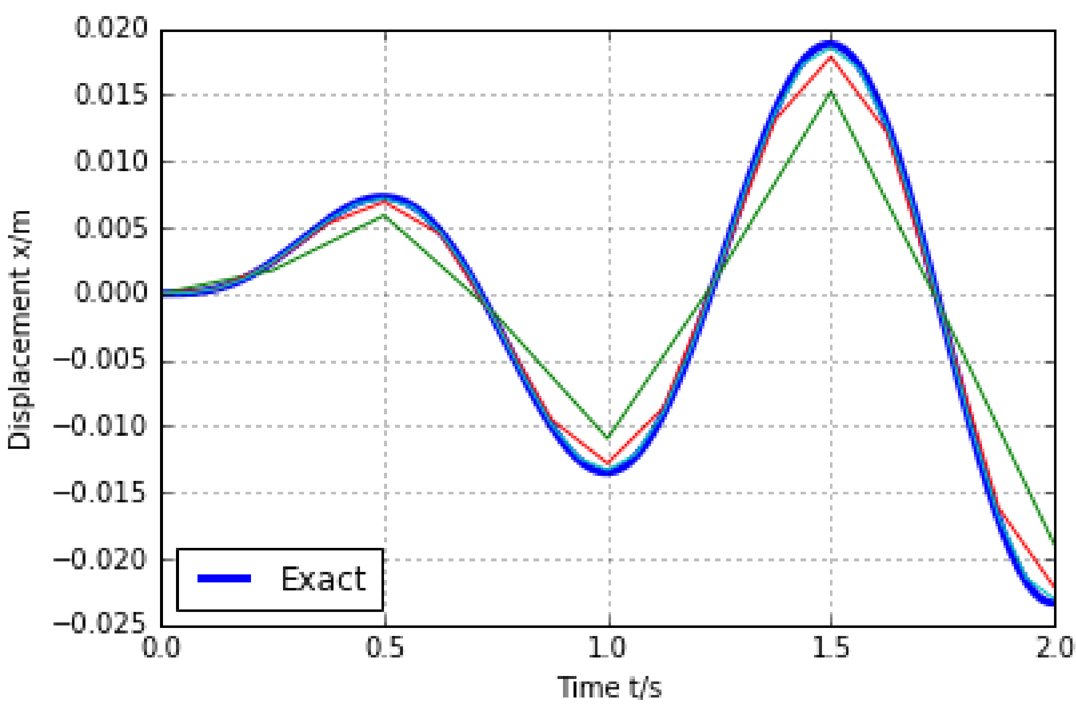
```
    cdh, sdh = initstep(h)
    x1=0. ; v1=0. ; p1=0

    for i, t in enumerate(T):
        X[i] = x1
        x0=x1 ; v0=v1 ; p0=p1 ; p1=P*sin(w*(t+h))
        x1,v1=step(x0,v0,p0,p1,h, cdh, sdh)
    return T, X
```

Let's compute the responses for different numbers of steps, and store them away too...

```
In [8]: t_x = {n:resp(n) for n in (4, 8, 16)}
```

Eventually we can plot the numerical responses along with the exact response

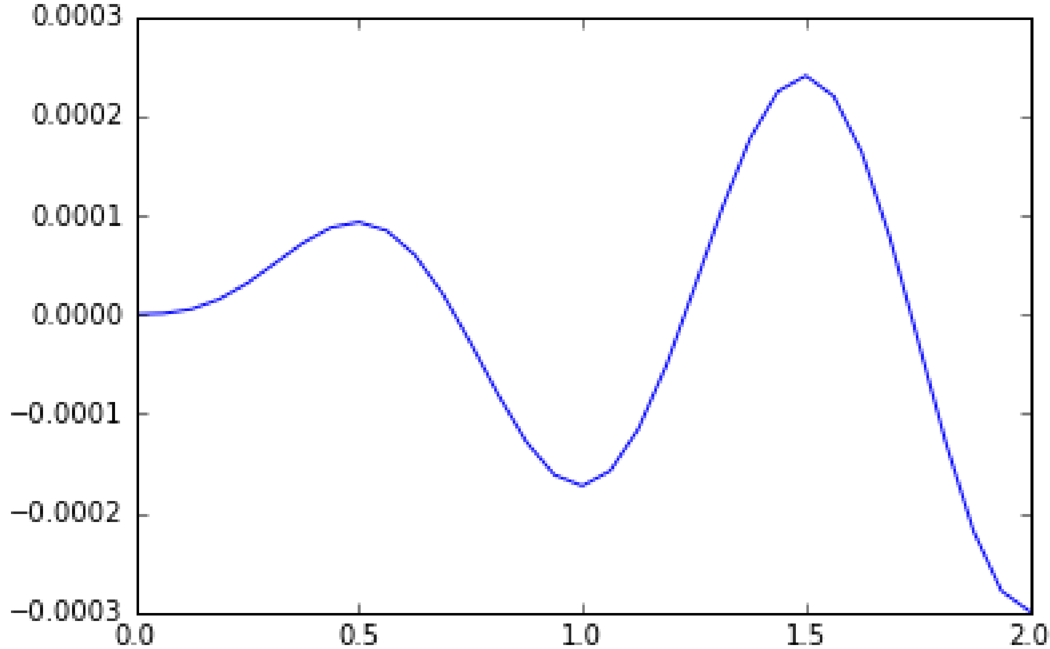
```
In [9]: plt.plot(t, exact(t), label='Exact', lw=3)
        plt.plot(*t_x[4], *t_x[8], *t_x[16])
        plt.grid()
        plt.legend(loc=3)
        plt.xlabel('Time t/s')
        plt.ylabel('Displacement x/m');
```



But... there are only two numerical curves and I've plotted three of them.

Let's plot the difference between the exact response and the response computed at 16 samples per period...

```
In [10]: t16, x16 = t_x[16]
         plt.plot(t16, exact(t16)-x16)

Out[10]: [<matplotlib.lines.Line2D at 0x7f18b89c6208>]
```



As you can see, the max difference is about 0.3 mm, to be compared with a max response of almost 25 mm, hence an error in the order of 1.2% that in the previous plot led to the apparent disappearance of the `NSTEP=16` curve.