Support Motion

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1 Support Motion

We add a DOF corresponding to the imposed support motion and collect the bending moments due to unit forces in `M`, we compute `F33` using the PVD, we compute the influence matrix `E` using the last column of `E33`, we compute the augmented stiffness `K33` by inversion of `F33` and the structural stiffness matrix `K` by partitioning. The structural mass matrix `M` is trivial...

```
L = [1, 3, 1]
M = [[p(+1, +0), p(+0, -1), p(+0, -1)],
     [p(+0, +0), p(+1, +0), p(+1, +3)],
     [p(+0, +0), p(+0, +0), p(+1, +0)]]

F33 = array([[sum(integrate(m1*m2, 0, l) for m1, m2, l in zip(M1, M2,L))
                  for M1 in M] for M2 in M])
E = F33[:2,2]/F33[2,2]

K33 = inv(F33)
K = K33[:2,:2]
M = array(((1, 0), (0, 1)))
```

$$\bar{\boldsymbol{F}} = \frac{1}{6}\frac{L^3}{EJ}\begin{bmatrix} 26.0 & -48.0 & -3.0 \\ -48.0 & 128.0 & 11.0 \\ -3.0 & 11.0 & 2.0 \end{bmatrix}, \qquad \bar{\boldsymbol{K}} = \frac{1}{153}\frac{EJ}{L^3}\begin{bmatrix} 135.0 & 63.0 & -144.0 \\ 63.0 & 43.0 & -142.0 \\ -144.0 & -142.0 & 1024.0 \end{bmatrix},$$

$$\boldsymbol{K} = \frac{1}{153}\frac{EJ}{L^3}\begin{bmatrix} 135.0 & 63.0 \\ 63.0 & 43.0 \end{bmatrix}, \qquad \boldsymbol{M} = m\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$\boldsymbol{E} = \frac{1}{2}\begin{bmatrix} -3.0 \\ 11.0 \end{bmatrix}.$$

Now, the free vibrations problem can be solved using `eigh`

```
Lambda2, evecs = eigh(K,M)
```

$$\boldsymbol{\Lambda}^2 = \begin{bmatrix} 0.0719 & 0.0000 \\ 0.0000 & 1.0915 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} 0.2681 & 0.0000 \\ 0.0000 & 1.0448 \end{bmatrix}, \quad \boldsymbol{\Psi} = \begin{bmatrix} 0.4529 & -0.8915 \\ -0.8915 & -0.4529 \end{bmatrix}.$$

We start from the equation of motion in terms of t:

$$m\boldsymbol{M}\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} + \frac{EJ}{L^3}\boldsymbol{K}\boldsymbol{x} = -m\boldsymbol{M}\boldsymbol{e}\delta\omega_0^2 f(\omega_0 t).$$

Changing the time variable from t to $a=\omega_0 t$, using the chain rule for derivation (the dot notation means derivation with respect to a), dividing both members by $m\omega_0^2$ and taking into account that $EJ/mL^3=\omega_0^2$, eventually we have

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{e}\delta f(a).$$

Using the modal transformation, with $\eta_i = q_i/\delta$, for every $i=1,...,\text{NDOF}$ we have

$$\ddot{\eta}_i + \lambda_i^2\eta_i = \gamma_i f(a), \qquad \text{with } \gamma_i = -\boldsymbol{\psi}_i^T\boldsymbol{M}\boldsymbol{e}.$$

```
gamma = -evecs.T@M@E
platex(r'\boldsymbol\gamma_=_', mat2lat(gamma[:,None].T, dlm='B'), '{}^T')
```

$$\boldsymbol{\gamma} = \begin{Bmatrix} 5.58289 & 1.15384 \end{Bmatrix}^T$$

Because $f(a)$ is a polynomial, $f(a) = \sum_0^m f_j a^j$

```
# the polynomial is here defined by the list of its roots
f = poly1d((0, 3, 5), r=1, variable='a')
print(f)
```

```
   3     2
1 a - 8 a + 15 a
```

we can

- write $\gamma_i f(a) = f_i(a) = \sum_0^m f_{i,j} a^j$ and
- assume that $h_i(a) = \sum_0^m h_{i,j} a^j$ is a particular integral of the i-th equation of motion.

The second derivative of h can be written $\ddot{h}_i = \sum_0^m (j+2)(j+1)h_{i,j+2}a^j$ with the understanding that the coefficients of a^{m-1} and a^m are equal to zero: $h_{i,m+1} = h_{i,m+2} = 0$. Substituting in the i-th eom and equating terms with the same power of a on both sides we have

$$\big((j+2)(j+1)h_{i,j+2} + \lambda_i^2 h_{i,j}\big)a^j = f_{i,j}a^j, \qquad j = m,...,0.$$

We have written $j = m,...,0$ because we can solve formally for $h_{i,j}$

$$h_{i,j} = \frac{f_{i,j} - (j+2)(j+1)h_{i,j+2}}{\lambda_i^2}, \qquad j = m, m-1, m-2, ...$$

and notice that the coefficient $h_{i,m}$ can be computed because $h_{i,m+2} = 0$, the coefficent $h_{i,m-1}$ can be computed because $h_{i,m+1} = 0$ and that all the remaining coefficients can be computed, *in inverse order*, because the terms with higher indices have already been computed.

The procedure sketched above can be easily programmed in terms of the coefficients of $f_i(a)$, `f_i` and of the value of λ_i^2, `l2_1`:

```
def part_int(f_i, l2_i):
    h_i = [0, 0]
    m = f.order
    for j in range(m, -1, -1): # j = m, m-1, m-2, ..., 0
        h_ij = (f_i[j]-(j+1)*(j+2)*h_i[-2]) / l2_i
        h_i.append(h_ij)
    return poly1d(h_i, variable='a')
```

Eventually we can compute the particular integrals for our modal eom's (and check the results too...):

```
h = [part_int(f*g_i, l2_i) for l2_i, g_i in zip(Lambda2, gamma)]
```

$$h_i = +77.698183a^3 - 621.585462a^2 - 5322.574538a + 17301.459409$$
$$\ddot{h}_i + \lambda_i^2 h_i = +5.582889a^3 - 44.663109a^2 + 83.743330a$$
$$f_i(a) = \gamma_i f(a) = +5.582889a^3 - 44.663109a^2 + 83.743330a$$
$$h_i = +1.057073a^3 - 8.456586a^2 + 10.045584a + 15.494707$$
$$\ddot{h}_i + \lambda_i^2 h_i = +1.153843a^3 - 9.230746a^2 + 17.307649a$$
$$f_i(a) = \gamma_i f(a) = +1.153843a^3 - 9.230746a^2 + 17.307649a$$

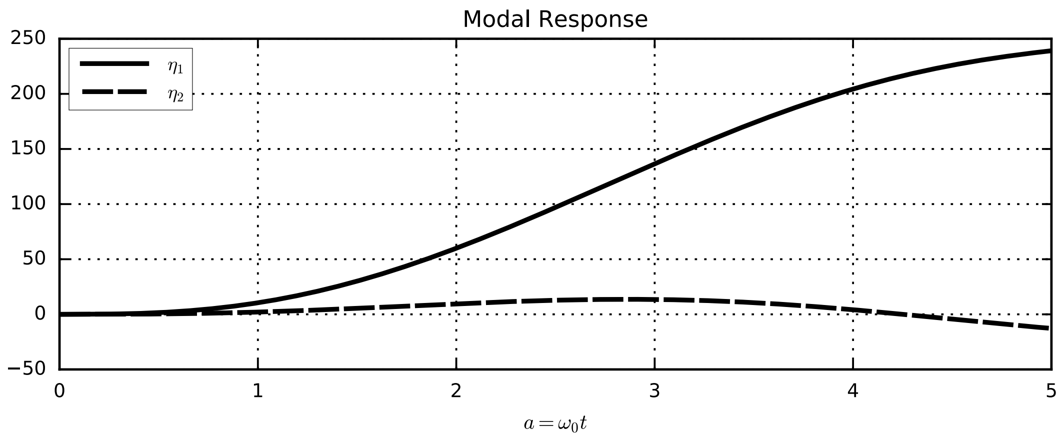
Our system starts from rest conditions, hence the initial conditions in modal coordinates are $q_i(0) = 0$ and $\dot{q}_i(0) = 0$. From the expression of the general integral $\eta_i = A_i \sin\lambda_i a + B_i \cos\lambda_i a + h_i(a)$ we have

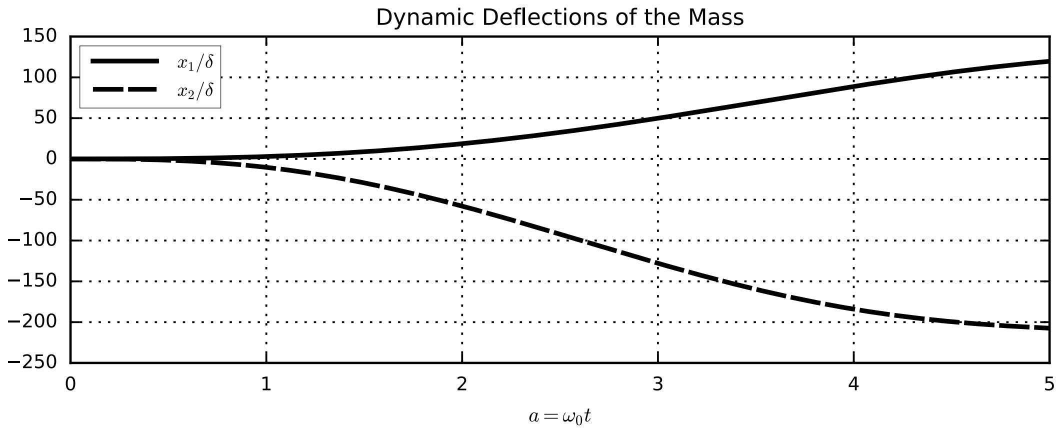
$$\begin{cases} B_i = -h_i(0) \\ A_i = -\dot{h}_i(0)/\lambda_i \end{cases}$$

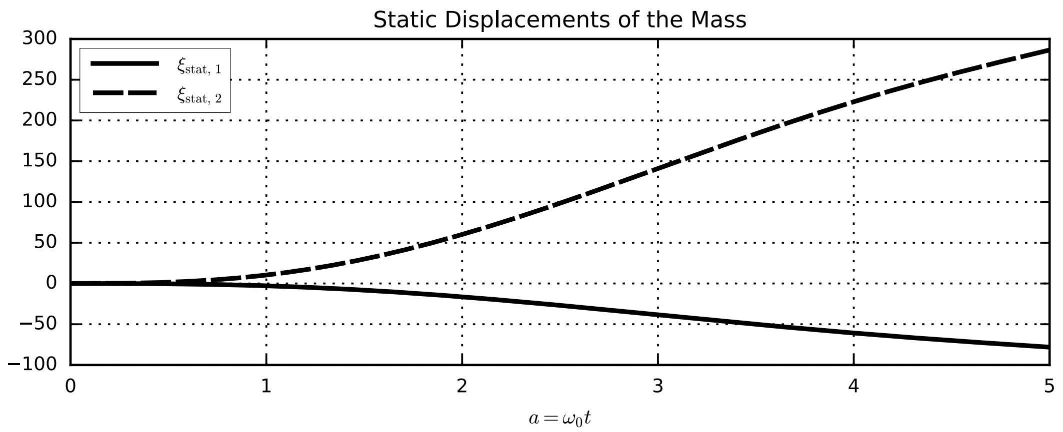
```
Lambda = sqrt(Lambda2)
B = -array([h_i(0) for h_i in h])
A = -array([h_i.deriv()(0) for h_i in h])/Lambda
```

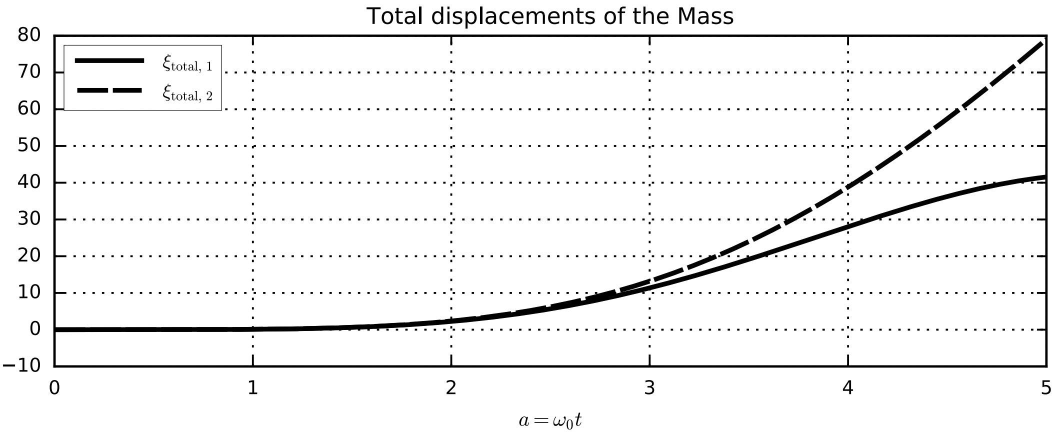
$$\eta_1(a) = 19856.271355\sin 0.268055a - 17301.459409\cos 0.268055a + 77.698183a^3 - 621.585462a^2 - 5322.574538a + 17301.459409$$
$$\eta_2(a) = -9.615112\sin 1.044770a - 15.494707\cos 1.044770a + 1.057073a^3 - 8.456586a^2 + 10.045584a + 15.494707$$

```
a = linspace(0, 5, 1001)
eta = col(A)*sin(col(Lambda)*a)+col(B)*cos(col(Lambda)*a)+[h_i(a) for h_i in h]
xi = evecs@eta
xi_stat = col(E)*f.integ(2)(a)
xi_tot  = xi_stat+xi
```


Modal Response
η_1
η_2
250
200
150
100
50
0
−50
0
1
2
3
4
5
$a = \omega_0 t$


Dynamic Deflections of the Mass
x_1/δ
x_2/δ
150
100
50
0
−50
−100
−150
−200
−250
0
1
2
3
4
5
$a = \omega_0 t$


Static Displacements of the Mass
$\xi_{\mathrm{stat},\,1}$
$\xi_{\mathrm{stat},\,2}$
300
250
200
150
100
50
0
−50
−100
0
1
2
3
4
5
$a = \omega_0 t$



Total displacements of the Mass
80
70
60
50
40
30
20
10
0
−10
0
1
2
3
4
5
$\xi_{\text{total, 1}}$
$\xi_{\text{total, 2}}$
$a = \omega_0 t$