Generalized Single Degree of Freedom Systems

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April 4, 2016

Outline

Introductory Remarks

Until now our *SDOF*'s were described as composed by a single mass connected to a fixed reference by means of a spring and a damper. While the mass-spring is a useful representation, many different, more complex systems can be studied as *SDOF* systems, either exactly or under some simplifying assumption.

Introductory Remarks

Until now our *SDOF*'s were described as composed by a single mass connected to a fixed reference by means of a spring and a damper.
While the mass-spring is a useful representation, many different, more complex systems can be studied as *SDOF* systems, either exactly or under some simplifying assumption.

1. *SDOF* rigid body assemblages, where the flexibility is concentrated in a number of springs and dampers, can be studied, e.g., using the Principle of Virtual Displacements and the D'Alembert Principle.
2. simple structural systems can be studied, in an approximate manner, assuming a fixed pattern of displacements, whose amplitude (the single degree of freedom) varies with time.

Further Remarks on Rigid Assemblages

Today we restrict our consideration to plane, 2-D systems.
In rigid body assemblages the limitation to a single shape of displacement is a consequence of the configuration of the system, i.e., the disposition of supports and internal hinges.
When the equation of motion is written in terms of a single parameter and its time derivatives, the terms that figure as coefficients in the equation of motion can be regarded as the *generalised* properties of the assemblage: generalised mass, damping and stiffness on left hand, generalised loading on right hand.

$$m^\star \ddot{x} + c^\star \dot{x} + k^\star x = p^\star(t)$$

Further Remarks on Continuous Systems

Continuous systems have an infinite variety of deformation patterns. By restricting the deformation to a single shape of varying amplitude, we introduce an infinity of internal contstraints that limit the infinite variety of deformation patterns, but under this assumption the system configuration is mathematically described by a single parameter, so that

- our *model* can be analysed in exactly the same way as a strict *SDOF* system,
- we can compute the *generalised* mass, damping, stiffness properties of the *SDOF model* of the continuous system.

Final Remarks on Generalised *SDOF* Systems

From the previous comments, it should be apparent that everything we have seen regarding the behaviour and the integration of the equation of motion of proper *SDOF* systems applies to rigid body assemblages and to *SDOF* models of flexible systems, provided that we have the means for determining the *generalised* properties of the dynamical systems under investigation.

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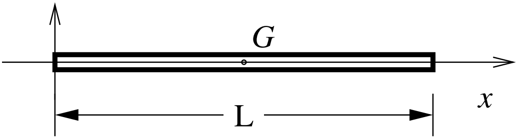
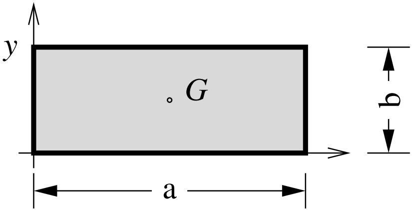
- inertial forces are distributed forces, acting on each material point of each rigid body, their resultant can be described by
  - a force applied to the centre of mass of the body, proportional to acceleration vector (of the centre of mass itself) and total mass $M = \int dm$
  - a couple, proportional to angular acceleration and the moment of inertia J of the rigid body, $J = \int (x^2 + y^2) dm$.

Rigid Bar



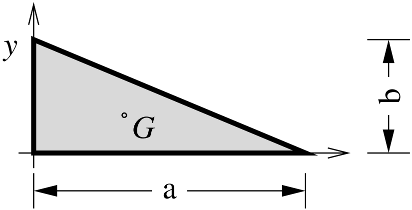
Unit mass	$\bar{m} = \text{constant},$
Length	$L,$
Centre of Mass	$x_G = L/2,$
Total Mass	$m = \bar{m}L,$
Moment of Inertia	$J = m\frac{L^2}{12} = \bar{m}\frac{L^3}{12}$

Rigid Rectangle



Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/2, \quad y_G = b/2$
Total Mass	$m = \gamma ab,$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{12} = \gamma\dfrac{a^3b+ab^3}{12}$

Rigid Triangle

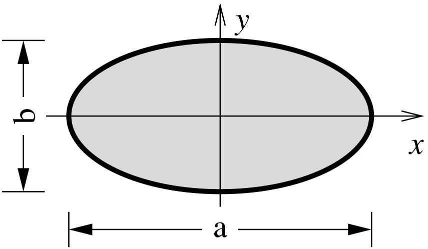


For a right triangle.

Unit mass	$\gamma = \text{constant},$
Sides	$a,\ b$
Centre of Mass	$x_G = a/3, \quad y_G = b/3$
Total Mass	$m = \gamma ab/2,$
Moment of Inertia	$J = m\dfrac{a^2+b^2}{18} = \gamma\dfrac{a^3b+ab^3}{36}$

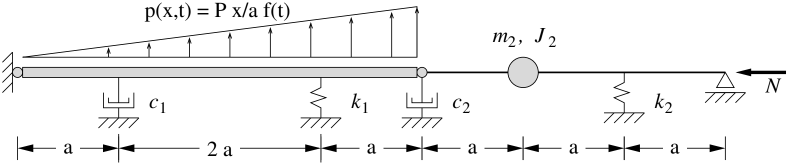
Rigid Oval

When $a = b = D = 2R$ the oval is a circle.



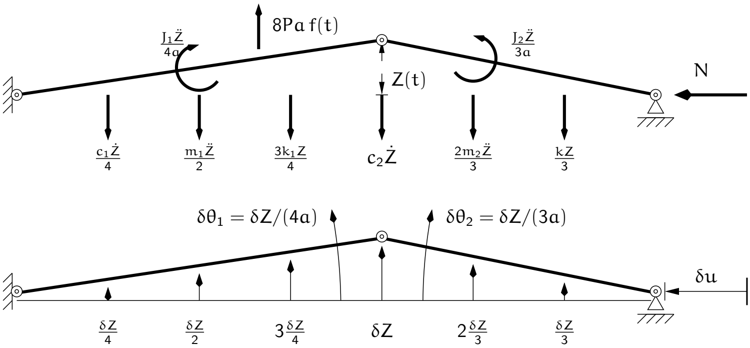
Unit mass	$\gamma = \text{constant},$
Axes	$a,\ b$
Centre of Mass	$x_G = y_G = 0$
Total Mass	$m = \gamma \dfrac{\pi a b}{4},$
Moment of Inertia	$J = m \dfrac{a^2 + b^2}{16}$

trabacolo1



The mass of the left bar is $m_1 = \bar{m}\,4a$ and its moment of inertia is $J_1 = m_1\frac{(4a)^2}{12} = 4a^2m_1/3$.
The maximum value of the external load is $P_{max} = P\,4a/a = 4P$ and the resultant of triangular load is $R = 4P \times 4a/2 = 8Pa$

Forces and Virtual Displacements



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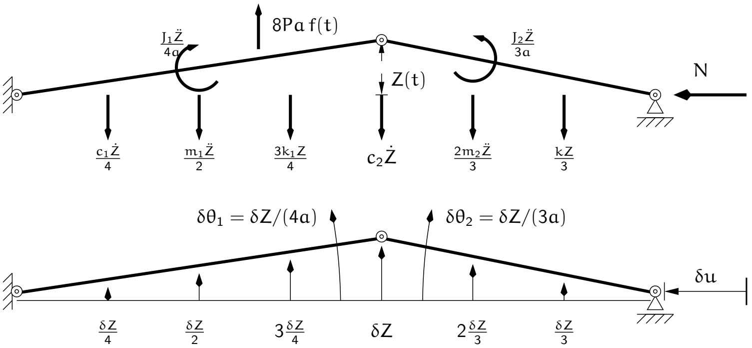


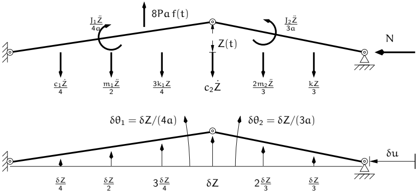
$$u = 7a - 4a\cos\theta_1 - 3a\cos\theta_2, \qquad \delta u = 4a\sin\theta_1\delta\theta_1 + 3a\sin\theta_2\delta\theta_2$$

$$\delta\theta_1 = \delta Z/(4a), \qquad \delta\theta_2 = \delta Z/(3a)$$

$$\sin\theta_1 \approx Z/(4a), \qquad \sin\theta_2 \approx Z/(3a)$$

$$\delta u = \left(\frac{1}{4a} + \frac{1}{3a}\right) Z\,\delta Z = \frac{7}{12a} Z\,\delta Z$$

Principle of Virtual Displacements



The virtual work of the Inertial forces:

$$\delta W_I = -m_1\frac{\ddot{Z}}{2}\frac{\delta Z}{2} - J_1\frac{\ddot{Z}}{4a}\frac{\delta Z}{4a} - m_2\frac{2\ddot{Z}}{3}\frac{2\delta Z}{3} - J_2\frac{\ddot{Z}}{3a}\frac{\delta Z}{3a}$$

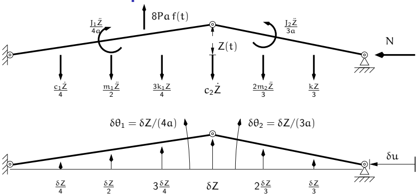
$$= -\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z}\,\delta Z$$

$$\delta W_D = -c_1\frac{\dot{Z}}{4}\frac{\delta Z}{4} - -c_2 Z\,\delta Z = -\left(c_2 + c_1/16\right)\dot{Z}\,\delta Z$$

$$\delta W_S = -k_1\frac{3Z}{4}\frac{3\delta Z}{4} - k_2\frac{Z}{3}\frac{\delta Z}{3} = -\left(\frac{9k_1}{16} + \frac{k_2}{9}\right)Z\,\delta Z$$

$$\delta W_{Ext} = 8Pa\,f(t)\frac{2\delta Z}{3} + N\frac{7}{12a}Z\,\delta Z$$

Principle of Virtual Displacements



The virtual work of the Damping forces:

$$\delta W_I = -m_1\frac{\ddot Z}{2}\frac{\delta Z}{2} - J_1\frac{\ddot Z}{4a}\frac{\delta Z}{4a} - m_2\frac{2\ddot Z}{3}\frac{2\delta Z}{3} - J_2\frac{\ddot Z}{3a}\frac{\delta Z}{3a}$$
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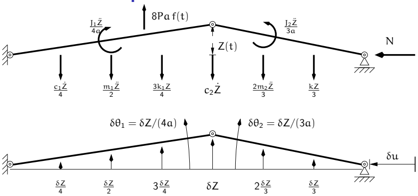

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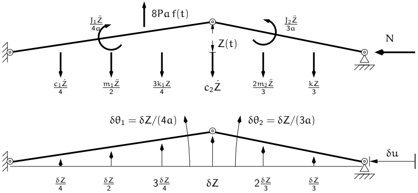
Principle of Virtual Displacements



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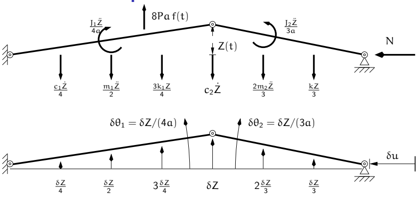

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Principle of Virtual Displacements

For a rigid body in condition of equilibrium the total virtual work must be equal to zero

$$\delta W_{\mathsf{I}} + \delta W_{\mathsf{D}} + \delta W_{\mathsf{S}} + \delta W_{\mathsf{Ext}} = 0$$

Substituting our expressions of the virtual work contributions and simplifying δZ, the equation of equilibrium is

$$\left(\frac{m_1}{4} + 4\frac{m_2}{9} + \frac{J_1}{16a^2} + \frac{J_2}{9a^2}\right)\ddot{Z} + \\ + \left(c_2 + c_1/16\right)\dot{Z} + \left(\frac{9k_1}{16} + \frac{k_2}{9}\right)Z = \\ 8Pa\,f(t)\frac{2}{3} + N\frac{7}{12a}Z$$

Principle of Virtual Displacements

Collecting Z and its time derivatives give us

$$m^\star\ddot{Z} + c^\star\dot{Z} + k^\star Z = p^\star f(t)$$

introducing the so called *generalised properties*, in our example it is

$$\begin{aligned}
m^\star &= \frac{1}{4}m_1 + \frac{4}{9}9m_2 + \frac{1}{16a^2}J_1 + \frac{1}{9a^2}J_2,\\
c^\star &= \frac{1}{16}c_1 + c_2,\\
k^\star &= \frac{9}{16}k_1 + \frac{1}{9}k_2 - \frac{7}{12a}N,\\
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Geometrical stiffness

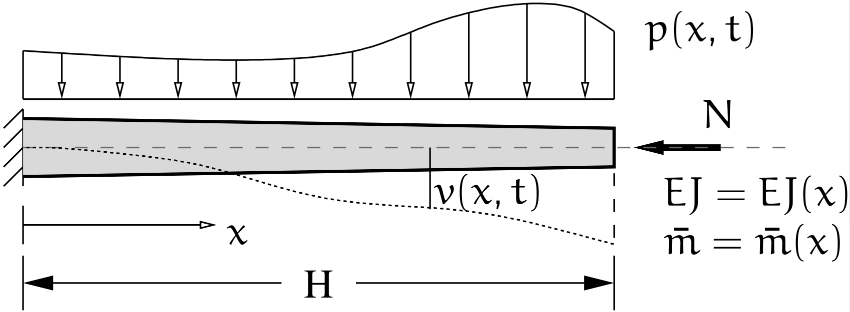
Let's start with an example...

Consider a cantilever, with varying properties $\bar{m}$ and EJ, subjected to a load that is function of both time t and position x,

$$p = p(x, t).$$

The transverse displacements v will be function of time and position,

$$v = v(x, t)$$



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Note that $\Psi(x)$, the *shape function*, is adimensional, while $Z(t)$ is dimensionally a generalised displacement, usually chosen to characterise the structural behaviour.
In our example we can use the displacement of the tip of the chimney, thus implying that $\Psi(H) = 1$ because

$$Z(t) = v(H,t) \quad \text{and}$$
$$v(H,t) = \Psi(H)\,Z(t)$$

Principle of Virtual Displacements

For a flexible system, the PoVD states that, at equilibrium,

$$\delta W_{\mathsf{E}} = \delta W_{\mathsf{I}}.$$

The virtual work of external forces can be easily computed, the virtual work of internal forces is usually approximated by the virtual work done by bending moments, that is

$$\delta W_{\mathsf{I}} \approx \int M \, \delta\chi$$

where χ is the curvature and $\delta\chi$ the virtual increment of curvature.

δW_E

The external forces are $p(x,t)$, N and the forces of inertia f_I; we have, by separation of variables, that $\delta v = \Psi(x)\delta Z$ and we can write

$$\delta W_p = \int_0^H p(x,t)\delta v\,dx = \left[\int_0^H p(x,t)\Psi(x)\,dx\right]\,\delta Z = p^\star(t)\,\delta Z$$

$$\begin{aligned}\delta W_{\text{Inertia}} &= \int_0^H -\bar{m}(x)\ddot{v}\delta v\,dx = \int_0^H -\bar{m}(x)\Psi(x)\ddot{Z}\Psi(x)\,dx\;\delta Z\\ &= \left[\int_0^H -\bar{m}(x)\Psi^2(x)\,dx\right]\,\ddot{Z}(t)\,\delta Z = m^\star\ddot{Z}\,\delta Z.\end{aligned}$$

The virtual work done by the axial force deserves a separate treatment...

δW_N

The virtual work of N is $\delta W_N = N\delta u$ where δu is the variation of the vertical displacement of the top of the chimney.
We start computing the vertical displacement of the top of the chimney in terms of the rotation of the axis line, $\phi \approx \Psi'(x)Z(t)$,

$$u(t) = H - \int_0^H \cos\phi\, dx = \int_0^H (1 - \cos\phi)\, dx,$$

substituting the well known approximation $\cos\phi \approx 1 - \frac{\phi^2}{2}$ in the above equation we have

$$u(t) = \int_0^H \frac{\phi^2}{2}\, dx = \int_0^H \frac{\Psi'^2(x)Z^2(t)}{2}\, dx$$

hence

$$\delta u = \int_0^H \Psi'^2(x)Z(t)\delta Z\, dx = \int_0^H \Psi'^2(x)\, dx\; Z\delta Z$$

and

$$\delta W_N = \left[\int_0^H \Psi'^2(x)\, dx\; N\right] Z\,\delta Z = k_G^\star\, Z\,\delta Z$$

δW_{Int}

Approximating the internal work with the work done by bending moments, for an infinitesimal slice of beam we write

$$\mathrm{d}W_{\mathsf{Int}} = \frac{1}{2} M v''(x,t)\,\mathrm{d}x = \frac{1}{2} M \Psi''(x) Z(t)\,\mathrm{d}x$$

with $M = EJ(x)v''(x)$

$$\delta(\mathrm{d}W_{\mathsf{Int}}) = EJ(x)\Psi''^{\,2}(x)Z(t)\delta Z\,\mathrm{d}x$$

integrating

$$\delta W_{\mathsf{Int}} = \left[\int_0^H EJ(x)\Psi''^{\,2}(x)\,\mathrm{d}x\right] Z\delta Z = k^\star\, Z\,\delta Z$$

Remarks

- the shape function *must* respect the geometrical boundary conditions of the problem, i.e., both

$$\Psi_1 = x^2 \qquad \text{and} \qquad \Psi_2 = 1 - \cos\frac{\pi x}{2H}$$

are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$

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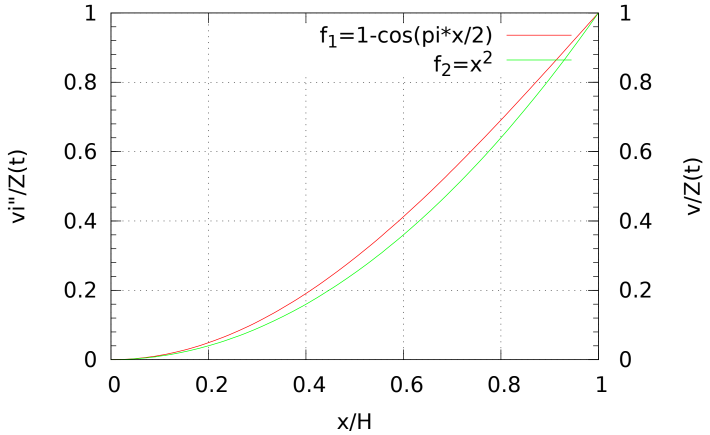

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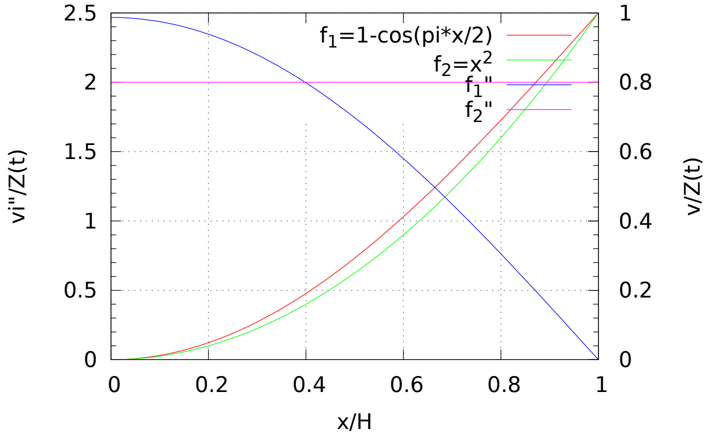
f1=1-cos(pi*x/2)
f2=x2
f1"
f2"
vi"/Z(t)
v/Z(t)
x/H
2.5
2
1.5
1
0.5
0
1
0.8
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  are accettable shape functions for our example, as $\Psi_1(0) = \Psi_2(0) = 0$ and $\Psi_1'(0) = \Psi_2'(0) = 0$

- better results are obtained when the second derivative of the shape function at least *resembles* the typical distribution of bending moments in our problem, so that between

$$\Psi_1'' = \text{constant} \qquad \text{and} \qquad \Psi_2\text{''} = \frac{\pi^2}{4H^2}\cos\frac{\pi x}{2H}$$

  the second choice is preferable.

Example

Using $\Psi(x) = 1 - \cos\frac{\pi x}{2H}$, with $\bar{m} =$ constant and $EJ =$ constant, with a load characteristic of seismic excitation, $p(t) = -\bar{m}\ddot{v}_g(t)$,

$$m^\star = \bar{m}\int_0^H (1 - \cos\frac{\pi x}{2H})^2\,dx = \bar{m}(\frac{3}{2} - \frac{4}{\pi})H$$

$$k^\star = EJ\frac{\pi^4}{16H^4}\int_0^H \cos^2\frac{\pi x}{2H}\,dx = \frac{\pi^4}{32}\frac{EJ}{H^3}$$

$$k_G^\star = N\frac{\pi^2}{4H^2}\int_0^H \sin^2\frac{\pi x}{2H}\,dx = \frac{\pi^2}{8H}N$$

$$p_g^\star = -\bar{m}\ddot{v}_g(t)\int_0^H 1 - \cos\frac{\pi x}{2H}\,dx = -\left(1 - \frac{2}{\pi}\right)\bar{m}H\,\ddot{v}_g(t)$$