August 2016 Homework

Dynamics of Structures 2015-2016

due not later than Wednesday, August 31st

Contents

Instructions

Submit your work by email[1] not later than Wednesday, August 31st, in the form of a typeset PDF attachment containing your solutions.

For every problem describe your procedure with sufficient detail and report the intermediate results necessary to derive the required answers — should you submit further materials (e.g., source code, spreadsheets, endless printouts by your programs) please provide them as *separate* attachments.[2]

You can score up to 25, 50 and 35 points for problems 1, 2, and 3 respectively, you have to score more than 60 points (over 110) to be admitted to September oral exams.

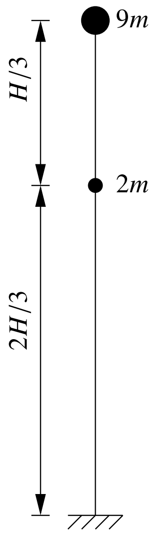
I recommend that you read *carefully* both the text and the notes of each problem before starting with the solution.

You can discuss the problems with your colleagues, with me, with other members of the Faculty but no one else — please note that *to discuss a problem* is quite different from *sharing parts of its solution*; any evidence of sharing implies no admission to September exams for all the involved parts.

[1] Address your email to `giacomo.boffi@polimi.it`

[2] Please do not attach a `.zip` or `.rar` archive with all your files within.

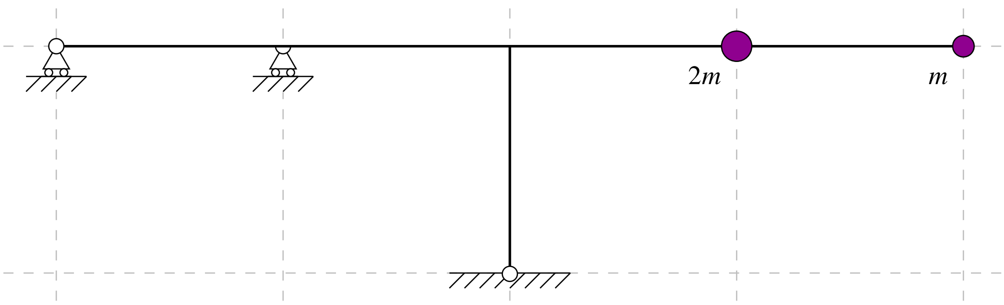
1 Rayleigh Quotient



A tower structure is modeled by a straight, uniform beam clamped at the base, supporting two massive bodies.
The beam is characterized by its length H, its negligible mass and its flexural stiffness EJ, the supported bodies are characterized by their masses and their positions (see figure).

1. Using the shape function $\varphi(z) = 1 - \cos\frac{\pi z}{2H}$ compute the Rayleigh Quotient estimate of the first eigenvalue of the structural model, R_{00}.
2. Compute the first refinement and the second refinement of the Rayleigh Quotient estimate, R_{01} and R_{11}.
3. Compare your results with the first eigenvalue of a 2 DOF model.

2 Multi DOF system



The structure above consists of two rigidly connected, uniform, slender beams (flexural stiffness EJ, lenghts $4L$ and L), supporting two *different* masses.

1. Model the system using 3 dynamical DOF expliciting the simplifying assumptions you have to introduce.

2.1 Earthquake Excitation

The structure is at rest when it is subjected to a horizontal ground acceleration, positive when rightward:

$$\ddot{u}_g(t) = \frac{\delta}{t_0^2}\begin{cases}120\tau^3 - 180\tau^2 + 60\tau & 0 \le t \le t_0 \text{ with } \tau = \frac{t}{t_0},\\ 0 & \text{otherwise.}\end{cases}$$

where δ is a displacement and t_0 is related to the structural characteristics by the relationship $\omega_0 t_0 = 3\pi$, where $\omega_0 = \sqrt{EJ/mL^3}$

2. Find the analytical expressions of the modal responses for $0 \le t \le 3t_0$.

3. Using the above analytical expressions, compute and plot the vertical displacement (positive when downward) of the smaller, rightmost mass in the same time interval.

2.2 Imposed Support Motion

The structure is at rest when it is subjected to a vertical motion of the bottom hinge, positive when downwards

$$v_{\text{hinge}}(t) = \delta \begin{cases} 0 & \omega_0 t < 0, \\ 6\tau^5 - 15\tau^4 + 10\tau^3 & 0 \le t \le t_0 \text{ with } \tau = \frac{t}{t_0}, \\ 1 & t_0 < t. \end{cases}$$

4. Plot $v_{\text{hinge}}(t)$, $\dot{v}_{\text{hinge}}(t)$ and $\ddot{v}_{\text{hinge}}(t)$ in the interval $0 \le t \le 3t_0$.

5. Compute and plot the total vertical displacement of the smaller mass in the same time interval using the modal superposition procedure.

Note: the particular integral for each mode i is a polynomial $\xi_i(t) = \sum_{j=0}^{3} C_{ij} t^j$.

Note: statically over-determined system You need the stiffness matrix associated with the 3 dynamical DOF but the system is statically over-determined. The 3×3 stiffness matrix can be determined using the following procedure.

1. Release one constraint (one of them is *much more convenient* than the others) and add the corresponding DOF to the 3 dynamic DOF.
2. Compute the flexibility matrix using the PVD for the resulting statically determined, 4 DOF system.
3. Compute the stiffness matrix of the statically determined, 4 DOF system.
4. The stiffness matrix of the over-determined system is just a partition of the stiffness matrix of the statically determined system.

The 4×4 flexibility matrix of step 2, for the obvious choice of the additional DOF, is:

$$\overline{\mathbf{F}} = \frac{L^3}{6EJ}\begin{bmatrix}72 & 40 & 21 & 14\\ 40 & 24 & 13 & 9\\ 21 & 13 & 10 & 5\\ 14 & 9 & 5 & 4\end{bmatrix}$$

Had you hopefully choosen the same additional DOF as me, your results may however differ from mine in terms of *a*) the signs of the off-diagonal terms, that depend on the choice of the signs of the DOF and *b*) the order of rows and columns, that depends on the numbering of the degrees of freedom.

3 Vibration Control

A machine of total mass $\mathcal{M}$ exerts a *steady-state* unbalanced force on its rigid supports,

$$p_{\text{s-s}}(t) = p_0 \sin \omega_0 t,$$

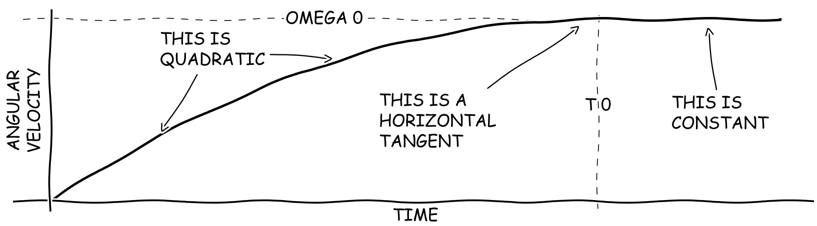
due to the rotation of an unbalanced part with constant angular velocity ω_0.

3.1 Suspension System

Using the damping ratio ζ as a free variable, design a suspension system to limit the steady-state transmitted force $f_{\text{s-s}}(t) = f_0 \sin(\omega_0 t - \varphi)$ so that it is $f_0 \le p_0/4$. Fulfilling this requirement implies a limit on the stiffness, $k(\zeta) \le \mathcal{M}\,\omega_0^2\,\kappa(\zeta)$.

1. Give the analytical expression of the *normalized stiffness* $\kappa(\zeta)$ for an under-critically damped suspension system, $0 \le \zeta < 1$.
2. Plot the normalized stiffness, $\kappa(\zeta)$.

3.2 Transient



At startup, for $0 \le t \le t_0$ the angular velocity $\omega(t)$ of the rotating part of the machine increases quadratically from 0 to ω_0, with a horizontal tangent in $t = t_0$ and later remains constant. The angular velocity, $\omega(t) = \dot{\vartheta}(t)$, is the time derivative of the *phase angle*, $\vartheta(t)$, which describes the position of the rotating part.

3. Give the analytical expressions of the phase angle, $\vartheta(t)$, the angular velocity $\omega(t)$ and the unbalanced force $p(t)$ for $0 \le t \le 2t_0$ (consider $\vartheta(0) = 0$).

4. Plot $\vartheta(t)$, $\omega(t)$ and $p(t)$ in the same time interval.

3.3 Transient Response

For $\omega_0 t_0 = 50\pi$, for $\zeta = 2.5\%, 10\%, 40\%$, for $f_0 = p_0/4$ and $k = \mathcal{M}\omega_0^2\kappa(\zeta)$.

5. Determine the peak value of the transmitted force during the transient.

6. Determine how long it takes for the response to be within 1% of the steady state response

Use the linear acceleration algorithm with a time step $h = T^\star/10$ ($T^\star = 2\pi/\omega_0$ being the period of the steady-state force) to derive the required results.

Note: non uniform circular motion Consider a mass m spinning around a fixed axis at a constant distance r.

The mass position in its plane can be written in terms of an angular variable using the exponential notation, $z = r\,\mathrm{e}^{\mathrm{i}\vartheta}$, where ϑ is the *phase angle*.

The mass velocity is $\dot{z} = \mathrm{i}\dot{\vartheta}\,r\,\mathrm{e}^{\mathrm{i}\vartheta} = \mathrm{i}\omega\,r\,\mathrm{e}^{\mathrm{i}\vartheta}$ and its acceleration (using Euler's formula) is

$$\begin{aligned}
\ddot{z} &= (\mathrm{i}\dot{\omega} - \omega^2)\,r\,\mathrm{e}^{\mathrm{i}\vartheta} \\
&= (\mathrm{i}\dot{\omega} - \omega^2)\,r\,(\cos\vartheta + \mathrm{i}\sin\vartheta) \\
&= \mathrm{i}\,(\dot{\omega}\cos\vartheta - \omega^2\sin\vartheta)\,r - (\omega^2\cos\vartheta + \dot{\omega}\sin\vartheta)\,r.
\end{aligned}$$

The reaction R of the axis is opposite to the force on the mass, $R = -m\ddot{z}$ and the unbalanced load is the component of R in the direction of gravity. Using the imaginary component, for a non uniform circular motion it is

$$p(t) = mr\,(\omega^2\sin\vartheta - \dot{\omega}\cos\vartheta).$$

For a uniform motion, $\omega(t) = \omega_0$ and $\dot{\omega} = 0$, the unbalanced load is simply $p(t) = mr\omega_0^2\sin(\omega_0 t - \varphi)$ and you have $p_0 = m\,\omega_0^2\,r$.

Note: machine mass and unbalanced mass I've used different symbols for the total mass of the machine and the the unbalanced mass.