Response by Superposition

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April 10, 2016

Eigenvector Expansion

For a *N-DOF* system, it is possible and often advantageous to represent the displacements $\boldsymbol{x}$ in terms of a linear combination of the free vibration modal shapes, the eigenvectors, by the means of a set of modal coordinates,

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- they possess orthogonality properties,
- we will see that it is usually possible to approximate the response using only a few low frequency terms.

Inverting Eigenvector Expansion

The columns of the eigenmatrix $\boldsymbol{\Psi}$ are the N linearly indipendent eigenvectors $\boldsymbol{\psi}_i$, hence the eigenmatrix is non-singular and it is always correct to write $\boldsymbol{q} = \boldsymbol{\Psi}^{-1}\boldsymbol{x}$.
However, it is not necessary to invert the eigenmatrix...

Inverting Eigenvector Expansion

The modal expansion is

$$\boldsymbol{x} = \sum \psi_i q_i = \boldsymbol{\Psi}\, \boldsymbol{q};$$

multiply each member by $\boldsymbol{\Psi}^T \boldsymbol{M}$, taking into account that $\boldsymbol{M}^\star = \boldsymbol{\Psi}^T \boldsymbol{M} \boldsymbol{\Psi}$:

$$\boldsymbol{\Psi}^T \boldsymbol{M}\boldsymbol{x} = \boldsymbol{\Psi}^T \boldsymbol{M}\boldsymbol{\Psi}\boldsymbol{q} \qquad \Rightarrow \qquad \boldsymbol{\Psi}^T \boldsymbol{M}\boldsymbol{x} = \boldsymbol{M}^\star \boldsymbol{q}$$

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Note: this formula works also when we don't know all the eigenvectors and the inversion of a partial, rectangular $\boldsymbol{\Psi}$ is not feasible.

Undamped System

Substituting the modal expansion $\boldsymbol{x} = \boldsymbol{\Psi}\,\boldsymbol{q}$ into the equation of motion, $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{p}(t)$,

$$\boldsymbol{M}\boldsymbol{\Psi}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{\Psi}\boldsymbol{q} = \boldsymbol{p}(t).$$

Premultiplying each term by $\boldsymbol{\Psi}^T$ and using the orthogonality of the eigenvectors with respect to the structural matrices, for each modal DOF we have an indipendent equation of dynamic equilibrium,

$$M_i\,\ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \quad i = 1, \ldots, N.$$

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The equations of motion written in terms of nodal coordinates constitute a system of N interdipendent, *coupled* differential equations, written in terms of modal coordinates constitute a set of N indipendent, *uncoupled* differential equations.

Damped System

For a damped system, the equation of motion is

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t)$$

and in modal coordinates

$$M_i\,\ddot{q}_i + \boldsymbol{\psi}^T\boldsymbol{C}\,\boldsymbol{\Psi}\,\dot{\boldsymbol{q}} + \omega_i^2 M_i q_i = p_i^\star(t).$$

With $\boldsymbol{\psi}_i^T\boldsymbol{C}\,\boldsymbol{\psi}_j = c_{ij}$ the i-th equation of dynamic equilibrium is

$$M_i\,\ddot{q}_i + \sum_j c_{ij}\,\dot{q}_j + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \ldots, N;$$

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The equations of motion in modal coordinates are uncoupled only if $c_{ij} = \delta_{ij} C_i$.
If we define the damping matrix as

$$\boldsymbol{C} = \sum_b \mathfrak{c}_b \boldsymbol{M}\left(\boldsymbol{M}^{-1}\boldsymbol{K}\right)^b,$$

we know that, as required,

$$c_{ij} = \delta_{ij} C_i \quad \text{with } C_i \; (= 2\zeta_i M_i \omega_i) = \sum_b \mathfrak{c}_b \left(\omega_i^2\right)^b.$$

Damped Systems, a Comment

If the response is computed by modal superposition, it is usually preferred a simpler but equivalent procedure: for each mode of interest the analyst imposes a given damping ratio and the integration of the modal equation of equilibrium is carried out as usual.

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The $\sum \mathfrak{c}_b \ldots$ procedure is useful when, e.g. for non-linear problems, the integration of the eq. of motion is carried out in nodal coordinates, because it is easier to specify damping properties globally as elastic modes properties (that can be measured or deduced from similar outsets) than to assign correct damping properties at the *FE* level and assembling $\boldsymbol{C}$ by the *FEM*.

Initial Conditions

For a damped system, the modal response can be evaluated, for rest initial conditions, using the Duhamel integral,

$$q_i(t) = \frac{1}{M_i \omega_i} \int_0^t p_i(\tau) e^{-\zeta_i \omega_i (t-\tau)} \sin \omega_{Di}(t-\tau)\, \mathrm{d}\tau$$

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For different initial conditions $\boldsymbol{x}_0$, $\dot{\boldsymbol{x}}_0$, we can easily have the initial conditions in modal coordinates:

$$\boldsymbol{q}_0 = \boldsymbol{M}^{\star-1}\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x}_0$$

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{M}^{\star-1}\boldsymbol{\Psi}^T\boldsymbol{M}\dot{\boldsymbol{x}}_0$$

and the total modal response can be obtained by superposition of Duhamel integral and free vibrations,

$$q_i(t) = e^{-\zeta_i\omega_i t}(q_{i,0}\cos\omega_{Di}t + \frac{\dot{q}_{i,0}+q_{i,0}\zeta_i\omega_i}{\omega_{Di}}\sin\omega_{Di}t) + \cdots$$

Truncated sum

Having computed all $q_i(t)$, we can sum all the modal responses using the eigenvectors,

$$\boldsymbol{x}(t) = \boldsymbol{\psi}_1 q_1(t) + \boldsymbol{\psi}_2 q_2(t) + \cdots + \boldsymbol{\psi}_N q_N(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i q_i(t)$$

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The importance of truncated sum approximation is twofold:

- less computational effort: less eigenpairs to calculate, less equation of motion to integrate etc
- in FEM models the higher modes are rough approximations to structural ones (mostly due to uncertainties in mass distribution details) and the truncated sum excludes potentially spurious contributions from the response.

Elastic Forces

Until now, we showed interest in displacements only, but we are interested in elastic forces too. We know that elastic forces can be expressed in terms of displacements and the stiffness matrix:

$$\boldsymbol{f}_S(t) = \boldsymbol{K}\,\boldsymbol{x}(t) = \boldsymbol{K}\psi_1 q_1(t) + \boldsymbol{K}\psi_2 q_2(t) + \cdots .$$

From the characteristic equation we know that

$$\boldsymbol{K}\psi_i = \omega_i^2 \boldsymbol{M}\psi_i$$

substituting in the previous equation

$$\boldsymbol{f}_S(t) = \omega_1^2 \boldsymbol{M}\psi_1 q_1(t) + \omega_2^2 \boldsymbol{M}\psi_2 q_2(t) + \cdots .$$

Elastic Forces, 2

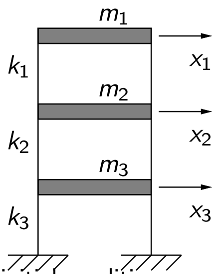
The high frequency modes contribution to the elastic forces, e.g.

$$\boldsymbol{f}_S(t) = \omega_1^2 \boldsymbol{M} \psi_1 q_1(t) + \cdots + \omega_{20}^2 \boldsymbol{M} \psi_{20} q_{20}(t) + \cdots ,$$

when compared to low frequency mode contributions are more important than their contributions to displacement, because of the multiplicative term ω_i^2.
From this fact follows that, to estimate internal forces within a given accuracy a greater number of modes must be considered in a truncated sum than the number required to estimate displacements within the same accuracy.

Example: problem statement

$$k_1 = 120\,\text{MN/m}, \quad m_1 = 200\,\text{t},$$
$$k_2 = 240\,\text{MN/m}, \quad m_2 = 300\,\text{t},$$
$$k_3 = 360\,\text{MN/m}, \quad m_3 = 400\,\text{t}.$$



1. The above structure is subjected to these initial conditions,

$$\boldsymbol{x}_0^T = \begin{Bmatrix} 5\,\text{mm} & 4\,\text{mm} & 3\,\text{mm} \end{Bmatrix},$$
$$\dot{\boldsymbol{x}}_0^T = \begin{Bmatrix} 0 & 9\,\text{mm/s} & 0 \end{Bmatrix}.$$

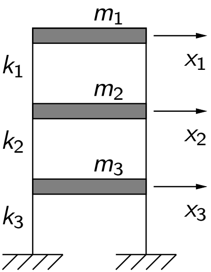
Write the equation of motion using modal superposition.

2. The above structure is subjected to a half-sine impulse,

$$\boldsymbol{p}^T(t) = \begin{Bmatrix} 1 & 2 & 2 \end{Bmatrix} 2.5\,\text{MN} \sin\frac{\pi t}{t_1}, \quad \text{with } t_1 = 0.02\,\text{s}.$$

Write the equation of motion using modal superposition.

Example: structural matrices



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The structural matrices can be written

$$\boldsymbol{K} = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} = k\overline{\boldsymbol{K}}, \qquad \text{with } k = 120\,\frac{\text{MN}}{\text{m}},$$

$$\boldsymbol{M} = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = m\overline{\boldsymbol{M}}, \qquad \text{with } m = 100000\,kg.$$

Example: adimensional eigenvalues

We want the solutions of the characteristic equation, so we start writing that the determinant of the equation must be zero:

$$\left\|\overline{\boldsymbol{K}} - \frac{\omega^2}{k/m}\overline{\boldsymbol{M}}\right\| = \left\|\overline{\boldsymbol{K}} - \Omega^2\overline{\boldsymbol{M}}\right\| = 0,$$

with $\omega^2 = 1200\left(\frac{\text{rad}}{\text{s}}\right)^2\Omega^2$.
Expanding the determinant

$$\left\|\begin{matrix} 1-2\Omega^2 & -1 & 0 \\ -1 & 3-3\Omega^2 & -2 \\ 0 & -2 & 5-4\Omega^2 \end{matrix}\right\| = 0$$

we have the following algebraic equation of 3rd order in Ω^2

$$24\left(\Omega^6 - \frac{11}{4}\Omega^4 + \frac{15}{8}\Omega^2 - \frac{1}{4}\right) = 0.$$

Example: table of eigenvalues etc

Here are the adimensional roots Ω_i^2, $i = 1, 2, 3$, the dimensional eigenvalues $\omega_i^2 = 1200\frac{\text{rad}^2}{\text{s}^2}\Omega_i^2$ and all the derived dimensional quantities:

$$\begin{aligned}
\Omega_1^2 &= 0.17573 & \Omega_2^2 &= 0.8033 & \Omega_3^2 &= 1.7710\\
\omega_1^2 &= 210.88 & \omega_2^2 &= 963.96 & \omega_3^2 &= 2125.2\\
\omega_1 &= 14.522 & \omega_2 &= 31.048 & \omega_3 &= 46.099\\
f_1 &= 2.3112 & f_2 &= 4.9414 & f_3 &= 7.3370\\
T_1 &= 0.43268 & T_3 &= 0.20237 & T_3 &= 0.1363
\end{aligned}$$

Example: eigenvectors and modal matrices

With $\psi_{1j} = 1$, using the 2nd and 3rd equations,

$$\begin{bmatrix} 3 - 3\Omega_j^2 & -2 \\ -2 & +5 - 4\Omega_j^2 \end{bmatrix} \begin{Bmatrix} \psi_{2j} \\ \psi_{3j} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The above equations must be solved for $j = 1, 2, 3$. The solutions are finally collected in the eigenmatrix

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 & 1 \\ +0.648535272183 & -0.606599092464 & -2.54193617967 \\ +0.301849953585 & -0.678977475113 & +2.43962752148 \end{bmatrix}.$$

The Modal Matrices are

$$\boldsymbol{M}^\star = \begin{bmatrix} 362.6 & 0 & 0 \\ 0 & 494.7 & 0 \\ 0 & 0 & 4519.1 \end{bmatrix} \times 10^3\,\text{kg},$$

$$\boldsymbol{K}^\star = \begin{bmatrix} 76.50 & 0 & 0 \\ 0 & 477.0 & 0 \\ 0 & 0 & 9603.9 \end{bmatrix} \times 10^6\,\frac{\text{N}}{\text{m}}$$

Example: initial conditions in modal coordinates

$$\boldsymbol{q}_0 = (\boldsymbol{M}^\star)^{-1}\boldsymbol{\Psi}^T\boldsymbol{M}\begin{Bmatrix}5\\4\\3\end{Bmatrix}\,\text{mm} = \begin{Bmatrix}+5.9027\\-1.0968\\+0.1941\end{Bmatrix}\,\text{mm},$$

$$\dot{\boldsymbol{q}}_0 = (\boldsymbol{M}^\star)^{-1}\boldsymbol{\Psi}^T\boldsymbol{M}\begin{Bmatrix}0\\9\\0\end{Bmatrix}\,\frac{\text{mm}}{\text{s}} = \begin{Bmatrix}+4.8288\\-3.3101\\-1.5187\end{Bmatrix}\,\frac{\text{mm}}{\text{s}}$$

Example: structural response

These are the displacements, in mm

$$x_1 = +5.91\cos(14.5t + .06) + 1.10\cos(31.0t - 3.04) + 0.20\cos(46.1t - 0.17)$$
$$x_2 = +3.83\cos(14.5t + .06) - 0.67\cos(31.0t - 3.04) - 0.50\cos(46.1t - 0.17)$$
$$x_3 = +1.78\cos(14.5t + .06) - 0.75\cos(31.0t - 3.04) + 0.48\cos(46.1t - 0.17)$$

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\end{aligned}
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and these the elastic/inertial forces, in kN

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\begin{aligned}
x_1 &= +249.\cos(14.5t+.06)+212.\cos(31.0t-3.04)+084.\cos(46.1t-0.17)\\
x_2 &= +243.\cos(14.5t+.06)-193.\cos(31.0t-3.04)-319.\cos(46.1t-0.17)\\
x_3 &= +151.\cos(14.5t+.06)-288.\cos(31.0t-3.04)+408.\cos(46.1t-0.17)
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As expected, the contributions of the higher modes are more important for the forces, less important for the displacements.