Truncation Errors, Correction Procedures

Giacomo Boffi

http://intranet.dica.polimi.it/people/boffi-giacomo

Dipartimento di Ingegneria Civile Ambientale e Territoriale
Politecnico di Milano

April 14, 2016

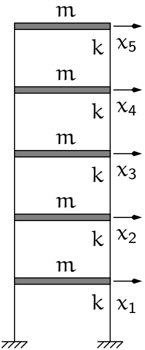
Outline

Rayleigh-Ritz Example

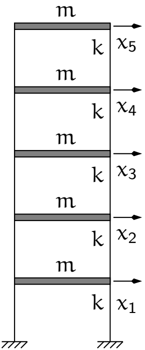
Subspace iteration

How many eigenvectors?
- Modal partecipation factor
- Dynamic magnification factor
- Static Correction

RR Example



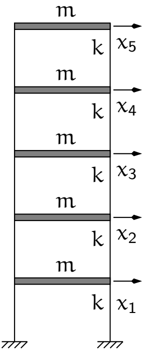
RR Example



The structural matrices

$$\mathbf{K} = k\begin{bmatrix} +2 & -1 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \mathbf{M} = m\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

RR Example



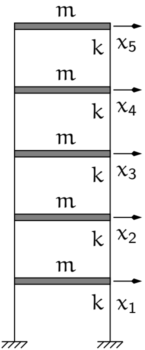
The structural matrices

$$\mathbf{K} = k \begin{bmatrix} +2 & -1 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Ritz base vectors and the reduced matrices,

$$\mathbf{\Phi} = \begin{bmatrix} 0.2 & -0.5 \\ 0.4 & -1.0 \\ 0.6 & -0.5 \\ 0.8 & +0.0 \\ 1.0 & 1.0 \end{bmatrix} \qquad \bar{\mathbf{K}} = k \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 2.0 \end{bmatrix} \qquad \bar{\mathbf{M}} = m \begin{bmatrix} 2.2 & 0.2 \\ 0.2 & 2.5 \end{bmatrix}$$

RR Example



The structural matrices

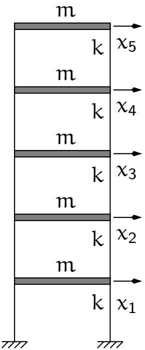
$$\mathbf{K} = k \begin{bmatrix} +2 & -1 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Ritz base vectors and the reduced matrices,

$$\mathbf{\Phi} = \begin{bmatrix} 0.2 & -0.5 \\ 0.4 & -1.0 \\ 0.6 & -0.5 \\ 0.8 & +0.0 \\ 1.0 & 1.0 \end{bmatrix} \quad \bar{\mathbf{K}} = k \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 2.0 \end{bmatrix} \quad \bar{\mathbf{M}} = m \begin{bmatrix} 2.2 & 0.2 \\ 0.2 & 2.5 \end{bmatrix}$$

Red. eigenproblem ($\rho = \omega^2\, m/k$): $\begin{bmatrix} 2 - 22\rho & 2 - 2\rho \\ 2 - 2\rho & 20 - 25\rho \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

RR Example



The structural matrices

$$\mathbf{K} = k\begin{bmatrix} +2 & -1 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \mathbf{M} = m\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Ritz base vectors and the reduced matrices,

$$\mathbf{\Phi} = \begin{bmatrix} 0.2 & -0.5 \\ 0.4 & -1.0 \\ 0.6 & -0.5 \\ 0.8 & +0.0 \\ 1.0 & 1.0 \end{bmatrix} \qquad \bar{\mathbf{K}} = k\begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 2.0 \end{bmatrix} \quad \bar{\mathbf{M}} = m\begin{bmatrix} 2.2 & 0.2 \\ 0.2 & 2.5 \end{bmatrix}$$

Red. eigenproblem ($\rho = \omega^2\, m/k$): $\begin{bmatrix} 2-22\rho & 2-2\rho \\ 2-2\rho & 20-25\rho \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

The roots are $\rho_1 = 0.0824$, $\rho_2 = 0.800$, the frequencies are $\omega_1 = 0.287\sqrt{k/m}\ [= 0.285]$, $\omega_2 = 0.850\sqrt{k/m}\ [= 0.831]$, while the k/m normalized exact eigenvalues are $[0.08101405, 0.69027853]$.

The first eigenvalue is estimated with good approximation.

Rayleigh-Ritz Example

The Ritz coordinates eigenvector matrix is $\mathbf{Z} = \begin{bmatrix} 1.329 & 0.03170 \\ -0.1360 & 1.240 \end{bmatrix}$.

The *RR* eigenvector matrix, $\mathbf{\Phi}$ and the exact one, $\mathbf{\Psi}$:

$$\mathbf{\Phi} = \begin{bmatrix} +0.3338 & -0.6135 \\ +0.6676 & -1.2270 \\ +0.8654 & -0.6008 \\ +1.0632 & +0.0254 \\ +1.1932 & +1.2713 \end{bmatrix}, \qquad \mathbf{\Psi} = \begin{bmatrix} +0.3338 & -0.8398 \\ +0.6405 & -1.0999 \\ +0.8954 & -0.6008 \\ +1.0779 & +0.3131 \\ +1.1932 & +1.0108 \end{bmatrix}.$$

The accuracy of the estimates for the 1st mode is very good, on the contrary the 2nd mode estimates are in the order of a few percents.

Rayleigh-Ritz Example

The Ritz coordinates eigenvector matrix is $\mathbf{Z} = \begin{bmatrix} 1.329 & 0.03170 \\ -0.1360 & 1.240 \end{bmatrix}$.

The *RR* eigenvector matrix, $\mathbf{\Phi}$ and the exact one, $\mathbf{\Psi}$:

$$\mathbf{\Phi} = \begin{bmatrix} +0.3338 & -0.6135 \\ +0.6676 & -1.2270 \\ +0.8654 & -0.6008 \\ +1.0632 & +0.0254 \\ +1.1932 & +1.2713 \end{bmatrix}, \qquad \mathbf{\Psi} = \begin{bmatrix} +0.3338 & -0.8398 \\ +0.6405 & -1.0999 \\ +0.8954 & -0.6008 \\ +1.0779 & +0.3131 \\ +1.1932 & +1.0108 \end{bmatrix}.$$

The accuracy of the estimates for the 1st mode is very good, on the contrary the 2nd mode estimates are in the order of a few percents.

It may be interesting to use $\hat{\mathbf{\Phi}} = \mathbf{K}^{-1}\mathbf{M}\,\mathbf{\Phi}$ as a new Ritz base to get a new estimate of the Ritz and of the structural eigenpairs.

Introduction to Subspace Iteration

Rayleigh-Ritz gives good estimates for $p \approx M/2$ modes, due also to the arbitrariness in the choice of the Ritz reduced base $\mathbf{\Phi}$.
Having to solve a $M = 2p$ order problem to find p eigenvalues is very costly, as the operation count is $\propto O(M^3)$.

Introduction to Subspace Iteration

Rayleigh-Ritz gives good estimates for $p \approx M/2$ modes, due also to the arbitrariness in the choice of the Ritz reduced base $\mathbf{\Phi}$.
Having to solve a $M = 2p$ order problem to find p eigenvalues is very costly, as the operation count is $\propto O(M^3)$.
Choosing *better* Ritz base vectors, we can use less vectors and solve a smaller (much smaller in terms of operations count) eigenvalue problem.

Introduction to Subspace Iteration

Rayleigh-Ritz gives good estimates for $p \approx M/2$ modes, due also to the arbitrariness in the choice of the Ritz reduced base $\mathbf{\Phi}$.
Having to solve a $M = 2p$ order problem to find p eigenvalues is very costly, as the operation count is $\propto O(M^3)$.
Choosing *better* Ritz base vectors, we can use less vectors and solve a smaller (much smaller in terms of operations count) eigenvalue problem.
If one thinks of it, with a $M = 1$ base we can always compute, within arbitrary accuracy, one eigenvector using the Matrix Iteration procedure, isn't it?
And the trick is to change the base at every iteration...

Introduction to Subspace Iteration

Rayleigh-Ritz gives good estimates for $p \approx M/2$ modes, due also to the arbitrariness in the choice of the Ritz reduced base $\mathbf{\Phi}$.
Having to solve a $M = 2p$ order problem to find p eigenvalues is very costly, as the operation count is $\propto O(M^3)$.
Choosing *better* Ritz base vectors, we can use less vectors and solve a smaller (much smaller in terms of operations count) eigenvalue problem.
If one thinks of it, with a $M = 1$ base we can always compute, within arbitrary accuracy, one eigenvector using the Matrix Iteration procedure, isn't it?
And the trick is to change the base at every iteration...
The *Subspace Iteration* procedure is a variant of the Matrix Iteration procedure, where we apply the same idea, to use the response to inertial loading in the next step, not to a single vector but to a set of different vectors at once.

Statement of the procedure

The first M eigenvalue equations can be written in matrix algebra, in terms of an $\mathrm{N}\times M$ matrix of eigenvectors $\mathbf{\Phi}$ and an $M\times M$ diagonal matrix $\mathbf{\Lambda}$ that collects the eigenvalues

$$\underset{\mathrm{N\times N}}{\mathbf{K}}\ \underset{\mathrm{N\times M}}{\mathbf{\Phi}} = \underset{\mathrm{N\times N}}{\mathbf{M}}\ \underset{\mathrm{N\times M}}{\mathbf{\Phi}}\ \underset{\mathrm{M\times M}}{\mathbf{\Lambda}}$$

Using again the hat notation for the unnormalized iterate, from the previous equation we can write

$$\mathbf{K}\hat{\mathbf{\Phi}}_1 = \mathbf{M}\mathbf{\Phi}_0$$

where $\mathbf{\Phi}_0$ is the matrix, $\mathrm{N}\times M$, of the zero order trial vectors, and $\hat{\mathbf{\Phi}}_1$ is the matrix of the non-normalized first order trial vectors.

Orthonormalization

To proceed with iterations,

1. the trial vectors in $\hat{\boldsymbol{\Phi}}_{n+1}$ must be orthogonalized, so that each trial vector converges to a *different* eigenvector instead of collapsing to the first eigenvector,
2. all the trial vectors must be normalized, so that the ratio between the normalized vectors and the unnormalized iterated vectors converges to the corresponding eigenvalue.

Orthonormalization

To proceed with iterations,

1. the trial vectors in $\hat{\mathbf{\Phi}}_{n+1}$ must be orthogonalized, so that each trial vector converges to a *different* eigenvector instead of collapsing to the first eigenvector,
2. all the trial vectors must be normalized, so that the ratio between the normalized vectors and the unnormalized iterated vectors converges to the corresponding eigenvalue.

These operations can be performed in different ways (e.g., ortho-normalization by Gram-Schmidt process).
Another possibility to do both tasks at once is to solve a Rayleigh-Ritz eigenvalue problem, defined in the Ritz base constituted by the vectors in $\hat{\mathbf{\Phi}}_{n+1}$.

Associated Eigenvalue Problem

Developing the procedure for $n = 0$, with the generalized matrices

$$\mathbf{K}_1^\star = \hat{\mathbf{\Phi}}_1{}^T \mathbf{K} \hat{\mathbf{\Phi}}_1$$

and

$$\mathbf{M}_1^\star = \hat{\mathbf{\Phi}}_1{}^T \mathbf{M} \hat{\mathbf{\Phi}}_1$$

the Rayleigh-Ritz eigenvalue problem associated with the orthonormalisation of $\hat{\mathbf{\Phi}}_1$ is

$$\mathbf{K}_1^\star \hat{\mathbf{Z}}_1 = \mathbf{M}_1^\star \hat{\mathbf{Z}}_1 \mathbf{\Omega}_1^2.$$

After solving for the Ritz coordinates mode shapes, $\hat{\mathbf{Z}}_1$ and the frequencies $\mathbf{\Omega}_1^2$, using any suitable procedure, it is usually convenient to normalize the shapes, so that $\hat{\mathbf{Z}}_1{}^T \mathbf{M}_1^\star \hat{\mathbf{Z}}_1 = \mathbf{I}$. The ortho-normalized set of trial vectors at the end of the iteration is then written as

$$\mathbf{\Phi}_1 = \hat{\mathbf{\Phi}}_1 \hat{\mathbf{Z}}_1.$$

The entire process can be repeated for $n = 1$, then $n = 2$, $n = \ldots$ until the eigenvalues converge within a prescribed tolerance.

Convergence

In principle, the procedure will converge to all the M lower eigenvalues and eigenvectors of the structural problem, but it was found that the subspace iteration method converges faster to the lower p eigenpairs, those required for dynamic analysis, if there is some additional trial vector; on the other hand, too many additional trial vectors slow down the computation without ulterior benefits.

Convergence

In principle, the procedure will converge to all the M lower eigenvalues and eigenvectors of the structural problem, but it was found that the subspace iteration method converges faster to the lower p eigenpairs, those required for dynamic analysis, if there is some additional trial vector; on the other hand, too many additional trial vectors slow down the computation without ulterior benefits. Experience has shown that the optimal total number M of trial vectors is the minimum of $2p$ and $p + 8$.

Convergence

In principle, the procedure will converge to all the M lower eigenvalues and eigenvectors of the structural problem, but it was found that the subspace iteration method converges faster to the lower p eigenpairs, those required for dynamic analysis, if there is some additional trial vector; on the other hand, too many additional trial vectors slow down the computation without ulterior benefits. Experience has shown that the optimal total number M of trial vectors is the minimum of $2p$ and $p + 8$.

The subspace iteration method makes it possible to compute simultaneosly a set of eigenpairs within any required level of approximation, and is the preferred method to compute the eigenpairs of a complex dynamic system.

Standard Form

In algebra textbooks, the eigenproblem is usually stated as

$$\mathbf{A}\,\boldsymbol{y} = \lambda\boldsymbol{y}$$

and all the relevant algorithms to actually compute the eigenthings (Jacobi method, $\mathbf{Q}\,\mathbf{R}$ method, etc) are referred to the above statement of the problem. Our problem is, instead, formulated as

$$\mathbf{K}\boldsymbol{x} = \lambda\mathbf{M}\,\boldsymbol{x}.$$

Of course one can premultiply both members by $\mathbf{M}^{-1}$,

$$\mathbf{M}\mathbf{K}\boldsymbol{x} = \lambda\,\boldsymbol{x},$$

but this procedure doesn't preserve the symmetry of the problem, leading to a more onerous solution.

Standard Form

If we want to preserve the symmetry of the structural matrices, we may proceed as follows.
Any symmetric, definite positive matrix $\mathbf{B}$ can be subjected to a unique *Choleski Decomposition (CD)*, $\mathbf{B} = \mathbf{L}\,\mathbf{L}^T$ where $\mathbf{L}$ is a lower triangular matrix. Applying *CD* to $\mathbf{M}$, the eigenvector equation is,

$$\mathbf{K}\,x = \mathbf{K}\,\underbrace{(\mathbf{L}^T)^{-1}\mathbf{L}^T}_{\mathbf{I}}x = \lambda\underbrace{\mathbf{L}\,\mathbf{L}^T}_{\mathbf{M}}x.$$

Premultiplying by $\mathbf{L}^{-1}$, with $y = \mathbf{L}^T x$

$$\underbrace{\mathbf{L}^{-1}\mathbf{K}(\mathbf{L}^T)^{-1}}_{\mathbf{A}}\underbrace{\mathbf{L}^T x}_{y} = \lambda\underbrace{\mathbf{L}^{-1}\mathbf{L}}_{\mathbf{I}}\underbrace{\mathbf{L}^T x}_{y} \qquad \rightarrow \qquad \mathbf{A}y = \lambda y.$$

Standard Form

If we want to preserve the symmetry of the structural matrices, we may proceed as follows.
Any symmetric, definite positive matrix $\mathbf{B}$ can be subjected to a unique *Choleski Decomposition (CD)*, $\mathbf{B} = \mathbf{L}\,\mathbf{L}^T$ where $\mathbf{L}$ is a lower triangular matrix. Applying *CD* to $\mathbf{M}$, the eigenvector equation is,

$$\mathbf{K}\,x = \mathbf{K}\,\underbrace{(\mathbf{L}^T)^{-1}\mathbf{L}^T}_{\mathbf{I}}x = \lambda\underbrace{\mathbf{L}\,\mathbf{L}^T}_{\mathbf{M}}x.$$

Premultiplying by $\mathbf{L}^{-1}$, with $y = \mathbf{L}^T x$

$$\underbrace{\mathbf{L}^{-1}\mathbf{K}(\mathbf{L}^T)^{-1}}_{\mathbf{A}}\underbrace{\mathbf{L}^T x}_{y} = \lambda\underbrace{\mathbf{L}^{-1}\mathbf{L}}_{\mathbf{I}}\underbrace{\mathbf{L}^T x}_{y} \qquad \rightarrow \qquad \mathbf{A}y = \lambda y.$$

It's worth to mention that, for a lumped mass matrix, $\mathbf{L}$ is a diagonal matrix, with

$$L_{ii} = \sqrt{m_{ii}},$$

How many eigenvectors?

To understand how many eigenvectors we have to use in a modal analysis, we must consider two factors, the loading shape and the excitation frequency.

Introduction

In the following, we'll consider *only* external loadings whose dependance on time and space can be separated, as in

$$\mathbf{p}(x, t) = \mathbf{r}\, f(t),$$

so that we can regard separately the two aspects of the problem.

Introduction

It is worth noting that earthquake loadings are precisely of this type:

$$\mathbf{p}(x, t) = \mathbf{M}\tilde{\mathbf{r}}\,\ddot{u}_g$$

where the vector $\tilde{\mathbf{r}}$ is used to choose the structural dof's that are *excited* by the ground motion component under consideration.

> $\tilde{\mathbf{r}}$ *is an incidence vector, often simply a vector of ones and zeroes where the ones stay for the inertial forces that are excited by a specific component of the earthquake ground acceleration.*

Introduction

It is worth noting that earthquake loadings are precisely of this type:

$$\mathbf{p}(x,t) = \mathbf{M}\tilde{\mathbf{r}}\,\ddot{u}_g$$

where the vector $\tilde{\mathbf{r}}$ is used to choose the structural dof's that are *excited* by the ground motion component under consideration.

> $\tilde{\mathbf{r}}$ *is an incidence vector, often simply a vector of ones and zeroes where the ones stay for the inertial forces that are excited by a specific component of the earthquake ground acceleration.*

Multiplication of $\mathbf{M}$ and division of $\ddot{u}_g$ by g, acceleration of gravity, serves to show a dimensional load vector multiplied by an adimensional function.

$$\begin{aligned}\mathbf{p}(x,t) &= g\,\mathbf{M}\tilde{\mathbf{r}}\,\frac{\ddot{u}_g(t)}{g}\\ &= \mathbf{r}^g f_g(t)\end{aligned}$$

Modal partecipation factor

Under the assumption of separability, we can write the i-th modal equation of motion as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \begin{cases} \frac{\boldsymbol{\psi}_i^T\mathbf{r}}{M_i} f(t) \\ \frac{g\,\boldsymbol{\psi}_i^T\mathbf{M}\hat{\mathbf{r}}}{M_i} f_g(t) \end{cases} = \Gamma_i f(t)$$

with the modal mass $M_i = \boldsymbol{\psi}_i^T\mathbf{M}\boldsymbol{\psi}_i$.
It is apparent that the modal response amplitude depends

- on the characteristics of the time dependency of loading, $f(t)$,
- on the so called *modal partecipation factor* Γ_i,

$$\begin{aligned}\Gamma_i &= \boldsymbol{\psi}_i^T\mathbf{r}/M_i \\ &= g\,\boldsymbol{\psi}_i^T\mathbf{M}\hat{\mathbf{r}}/M_i = \boldsymbol{\psi}_i^T\mathbf{r}^g/M_i\end{aligned}$$

Modal partecipation factor

Under the assumption of separability, we can write the i-th modal equation of motion as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \begin{cases} \frac{\boldsymbol{\psi}_i^T\mathbf{r}}{M_i} f(t) \\ \frac{g\,\boldsymbol{\psi}_i^T\mathbf{M}\hat{\mathbf{r}}}{M_i} f_g(t) \end{cases} = \Gamma_i f(t)$$

with the modal mass $M_i = \boldsymbol{\psi}_i^T\mathbf{M}\boldsymbol{\psi}_i$.

It is apparent that the modal response amplitude depends

- on the characteristics of the time dependency of loading, $f(t)$,
- on the so called *modal partecipation factor* Γ_i,

$$\begin{aligned}\Gamma_i &= \boldsymbol{\psi}_i^T\mathbf{r}/M_i \\ &= g\,\boldsymbol{\psi}_i^T\mathbf{M}\hat{\mathbf{r}}/M_i = \boldsymbol{\psi}_i^T\mathbf{r}^g/M_i\end{aligned}$$

Note that both the definitions of modal partecipation give it the dimensions of an acceleration.

Partecipation Factor Amplitudes

For a given loading $\mathbf{r}$ the modal partecipation factor Γ_i is proportional to the work done by the modal displacement $q_i\psi_i^T$ for the given loading $\mathbf{r}$:

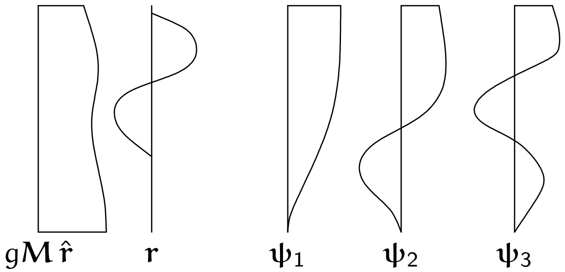
- if the mode shape and the loading shape are approximately equal (equal signs, component by component), the work (dot product) is maximized,
- if the mode shape is significantly different from the loading (different signs), there is some amount of cancellation and the value of the Γ's will be reduced.

Example

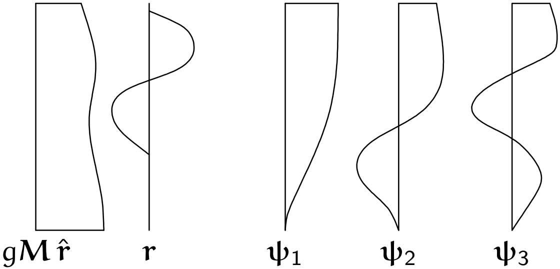
Consider a shear type building, with mass distribution approximately constant over its height:

$$\hat{\mathbf{r}} = \{1, 1, \ldots, 1\}^T \quad \text{and} \quad g\,\mathbf{M}\hat{\mathbf{r}} \approx mg\{1, 1, \ldots, 1\}^T.$$

an external loading and the first 3 eigenvectors as sketched below:



Example, cont.



For *EQ* loading, Γ_1 is relatively large for the first mode, as loading components and displacements have the same sign, with respect to other Γ_i's, where the oscillating nature of the higher eigenvectors will lead to increasing cancellation.
On the other hand, consider the external loading, whose peculiar shape is similar to the 3rd mode. Γ_3 will be more relevant than Γ_i's for lower or higher modes.

Modal Loads Expansion

We define the modal load contribution as

$$\mathbf{r}_i = \mathbf{M}\,\boldsymbol{\psi}_i a_i$$

and express the load vector as a linear combination of the modal contributions

$$\mathbf{r} = \sum_i \mathbf{M}\,\boldsymbol{\psi}_i a_i = \sum_i \mathbf{r}_i.$$

If we premultiply by $\boldsymbol{\psi}_j^T$ the above equation, we see how we can compute the coefficient a_iK

$$\boldsymbol{\psi}_j^T \mathbf{r} = \boldsymbol{\psi}_j^T \sum_i \mathbf{M}\,\boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i$$

Modal Loads Expansion

1. A modal load component works *only* for the displacements associated with the corresponding eigenvector,

$$\boldsymbol{\psi}_j^T \mathbf{r}_i = a_i\, \boldsymbol{\psi}_j^T \mathbf{M} \boldsymbol{\psi}_i = \delta_{ij} a_i M_i.$$

2. Comparing $\boldsymbol{\psi}_j^T \mathbf{r} = \boldsymbol{\psi}_j^T \sum_i \mathbf{M}\, \boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i$ with the definition of $\Gamma_i = \boldsymbol{\psi}_i^T \mathbf{r} / M_i$, we conclude that $a_i \equiv \Gamma_i$ and finally write

$$\mathbf{r}_i = \Gamma_i \mathbf{M}\, \boldsymbol{\psi}_i,$$

   it is possible to collect all the modal load contributions in a matrix: with $\boldsymbol{\Gamma} = \operatorname{diag} \Gamma_i$ we have

$$\mathbf{R} = \mathbf{M}\, \boldsymbol{\Psi}\, \boldsymbol{\Gamma}.$$

Equivalent Static Forces

For mode i, the equation of motion is

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \Gamma_i f(t)$$

with $q_i = \Gamma_i D_i$, we can write, to single out the dependency on the modulating function,

$$\ddot{D}_i + 2\zeta_i\omega_i\dot{D}_i + \omega_i^2 D_i = f(t)$$

The modal contribution to displacement is

$$\mathbf{x}_i = \Gamma_i \boldsymbol{\psi}_i D_i(t)$$

and the modal contribution to elastic forces $\mathbf{f}_i = \mathbf{K}\mathbf{x}_i$ can be written (being $\mathbf{K}\boldsymbol{\psi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\psi}_i$) as

$$\mathbf{f}_i = \mathbf{K}\mathbf{x}_i = \Gamma_i \mathbf{K}\boldsymbol{\psi}_i D_i = \omega_i^2 (\Gamma_i \mathbf{M}\boldsymbol{\psi}_i) D_i = \mathbf{r}_i \omega_i^2 D_i$$

Equivalent Static Forces

For mode i, the equation of motion is

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \Gamma_i f(t)$$

with $q_i = \Gamma_i D_i$, we can write, to single out the dependency on the modulating function,

$$\ddot{D}_i + 2\zeta_i\omega_i\dot{D}_i + \omega_i^2 D_i = f(t)$$

The modal contribution to displacement is

$$\mathbf{x}_i = \Gamma_i \boldsymbol{\psi}_i D_i(t)$$

and the modal contribution to elastic forces $\mathbf{f}_i = \mathbf{K}\,\mathbf{x}_i$ can be written (being $\mathbf{K}\boldsymbol{\psi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\psi}_i$) as

$$\mathbf{f}_i = \mathbf{K}\,\mathbf{x}_i = \Gamma_i \mathbf{K}\,\boldsymbol{\psi}_i D_i = \omega_i^2(\Gamma_i \mathbf{M}\,\boldsymbol{\psi}_i)D_i = \mathbf{r}_i\omega_i^2 D_i$$

Equivalent Static Response

The response can be determined by superposition of the effects of these pseudo-static forces $\mathbf{f}_i = \mathbf{r}_i \omega_i^2 D_i(t)$.
If a required response quantity (be it a nodal displacement, a bending moment in a beam, the total shear force in a building storey, etc etc) is indicated by $s(t)$, we can compute with a *static calculation* (usually using the *FEM* model underlying the dynamic analysis) the modal static contribution s_i^{st} and write

$$s(t) = \sum s_i^{\mathsf{st}}(\omega_i^2 D_i(t)) = \sum s_i(t),$$

where the modal contribution to response $s_i(t)$ is given by

1. static analysis using $\mathbf{r}_i$ as the static load vector,
2. dynamic amplification using the factor $\omega_i^2 D_i(t)$.

Equivalent Static Response

The response can be determined by superposition of the effects of these pseudo-static forces $\mathbf{f}_i = \mathbf{r}_i \omega_i^2 D_i(t)$.
If a required response quantity (be it a nodal displacement, a bending moment in a beam, the total shear force in a building storey, etc etc) is indicated by $s(t)$, we can compute with a *static calculation* (usually using the *FEM* model underlying the dynamic analysis) the modal static contribution s_i^{st} and write

$$s(t) = \sum s_i^{\text{st}}(\omega_i^2 D_i(t)) = \sum s_i(t),$$

where the modal contribution to response $s_i(t)$ is given by

1. static analysis using $\mathbf{r}_i$ as the static load vector,
2. dynamic amplification using the factor $\omega_i^2 D_i(t)$.

This formulation is particularly apt to our discussion of different contributions to response components.

Modal Contribution Factors

Say that the static response due to $\mathbf{r}$ is denoted by s^{st}, then $s_i(t)$, the modal contribution to response $s(t)$, can be written

$$s_i(t) = s_i^{\text{st}} \omega_i^2 D_i(t) = s^{\text{st}} \frac{s_i^{\text{st}}}{s^{\text{st}}} \omega_i^2 D_i(t) = \bar{s}_i s^{\text{st}} \omega_i^2 D_i(t).$$

We have introduced $\bar{s}_i = \frac{s_i^{\text{st}}}{s^{\text{st}}}$, the *modal contribution factor*, the ratio of the modal static contribution to the total static response. The $\bar{s}_i$ are dimensionless, are indipendent on the eigenvector scaling procedure and their sum is unity, $\sum \bar{s}_i = 1$.

Maximum Response

Denote by D_{i0} the maximum absolute value (or *peak*) of the pseudo displacement time history,

$$D_{i0} = \max_t \{|D_i(t)|\}.$$

It will be

$$s_{i0} = \bar{s}_i s^{st} \omega_i^2 D_{i0}$$

The dynamic response factor for mode i, $\mathfrak{R}_{di}$ is defined by

$$\mathfrak{R}_{di} = \frac{D_{i0}}{D_{i0}^{st}}$$

where D_{i0}^{st} is the peak value of the static pseudo displacement $D_i^{st} = \frac{f(t)}{\omega_i^2}$,

$$D_{i0}^{st} = \frac{f_0}{\omega_i^2}$$

Maximum Response

With $f_0 = \max\{|f(t)|\}$ the peak pseudo displacement is

$$D_{i0} = \mathfrak{R}_{di} f_0/\omega_i^2$$

and the peak of the modal contribution is

$$s_{i0}(t) = \bar{s}_i s^{st}\, \omega_i^2 D_{i0}(t) = f_0 s^{st}\ \ \bar{s}_i \mathfrak{R}_{di}$$

Maximum Response

With $f_0 = \max\{|f(t)|\}$ the peak pseudo displacement is

$$D_{i0} = \mathfrak{R}_{di} f_0/\omega_i^2$$

and the peak of the modal contribution is

$$s_{i0}(t) = \bar{s}_i s^{st} \omega_i^2 D_{i0}(t) = f_0 s^{st} \;\; \bar{s}_i \mathfrak{R}_{di}$$

The first two terms are independent of the mode, the last are independent from each other and their product is the factor that influences the modal contributions.
Note that this product has the sign of $\bar{s}_i$, as the dynamic response factor is always positive.

MCF's example

The following table (from Chopra, 2nd ed.) displays the $\bar{s}_i$ and their partial sums for a shear-type, 5 floors building where all the storey masses are equal and all the storey stiffnesses are equal too.
The response quantities chosen are $\bar{x}_{5n}$, the *MCF*'s to the top displacement and $\bar{V}_n$, the *MCF* 's to the base shear, for two different load shapes.

MCF's example

The following table (from Chopra, 2nd ed.) displays the $\bar{s}_i$ and their partial sums for a shear-type, 5 floors building where all the storey masses are equal and all the storey stiffnesses are equal too.
The response quantities chosen are $\bar{x}_{5n}$, the *MCF*'s to the top displacement and $\bar{V}_n$, the *MCF* 's to the base shear, for two different load shapes.

	$\mathbf{r} = \{0, 0, 0, 0, 1\}^T$				$\mathbf{r} = \{0, 0, 0, -1, 2\}^T$			
	Top Displacement		Base Shear		Top Displacement		Base Shear	
n or J	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	-0.362	0.890	0.123	0.915	-0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.043	1.172
4	0.008	0.998	-0.063	0.985	0.024	0.994	-0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

MCF's example

The following table (from Chopra, 2nd ed.) displays the $\bar{s}_i$ and their partial sums for a shear-type, 5 floors building where all the storey masses are equal and all the storey stiffnesses are equal too.
The response quantities chosen are $\bar{x}_{5n}$, the *MCF*'s to the top displacement and $\bar{V}_n$, the *MCF* 's to the base shear, for two different load shapes.

	$\mathbf{r} = \{0, 0, 0, 0, 1\}^T$				$\mathbf{r} = \{0, 0, 0, -1, 2\}^T$			
	Top Displacement		Base Shear		Top Displacement		Base Shear	
n or J	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	-0.362	0.890	0.123	0.915	-0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.043	1.172
4	0.008	0.998	-0.063	0.985	0.024	0.994	-0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

Note that

1. for any given $\mathbf{r}$, the base shear is more influenced by higher modes, and
2. for any given reponse quantity, the second, *skewed* $\mathbf{r}$ gives greater modal contributions for higher modes.

Dynamic Response Ratios

Dynamic Response Ratios are the same that we have seen for *SDOF* systems. Next page, for an undamped system, harmonically excited,

- solid line, the ratio of the modal elastic force $F_{S,i} = K_i q_i \sin\omega t$ to the harmonic applied modal force, $P_i \sin\omega t$, plotted against the frequency ratio $\beta = \omega/\omega_i$.

  For $\beta = 0$ the ratio is 1, the applied load is fully balanced by the elastic resistance.

  For fixed excitation frequency, $\beta \to 0$ for high modal frequencies.
- dashed line,the ratio of the modal inertial force, $F_{I,i} = -\beta^2 F_{S,i}$ to the load.

Dynamic Response Ratios

Dynamic Response Ratios are the same that we have seen for *SDOF* systems. Next page, for an undamped system, harmonically excited,

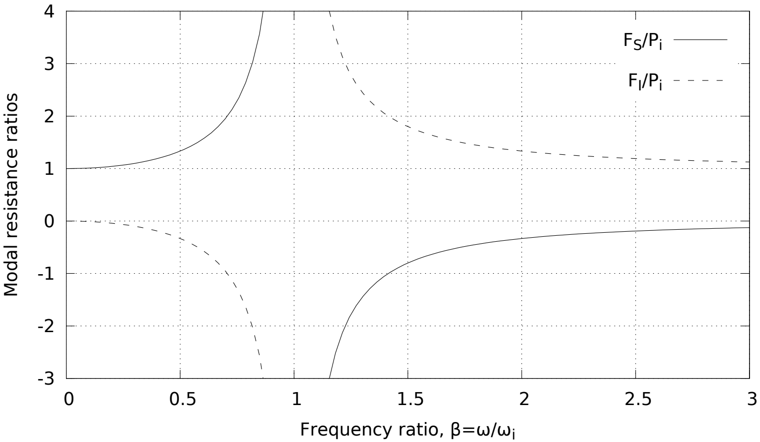
- solid line, the ratio of the modal elastic force $F_{S,i} = K_i q_i \sin \omega t$ to the harmonic applied modal force, $P_i \sin \omega t$, plotted against the frequency ratio $\beta = \omega/\omega_i$.

  For $\beta = 0$ the ratio is 1, the applied load is fully balanced by the elastic resistance.

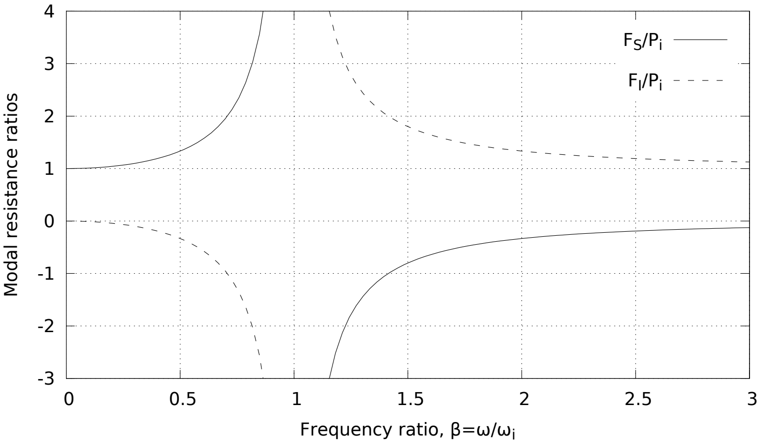
  For fixed excitation frequency, $\beta \to 0$ for high modal frequencies.
- dashed line,the ratio of the modal inertial force, $F_{I,i} = -\beta^2 F_{S,i}$ to the load.

Note that for steady-state motion the sum of the elastic and inertial force ratios is constant and equal to 1, as in

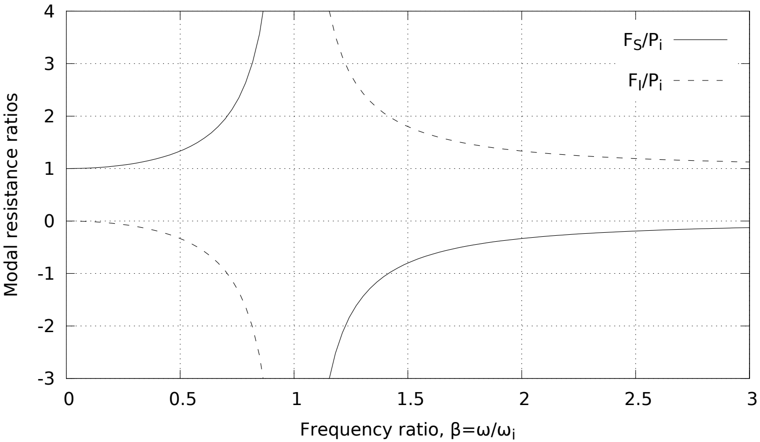
$$(F_{S,i} + F_{I,i}) \sin \omega t = P_i \sin \omega t.$$



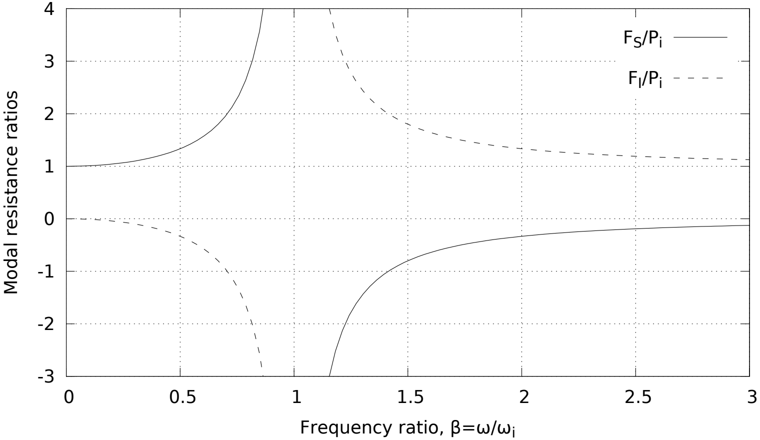
- For a fixed excitation frequency and high modal frequencies the frequency ratio $\beta \to 0$.



- For a fixed excitation frequency and high modal frequencies the frequency ratio $\beta \to 0$.
- For $\beta \to 0$ the response is *quasi-static*.



- For a fixed excitation frequency and high modal frequencies the frequency ratio $\beta \to 0$.
- For $\beta \to 0$ the response is *quasi-static*.
- Hence, for higher modes the response is *pseudo-static*.



- For a fixed excitation frequency and high modal frequencies the frequency ratio $\beta \to 0$.
- For $\beta \to 0$ the response is *quasi-static*.
- Hence, for higher modes the response is *pseudo-static*.
- On the contrary, for excitation frequencies high enough the lower modes respond with purely inertial forces.

Static Correction

The preceding discussion indicates that higher modes contributions to the response could be approximated with the static response, leading to a *Static Correction* of the dynamic response

Static Correction

The preceding discussion indicates that higher modes contributions to the response could be approximated with the static response, leading to a *Static Correction* of the dynamic response

For a system where $q_i(t) \approx \dfrac{p_i(t)}{K_i}$ for $i > n_{dy}$,

n_{dy} being the number of dynamically responding modes,
we can write

$$x(t) \approx x_{dy}(t) + x_{st}(t) = \sum_{1}^{n_{dy}} \psi_i q_i(t) + \sum_{n_{dy}+1}^{N} \psi_i \frac{p_i(t)}{K_i}$$

where the response for each of the first n_{dy} modes can be computed as usual.

Static Modal Components

The static modal displacement component $x_j, j > n_{dy}$ can be written

$$x_j(t) = \psi_j q_j(t) \approx \frac{\psi_j \psi_j^T}{K_j} \mathbf{p}(t) = \mathbf{F}_j \mathbf{p}(t)$$

The *modal flexibility matrix* is defined by

$$\mathbf{F}_j = \frac{\psi_j \psi_j^T}{K_j}$$

and is used to compute the j-th mode static deflections due to the applied load vector.
The total displacements, the dynamic contributions and the static correction, for $\mathbf{p}(t) = \mathbf{r}\, f(t)$, are then

$$\mathbf{x} \approx \sum_{1}^{n_{dy}} \psi_j q_j(t) + f(t) \sum_{n_{dy}+1}^{N} \mathbf{F}_j \mathbf{r}.$$

Alternative Formulation

Our last formula for static correction is

$$\mathbf{x} \approx \sum_{1}^{n_{\mathbf{dy}}} \boldsymbol{\psi}_j q_j(t) + f(t) \sum_{n_{\mathbf{dy}}+1}^{N} \mathbf{F}_j \mathbf{r}.$$

To use the above formula all mode shapes, all modal stiffnesses and all modal flexibility matrices must be computed, undermining the efficiency of the procedure.

Alternative Formulation

This problem can be obviated computing the total static displacements, $x_{st}^{total} = \mathbf{K}^{-1}\mathbf{p}(t)$, and subtracting the static displacements due to the first n_{dy} modes...

$$\sum_{n_{dy}}^{N} \mathbf{F}_j \mathbf{r} f(t) = \mathbf{K}^{-1}\mathbf{r} f(t) - \sum_{1}^{n_{dy}} \mathbf{F}_j \mathbf{r} f(t) = f(t) \left(\mathbf{K}^{-1} - \sum_{1}^{n_{dy}} \mathbf{F}_j \right) \mathbf{r},$$

so that the corrected total displacements have the expression

$$x \approx \sum_{1}^{n_{dy}} \psi_i q_i(t) + f(t) \left(\mathbf{K}^{-1} - \sum_{1}^{n_{dy}} \mathbf{F}_i \right) \mathbf{r},$$

The constant term (a generalized displacement vector) following $f(t)$ can be computed with the information in our posses at the moment we begin the integration of the modal equations of motion.

Effectiveness of Static Correction

In these circumstances, few modes with static correction give results comparable to the results obtained using much more modes in a straightforward modal displacement superposition analysis.

Effectiveness of Static Correction

In these circumstances, few modes with static correction give results comparable to the results obtained using much more modes in a straightforward modal displacement superposition analysis.

- An high number of modes is required to account for the spatial distribution of the loading but only a few lower modes are subjected to significant dynamic amplification.

Effectiveness of Static Correction

In these circumstances, few modes with static correction give results comparable to the results obtained using much more modes in a straightforward modal displacement superposition analysis.

- An high number of modes is required to account for the spatial distribution of the loading but only a few lower modes are subjected to significant dynamic amplification.
- Refined stress analysis is required even if the dynamic response involves only a few lower modes.