Rayleigh's Quotient

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Outline

Vibration Analysis

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- We can use our previous results for flexible systems, based on the *SDOF* model, to give an estimate of the natural frequency $\omega^2 = k^\star/m^\star$
- A different approach, proposed by Lord Rayleigh, starts from different premises to give the same results but the *Rayleigh's Quotient* method is important because it offers a better understanding of the vibrational behaviour, eventually leading to successive refinements of the first estimate of ω^2.

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- the displacement and the velocity are in quadrature: when v is at its maximum $\dot{v} = 0$ (hence $V = V_{max}$, $T = 0$) and when $v = 0$ $\dot{v}$ is at its maximum (hence $V = 0$, $T = T_{max}$,

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- disregarding damping, the energy of the system is constant during free vibrations,

$$V_{\text{max}} + 0 = 0 + T_{\text{max}}$$

Rayleigh' s Quotient Method

Now we write the expressions for V_{max} and T_{max},

$$V_{max} = \frac{1}{2} Z_0^2 \int_S EJ(x) \Psi''^2(x)\, dx,$$
$$T_{max} = \frac{1}{2} \omega^2 Z_0^2 \int_S \bar{m}(x) \Psi^2(x)\, dx,$$

equating the two expressions and solving for ω^2 we have

$$\omega^2 = \frac{\int_S EJ(x)\Psi''^2(x)\, dx}{\int_S \bar{m}(x)\Psi^2(x)\, dx}.$$

Recognizing the expressions we found for $k^\star$ and $m^\star$ we could question the utility of Rayleigh's Quotient...

Rayleigh's Quotient Method

- in Rayleigh's method we know the specific time dependency of the inertial forces

  $$f_I = -\bar{m}(x)\ddot{v} = \bar{m}(x)\omega^2 Z_0 \Psi(x) \sin \omega t$$

  f_I has the same *shape* we use for displacements.
- if Ψ were the real shape assumed by the structure in free vibrations, the displacements v due to a loading $f_I = \omega^2 \bar{m}(x)\Psi(x)Z_0$ should be proportional to $\Psi(x)$ through a constant factor, with equilibrium respected in every point of the structure during free vibrations.

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- starting from a shape function $\Psi_0(x)$, a new shape function Ψ_1 can be determined normalizing the displacements due to the inertial forces associated with $\Psi_0(x)$, $f_I = \bar{m}(x)\Psi_0(x)$,

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- we are going to demonstrate that the new shape function is a better approximation of the true mode shape

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- the frequency of vibration of a structure with additional constraints is higher than the true natural frequency,
- the criterium to discriminate between different shape functions is: better shape functions give lower estimates of the natural frequency, the true natural frequency being a lower bound of all estimates.

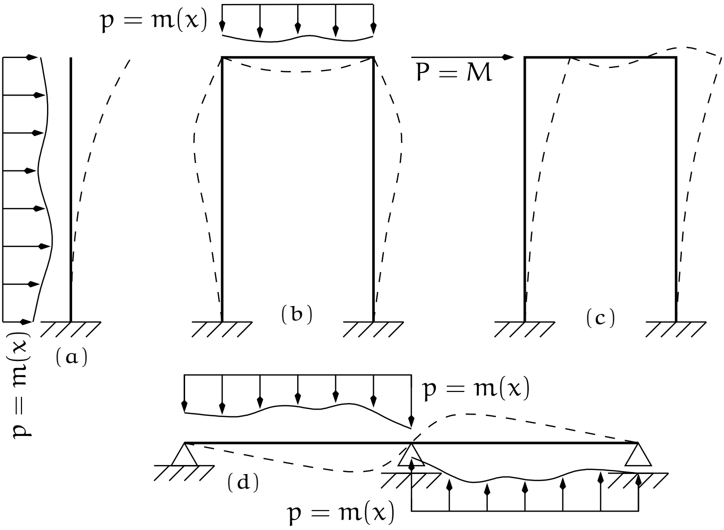
Selection of mode shapes 2

In general the selection of trial shapes goes through two steps,

1. the analyst considers the flexibilities of different parts of the structure and the presence of symmetries to devise an approximate shape,
2. the structure is loaded with constant loads directed as the assumed displacements, the displacements are computed and used as the shape function,

of course a little practice helps a lot in the the choice of a proper pattern of loading...

Selection of mode shapes 3



Refinement R_{00}

Choose a trial function $\Psi^{(0)}(x)$ and write

$$v^{(0)} = \Psi^{(0)}(x) Z^{(0)} \sin \omega t$$
$$V_{\text{max}} = \frac{1}{2} Z^{(0)2} \int EJ \Psi^{(0)\prime\prime 2}\, dx$$
$$T_{\text{max}} = \frac{1}{2} \omega^2 Z^{(0)2} \int \bar{m} \Psi^{(0)2}\, dx$$

our first estimate R_{00} of ω^2 is

$$\omega^2 = \frac{\int EJ \Psi^{(0)\prime\prime 2}\, dx}{\int \bar{m} \Psi^{(0)2}\, dx}.$$

Refinement R_{01}

We try to give a better estimate of V_{max} computing the external work done by the inertial forces,

$$p^{(0)} = \omega^2 \bar{m}(x) v^{(0)} = Z^{(0)} \omega^2 \Psi^{(0)}(x)$$

the deflections due to $p^{(0)}$ are

$$v^{(1)} = \omega^2 \frac{v^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \frac{Z^{(1)}}{\omega^2} = \omega^2 \Psi^{(1)} \bar{Z}^{(1)},$$

where we write $\bar{Z}^{(1)}$ because we need to keep the unknown ω^2 in evidence. The maximum strain energy is

$$V_{max} = \frac{1}{2} \int p^{(0)} v^{(1)} \, dx = \frac{1}{2} \omega^4 Z^{(0)} \bar{Z}^{(1)} \int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, dx$$

Equating to our previus estimate of T_{max} we find the R_{01} estimate

$$\omega^2 = \frac{Z^{(0)}}{\bar{Z}^{(1)}} \frac{\int \bar{m}(x) \Psi^{(0)} \Psi^{(0)} \, dx}{\int \bar{m}(x) \Psi^{(0)} \Psi^{(1)} \, dx}$$

Refinement R_{11}

With little additional effort it is possible to compute T_{max} from $v^{(1)}$:

$$T_{max} = \frac{1}{2}\omega^2 \int \bar{m}(x) v^{(1)2}\, dx = \frac{1}{2}\omega^6 \bar{Z}^{(1)2} \int \bar{m}(x) \Psi^{(1)2}\, dx$$

equating to our last approximation for V_{max} we have the R_{11} approximation to the frequency of vibration,

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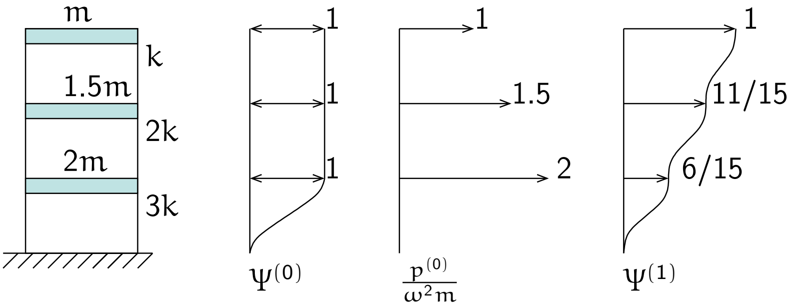
Of course the procedure can be extended to compute better and better estimates of ω^2 but usually the refinements are not extended beyond R_{11}, because it would be contradictory with the quick estimate nature of the Rayleigh's Quotient method and also because R_{11} estimates are usually very good ones.

Refinement Example



$$T = \frac{1}{2}\omega^2 \times 4.5 \times m\, Z_0$$

$$V = \frac{1}{2} \times 1 \times 3k\, Z_0$$

$$\omega^2 = \frac{3}{9/2}\frac{k}{m} = \frac{2}{3}\frac{k}{m}$$

$$v^{(1)} = \frac{15}{4}\frac{m}{k}\omega^2\Psi^{(1)}$$

$$\bar{Z}^{(1)} = \frac{15}{4}\frac{m}{k}$$

$$V^{(1)} = \frac{1}{2}m\frac{15}{4}\frac{m}{k}\omega^4\,(1 + 33/30 + 4/5)$$

$$= \frac{1}{2}m\frac{15}{4}\frac{m}{k}\omega^4\,\frac{87}{30}$$

$$\omega^2 = \frac{\frac{9}{2}m}{m\frac{87}{8}\frac{m}{k}} = \frac{12}{29}\frac{k}{m} = 0.4138\frac{k}{m}$$