Continuous Systems, Infinite Degrees of Freedom

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Outline

Intro

Discrete models

Until now the dynamical behaviour of structures has been modeled using discrete degrees of freedom, as in the Finite Element Method procedure, and in many cases we have found that we are able to reduce the number of *dynamical degrees of freedom* using the static condensation procedure (multistory buildings are an excellent example of structures for which a few dynamical degrees of freedom can describe the dynamical response).

Intro

Continuous models

For different type of structures (e.g., bridges, chimneys), a lumped mass model is not an option. While a *FE* model is always appropriate, there is no apparent way of lumping the structural masses in a way that is obviously correct, and a great number of degrees of freeedom must be retained in the dynamic analysis.
An alternative to detailed *FE* models is deriving the equation of motion, in terms of partial derivatives differential equation, directly for the continuous systems.

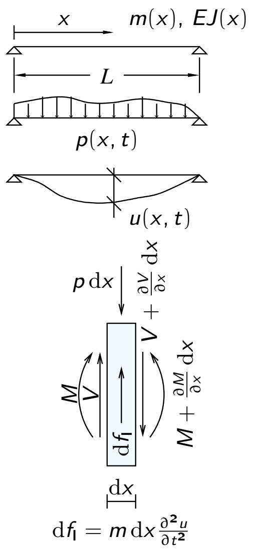
Continuous Systems

There are many different continuous systems whose dynamics are approachable with the instruments of classical mechanics,

- taught strings,
- axially loaded rods,
- beams in flexure,
- plates and shells,
- 3D solids.

In the following, we will focus our interest on beams in flexure.

EoM for an undamped beam



At the left, a straight beam with characteristic depending on position x: $m = m(x)$ and $EJ = EJ(x)$; with the signs conventions for displacements, accelerations, forces and bending moments reported left, the equation of vertical equilibrium for an infinitesimal slice of beam is

$$V - (V + \frac{\partial V}{\partial x}\mathrm{d}x) + m\,\mathrm{d}x\frac{\partial^2 u}{\partial t^2} - p(x,t)\,\mathrm{d}x = 0.$$

Rearranging and simplifying $\mathrm{d}x$,

$$\frac{\partial V}{\partial x} = m\frac{\partial^2 u}{\partial t^2} - p(x,t).$$

Equation of motion, 2

The rotational equilibrium, neglecting rotational inertia and simplifying $\mathrm{d}x$ is

$$\frac{\partial M}{\partial x} = V.$$

Deriving with respect to x both members of the rotational equilibrium equation, it is

$$\frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2}$$

Substituting in the equation of vertical equilibrium and rearranging

$$m(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 M}{\partial x^2} = p(x,t)$$

Equation of motion, 3

Using the moment-curvature relationship,

$$M = -EJ\frac{\partial^2 u}{\partial x^2}$$

and substituting in the equation above we have the equation of dynamic equilibrium

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t).$$

Partial Derivatives Differential Equation

In this formulation of the equation of equilibrium we have

- one equation of equilibrium
- one unknown, $u(x,t)$.

It is a partial derivatives differential equation because we have the derivatives of u with respect to x and t simultaneously in the same equation.

Effective Earthquake Loading

If our continuous structure is subjected to earthquake excitation, we will write, as usual, $u_{\mathrm{TOT}} = u(x,t) + u_{\mathrm{g}}(t)$ and, consequently,

$$\ddot{u}_{\mathrm{TOT}} = \ddot{u}(x,t) + \ddot{u}_{\mathrm{g}}(t)$$

and, using the usual considerations,

$$p_{\mathrm{eff}}(x,t) = -m(x)\ddot{u}_{\mathrm{g}}(t).$$

In p_{eff} we have a separation of variables: in the case of earthquake excitation all the considerations we have done on expressing the response in terms of static modal responses and pseudo/acceleration response will be applicable.
Only a word of caution, in every case we must consider the component of earthquake acceleration *parallel* to the transverse motion of the beam.

Free Vibrations

For free vibrations, $p(x,t) \equiv 0$ and the equation of equilibrium is

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 u}{\partial x^2}\right] = 0.$$

Using separation of variables, with the following notations,

$$u(x,t) = q(t)\phi(x), \;\; \frac{\partial u}{\partial t} = \dot{q}\phi, \;\; \frac{\partial u}{\partial x} = q\phi'$$

etc for higher order derivatives, we have

$$m(x)\ddot{q}(t)\phi(x) + q(t)\left[EJ(x)\phi''(x)\right]'' = 0.$$

Free Vibrations, 2

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Dividing both terms in

$$m(x)\ddot{q}(t)\phi(x) + q(t)\left[EJ(x)\phi''(x)\right]'' = 0.$$

by $m(x)u(x,t) = m(x)q(t)\phi(x)$ and rearranging, we have

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{[EJ(x)\phi''(x)]''}{m(x)\phi(x)}.$$

The left member is a function of time only, the right member a function of position only, and they are equal... this is possible if and only if both terms are constant, let's name this constant ω^2 and write

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{[EJ(x)\phi''(x)]''}{m(x)\phi(x)} = \omega^2,$$

Free Vibrations, 3

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

From the previous equations we can derive the following two

$$\ddot{q} + \omega^2 q = 0$$
$$\left[EJ(x)\phi''(x)\right]'' = \omega^2 m(x)\phi(x)$$

The first equation, $\ddot{q} + \omega^2 q = 0$, has the homogeneous integral

$$q(t) = A\sin\omega t + B\cos\omega t$$

so that our free vibration solution is

$$u(x,t) = \phi(x)\left(A\sin\omega t + B\cos\omega t\right),$$

the free vibration shape $\phi(x)$ will be modulated by a harmonic function of time.
To find something about ω's and ϕ's (that is, the eigenvalues and the *eigenfunctions* of our problem), we have to introduce an important simplification.

Eigenpairs of a uniform beam

With $EJ = \text{const.}$ and $m = \text{const.}$, we have from the second equation in previous slide,

$$EJ\phi^{IV} - \omega^2 m\phi = 0,$$

with $\beta^4 = \frac{\omega^2 m}{EJ}$ it is

$$\phi^{IV} - \beta^4\phi = 0$$

a differential equation of 4^{th} order with constant coefficients.
Substituting $\phi = \exp st$ and simplyfing,

$$s^4 - \beta^4 = 0,$$

the roots of the associated polynomial are

$$s_1 = \beta,\ s_2 = -\beta,\ s_3 = i\beta,\ s_4 = -i\beta$$

and the general integral is

$$\phi(x) = \mathcal{A}\sin\beta x + \mathcal{B}\cos\beta x + \mathcal{C}\sinh\beta x + \mathcal{D}\cosh\beta x$$

Constants of Integration

For a uniform beam in free vibration, the general integral is

$$\phi(x) = \mathcal{A}\sin\beta x + \mathcal{B}\cos\beta x + \mathcal{C}\sinh\beta x + \mathcal{D}\cosh\beta x$$

In this expression we see 5 parameters, the 4 constants of integration and the wave number β (further consideration shows that the constants can be arbitrarily scaled).
In general for the transverse motion of a segment of beam supported at the extremes we can write exactly 4 equations expressing boundary conditions, either from kinematc or static considerations.
All these boundary conditions

- lead to linear, homogeneous equation where
- the coefficients of the equations depend on the parameter β.

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Simply Supported Beam
Cantilever Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Eigenvalues and eigenfunctions

Imposing the boundary conditions give a homogeneous linear system with coefficients depending on β, hence:

- a non trivial solution is possible only for particular values of β, for which the determinant of the matrix of cofficients is equal to zero and
- the constants are known within a proportionality factor.

In the case of *MDOF* systems, the determinantal equation is an algebraic equation of order N, giving exactly N eigenvalues, now the equation to be solved is a trascendental equation (examples from the next slide), with an infinity of solutions.

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Simply Supported Beam
Cantilever Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Simply supported beam

Consider a simply supported uniform beam of lenght L, displacements at both ends must be zero, as well as the bending moments:

$$\phi(0) = \mathcal{B} + \mathcal{D} = 0, \qquad \phi(L) = 0,$$
$$-EJ\phi''(0) = \beta^2 EJ(\mathcal{B} - \mathcal{D}) = 0, \qquad -EJ\phi''(L) = 0.$$

The conditions for the left support require that $\mathcal{B} = \mathcal{D} = 0$
Now, we can write the equations for the right support as

$$\phi(L) = \mathcal{A}\sin\beta L + \mathcal{C}\sinh\beta L = 0$$
$$-EJ\phi''(L) = \beta^2 EJ(\mathcal{A}\sin\beta L - \mathcal{C}\sinh\beta L) = 0$$

or

$$\begin{bmatrix} +\sin\beta L & +\sinh\beta L \\ +\sin\beta L & -\sinh\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{C} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Simply supported beam, 2

For a simply supported beam we have $$\begin{bmatrix} +\sin\beta L & +\sinh\beta L \\ +\sin\beta L & -\sinh\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{C} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The determinant is $-2\sin\beta L\sinh\beta L$, equating to zero with the understanding that $\sinh\beta L \neq 0$ if $\beta \neq 0$ results in

$$\sin\beta L = 0.$$

All positive β solutions are given by

$$\beta L = n\pi$$

with $n = 1, \ldots, \infty$. We have an infinity of eigenvalues,

$$\beta_n = \frac{n\pi}{L} \text{ and } \omega_n = \beta^2\sqrt{\frac{EJ}{m}} = n^2\pi^2\sqrt{\frac{EJ}{mL^4}}$$

and of eigenfunctions

$$\phi_1 = \sin\frac{\pi x}{L}, \ \phi_2 = \sin\frac{2\pi x}{L}, \ \phi_3 = \sin\frac{3\pi x}{L}, \ \cdots$$

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Simply Supported Beam
Cantilever Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Cantilever beam

For $x = 0$, we have zero displacement and zero rotation

$$\phi(0) = \mathcal{B} + \mathcal{D} = 0, \qquad \phi'(0) = \beta(\mathcal{A} + \mathcal{C}) = 0,$$

for $x = L$, both bending moment and shear must be zero

$$-EJ\phi''(L) = 0, \qquad -EJ\phi'''(L) = 0.$$

Substituting the expression of the general integral, with $\mathcal{D} = -\mathcal{B}$, $\mathcal{C} = -\mathcal{A}$ from the left end equations, in the right end equations and simplifying

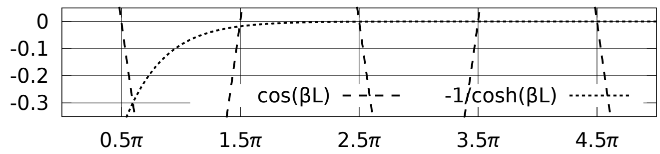
$$\begin{bmatrix} \sinh\beta L + \sin\beta L & \cosh\beta L + \cos\beta L \\ \cosh\beta L + \cos\beta L & \sinh\beta L - \sin\beta L \end{bmatrix} \begin{Bmatrix} \mathcal{A} \\ \mathcal{B} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Cantilever beam, 2

Imposing a zero determinant results in

$$(\cosh^2\beta L - \sinh^2\beta L) + (\sin^2\beta L + \cos^2\beta L) + 2\cos\beta L\cosh\beta L = \\ = 2(1+\cos\beta L\cosh\beta L) = 0$$

Rearranging, $\cos\beta L = -(\cosh\beta L)^{-1}$ and plotting these functions on the same graph



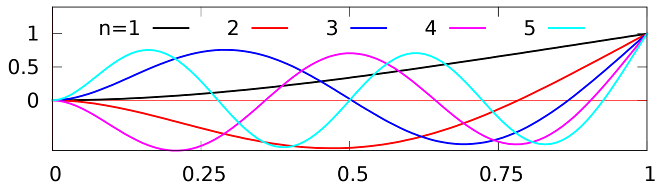
it is $\beta_1 L = 1.8751$ and $\beta_2 L = 4.6941$, while for $n = 3, 4, \ldots$ with good approximation it is $\beta_n L \approx \frac{2n-1}{2}\pi$.

Cantilever beam, 3

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Simply Supported Beam
Cantilever Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Eigenvectors are given by

$$\phi_n(x) = C_n\left[(\cosh\beta_n x - \cos\beta_n x) - \frac{\cosh\beta_n L + \cos\beta_n L}{\sinh\beta_n L + \sin\beta_n L}(\sinh\beta_n x - \sin\beta_n x)\right]$$



Above, in abscissas x/L and in ordinates $\phi_n(x)$ for the first 5 modes of vibration of the cantilever beam.

n	1	2	3	4	5
$\beta_n L$	1.8751	4.6941	7.8548	10.9962	$\approx 4.5\pi$
$\omega\sqrt{\frac{mL^4}{EJ}}$	3.516	22.031	61.70	120.9	$\cdots$

Other Boundary Conditions

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

It is possible that

- the beam is supported not by a fixed constraint but by a spring, either extensional or flexural,
- the beam at its end supports a lumped mass, with inertia and possibly rotatory inertia.

Elastic Support

Consider the right end, $x = L$, supported by an extensional spring k. An infinitesimal slice of beam is subjected to two discrete forces, the shear $V(L,t) = -EJ\phi'''(L)q(t)$ and the spring reaction, $ku(L,t) = k\phi(L)q(t)$. With our sign conventions, the equilibrium is written $-V - ku = 0$ or, simplifying the time dependency,

$$\phi(L) = \frac{EJ}{k}\phi'''(L) = \frac{EJ}{kL^3}(\beta L)^3 f(\beta L; \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$$

where we have shown that the right member depends only on βL. The equation of equilibrium is an homogeneous equation in $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$.

Supported Mass

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

The beam end supports a lumped mass M and it is subjected to shear $V(L,t)$ and an inertial force, $f_I = -M\frac{\partial^2 u(L,t)}{\partial t^2}$. Considering that in free vibrations we have harmonic time dependency, it is

$$f_I = -M\phi(L)\frac{\partial^2 q(t)}{\partial t^2} = M\omega^2\,\phi(L)q(t) = M\beta^4\frac{EJ}{m}\phi(L)q(t).$$

and the equation of equilibrium is, simplifying EJ and the time dependency

$$\beta^3 f(\ldots) + \frac{M}{mL}\beta^4 L\phi(\ldots) = 0$$

eventually dividing by β^3 we have an homegeneous equation in $\mathcal{A}\ldots$ as well,

$$f(\ldots) + \frac{M}{mL}\beta L\phi(\ldots) = 0/$$

Mode Orthogonality

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

We will demonstrate mode orhogonality for a restricted set of of boundary conditions, i.e., disregarding elastic supports and supported masses. In the beginning we have, for $n = r$,

$$\left[EJ(x)\phi_r''(x)\right]'' = \omega_r^2 m(x)\phi_r(x).$$

Premultiply both members by $\phi_s(x)$ and integrate over the length of the beam gives you

$$\int_0^L \phi_s(x)\left[EJ(x)\phi_r''(x)\right]''\,\mathrm{d}x = \omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x.$$

Mode Orthogonality, 2

The left member can be integrated by parts, two times, as in

$$\int_0^L \phi_s(x)\left[EJ(x)\phi_r''(x)\right]''\,\mathrm{d}x = \left[\phi_s(x)\left[EJ(x)\phi_r''(x)\right]'\right]_0^L - \left[\phi_s'(x)EJ(x)\phi_r''(x)\right]_0^L + \int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x$$

but the terms in brackets are always zero, the first being the product of end displacement by end shear, the second the product of end rotation by bending moment, and for fixed constraints or free end one of the two terms must be zero. So it is

$$\int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x = \omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x.$$

Mode Orthogonality, 3

We write the last equation exchanging the roles of r and s and subtract from the original,

$$\int_0^L \phi_s''(x)EJ(x)\phi_r''(x)\,\mathrm{d}x - \int_0^L \phi_r''(x)EJ(x)\phi_s''(x)\,\mathrm{d}x = \omega_r^2\int_0^L \phi_s(x)m(x)\phi_r(x)\,\mathrm{d}x - \omega_s^2\int_0^L \phi_r(x)m(x)\phi_s(x)\,\mathrm{d}x.$$

This obviously can be simplyfied giving

$$(\omega_r^2-\omega_s^2)\int_0^L \phi_r(x)m(x)\phi_s(x)\,\mathrm{d}x = 0$$

implying that, for $\omega_r^2 \neq \omega_s^2$ the modes are orthogonal with respect to the mass distribution, $\int \phi_s\phi_r\, m\,\mathrm{d}x = \delta_{rs}m_r$.
It is then easy to show that $\int \phi_s''\phi_r''EJ\,\mathrm{d}x = \delta_{rs}m_r\omega_r^2$.

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Forced dynamic response

With $u(x,t)=\sum_1^\infty \phi_m(x)q_m(t)$, the equation of motion can be written

$$\sum_1^\infty m(x)\phi_m(x)\ddot{q}_m(t) + \sum_1^\infty \left[EJ(x)\phi_m''(x)\right]'' q_m(t) = p(x,t)$$

premultiplying by ϕ_n and integrating each sum and the loading term

$$\sum_1^\infty \int_0^L \phi_n(x)m(x)\phi_m(x)\ddot{q}_m(t)\,\mathrm{d}x + \sum_1^\infty \int_0^L \phi_n(x)\left[EJ(x)\phi_m''(x)\right]'' q_m(t)\,\mathrm{d}x = \int_0^L \phi_n(x)p(x,t)\,\mathrm{d}x$$

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response

Forced dynamic response, 2

By the orthogonality of the eigenfunctions this can be simplyfied to

$$m_n \ddot{q}_n(t) + k_n q_n(t) = p_n(t), \qquad n = 1, 2, \ldots, \infty$$

with

$$m_n = \int_0^L \phi_n m \phi_n \, \mathrm{d}x, \qquad k_n = \int_0^L \phi_n \left[EJ\phi_n'' \right]'' \mathrm{d}x,$$

and $$p_n(t) = \int_0^L \phi_n p(x,t) \, \mathrm{d}x.$$

For free ends and/or fixed supports, $k_n = \int_0^L \phi_n'' EJ \phi_n'' \, \mathrm{d}x$.

Earthquake response

Consider an effective earthquake load, $p(x,t) = m(x)\ddot{u}_\mathrm{g}(t)$, with

$$\mathcal{L}_n = \int_0^L \phi_n(x) m(x) \, \mathrm{d}x, \qquad \Gamma_n = \frac{\mathcal{L}_n}{m_n},$$

the modal equation of motion can be written (with an obvious generalisation)

$$\ddot{q}_n + 2\omega_n \zeta_n \dot{q}_n + \omega_n^2 q = -\Gamma_n \ddot{u}_\mathrm{g}(t).$$

The modal response, analogously to the case of discrete models, is the product of the modal partecipation factor and the pseudo-displacement response,

$$q_n(t) = \Gamma_n D_n(t).$$

Earthquake response, 2

Modal contributions can be computed directly, e.g.

$$u_n(x,t) = \Gamma_n \phi_n(x) D_n(t),$$
$$M_n(x,t) = -\Gamma_n EJ(x) \phi_n''(x) D_n(t),$$

or can be computed from the equivalent static forces,

$$f_s(x,t) = \left[EJ(x) u(x,t)'' \right]''.$$

Earthquake response, 3

The modal contributions to equiv. static forces are

$$f_{sn}(x,t) = \Gamma_n \left[EJ(x)\phi_n(x)''\right]'' D_n(t),$$

that, because it is

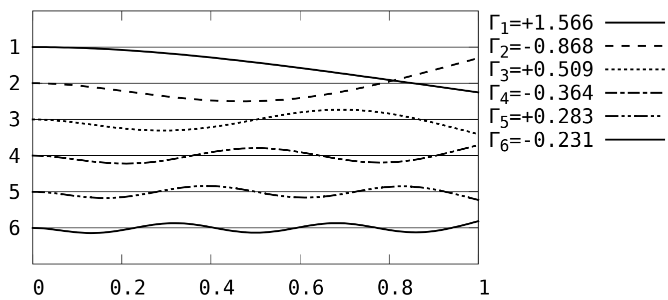
$$\left[EJ(x)\phi''(x)\right]'' = \omega^2 m(x)\phi(x)$$

can be written in terms of the mass distribution and of the pseudo-acceleration response $A_n(t) = \omega_n^2 D_n(t)$

$$f_{sn}(x,t) = \Gamma_n m(x)\phi_n(x)\omega_n^2 D_n(t) = \Gamma_n m(x)\phi_n(x)A_n(t).$$

Earthquake response, 4

The effective load is proportional to the mass distribution, and we can do a modal mass decomposition in the same way that we had for *MDOF* systems, $m(x) = \sum r_n(x) = \sum \Gamma_n m(x)\phi_n(x)$



Above, the modal mass decomposition $r_n = \Gamma_n m\phi_n$,for the first six modes of a uniform cantilever, in abscissa x/L.

EQ example, cantilever

For a cantilever, it is possible to derive explicitly some response quantities,

$$V(x), \qquad V_B, \qquad\qquad M(x), \qquad M_B,$$

that is, the shear force and the base shear force, the bending moment and the base bending moment.

$$V_n^{st}(x) = \int_x^L r_n(s)\,ds, \qquad V_B^{st} = \int_0^L r_n(s)\,ds = \Gamma_n\mathcal{L}_n = M_n^\star,$$

$$M_n^{st}(x) = \int_x^L r_n(s)(s-x)\,ds, \qquad M_B^{st} = \int_0^L s r_n(s)\,ds = M_n^\star h_n^\star.$$

$M_n^\star$ is the *partecipating modal mass* and expresses the partecipation of the different modes to the base shear, it is $\sum M_n^\star = \int_0^L m(x)\,dx$.
$M_n^\star h_n^\star$ expresses the modal partecipation to base moment, $h_n^\star$ is the height where the partecipating modal mass $M_n^\star$ must be placed so that its effects on the base are the same of the static modal forces effects, or $M_n^\star$ is the resultant of s.m.f. and $h_n^\star$ is the position of this resultant.

EQ example, cantilever, 2

Starting with the definition of total mass and operating a chain of substitutions,

$$\begin{aligned} M_{\text{TOT}} &= \int_0^L m(x)\,\mathrm{d}x = \sum \int_0^L r_n(x)\,\mathrm{d}x \\ &= \sum \int_0^L \Gamma_n m(x)\phi_n(x)\,\mathrm{d}x = \sum \Gamma_n \int_0^L m(x)\phi_n(x)\,\mathrm{d}x \\ &= \sum \Gamma_n \mathcal{L}_n = \sum M_n^\star, \end{aligned}$$

we have demonstrated that the sum of the partecipating modal mass is equal to the total mass.
The demonstration that $M_{\text{B,TOT}} = \sum M_n^\star h_n^\star$ is similar and is left as an exercise.

EQ example, cantilever, 3

Continuous Systems, Infinite Degrees of Freedom
Giacomo Boffi
Continous Systems
Beams in Flexure
Equation of motion
Earthquake Loading
Free Vibrations
Eigenpairs of a Uniform Beam
Other Boundary Conditions
Mode Orthogonality
Forced Response
Earthquake Response
Example

For the first 6 modes of a uniform cantilever,

n	$\mathcal{L}_n$	m_n	Γ_n	$V_{B,n}$	h_n	$M_{B,n}$
1	0.391496	0.250	1.565984	0.613076	0.726477	0.445386
2	-0.216968	0.250	-0.867872	0.188300	0.209171	0.039387
3	0.127213	0.250	0.508851	0.064732	0.127410	0.008248
4	-0.090949	0.250	-0.363796	0.033087	0.090943	0.003009
5	0.070735	0.250	0.282942	0.020014	0.070736	0.001416
6	-0.057875	0.250	-0.231498	0.013398	0.057875	0.000775
7	0.048971	0.250	0.195883	0.009593	0.048971	0.000470
8	-0.042441	0.250	-0.169765	0.007205	0.042442	0.000306

The convergence for MB is faster than for V_{B}, because the latter is proportional ton an higher derivative of displacements.