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# Multi Degrees of Freedom Systems
## MDOF's

### Giacomo Boffi

http://intranet.dica.polimi.it/people/boffi-giacomo

Dipartimento di Ingegneria Civile Ambientale e Territoriale
Politecnico di Milano

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# Outline

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# Introductory Remarks

Consider an undamped system with two masses and two degrees of freedom.

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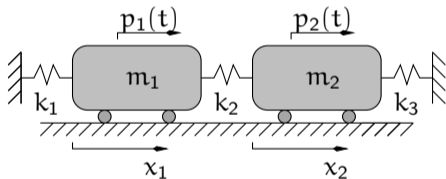
We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally. write an equation of dynamic equilibrium for each mass.



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t)$$



$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t)$$

# The equation of motion of a 2DOF system

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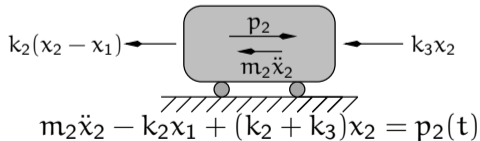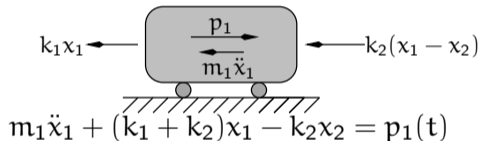Structural
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With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = p_1(t), \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

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Introducing the loading vector $\mathbf{p}$, the vector of inertial forces $\mathbf{f}_I$ and the vector of elastic forces $\mathbf{f}_S$,

$$\mathbf{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \mathbf{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\mathbf{f}_I + \mathbf{f}_S = \mathbf{p}(t).$$

# $\mathbf{f}_S = \mathbf{K}\mathbf{x}$

It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements $\mathbf{x} = \left\{x_1 x_2\right\}^{\mathsf{T}}$ in terms of a matrix product, introducing the so called *stiffness matrix* $\mathbf{K}$.

# $\mathbf{f}_S = \mathbf{K}\,x$

It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements $x = \{x_1 x_2\}^\mathsf{T}$ in terms of a matrix product, introducing the so called *stiffness matrix* $\mathbf{K}$.
In our example it is

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} x = \mathbf{K}\,x$$

# $\mathbf{f}_S = \mathbf{K}\mathbf{x}$

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It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements $\mathbf{x} = \left\{ x_1 x_2 \right\}^\mathsf{T}$ in terms of a matrix product, introducing the so called *stiffness matrix* $\mathbf{K}$.
In our example it is

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{x} = \mathbf{K}\mathbf{x}$$

The stiffness matrix $\mathbf{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\mathbf{K}$ is hence a *square matrix* $\underset{\text{ndof} \times \text{ndof}}{\mathbf{K}}$

# $\mathbf{f}_I = \mathbf{M}\ddot{\mathbf{x}}$

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Analogously, introducing the *mass matrix* $\mathbf{M}$ that, for our example, is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\mathbf{f}_I = \mathbf{M}\ddot{\mathbf{x}}.$$

Also the mass matrix $\mathbf{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

# Matrix Equation

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Finally it is possible to write the equation of motion in matrix format:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t).$$

# Matrix Equation

Finally it is possible to write the equation of motion in matrix format:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t).$$

*Of course it is possible to take into consideration also the
damping forces, taking into account the velocity vector $\dot{\mathbf{x}}$ and
introducing a damping matrix $\mathbf{C}$ too, so that we can eventually
write*

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t).$$

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write*

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t).$$

*But today we are focused on undamped systems...*

# Properties of **K**

- ▶ **K** is symmetrical.
  The elastic force exerted on mass i due to an unit displacement
  of mass j, $f_{S,i} = k_{ij}$ is equal to the force $k_{ji}$ exerted on mass j
  due to an unit diplacement of mass i, in virtue of *Betti's
  theorem* (also known as Maxwell-Betti reciprocal work theorem).

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▶ **K** is symmetrical.
  The elastic force exerted on mass $i$ due to an unit displacement of mass $j$, $f_{S,i} = k_{ij}$ is equal to the force $k_{ji}$ exerted on mass $j$ due to an unit diplacement of mass $i$, in virtue of *Betti's theorem* (also known as Maxwell-Betti reciprocal work theorem).

▶ **K** is a positive definite matrix.
  The strain energy $V$ for a discrete system is

$$V = \frac{1}{2}\boldsymbol{x}^\mathsf{T}\boldsymbol{f}_S,$$

  and expressing $\boldsymbol{f}_S$ in terms of **K** and $\boldsymbol{x}$ we have

$$V = \frac{1}{2}\boldsymbol{x}^\mathsf{T}\boldsymbol{K}\boldsymbol{x},$$

  and because the strain energy is positive for $\boldsymbol{x} \neq 0$ it follows that **K** is definite positive.

# Properties of $\mathbf{M}$

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive.
Both the mass and the stiffness matrix are symmetrical and definite positive.

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# Properties of $\mathbf{M}$

Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive. Both the mass and the stiffness matrix are symmetrical and definite positive.

*Note that the kinetic energy for a discrete system can be written*

$$T = \frac{1}{2}\dot{x}^\mathsf{T} \mathbf{M} \dot{x}.$$

# Generalisation of previous results

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The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.

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# Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.

1. For a general structural system, in which not all DOFs are related to a mass, $\mathbf{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy is zero.

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# Generalisation of previous results

The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.
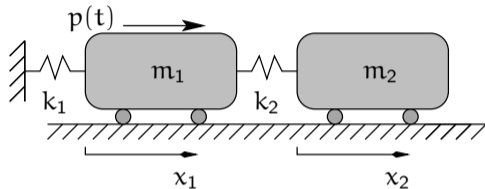
1. For a general structural system, in which not all DOFs are related to a mass, $\mathbf{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy is zero.

2. For a general structural system subjected to axial loads, due to the presence of *geometrical stiffness* it is possible that for some particular displacement vector the strain energy is zero and $\mathbf{K}$ is *semi-definite* positive.

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Graphical statement of the problem



$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = m;$$
$$p(t) = p_0 \sin \omega t.$$

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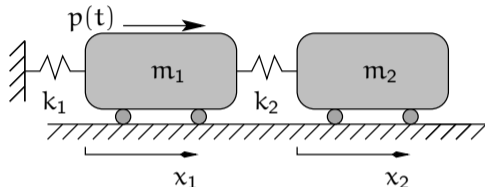Properties of
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Graphical statement of the problem



$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = m;$
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The equations of motion

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = p_0 \sin \omega t,$$
$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0.$$

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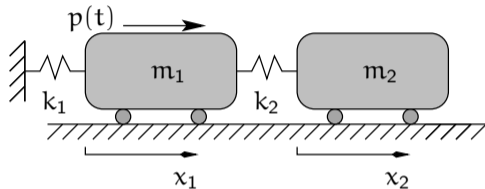Matrices
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... but we prefer the matrix notation ...

# The steady state solution

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We prefer the matrix notation because we can find the steady-state response of a *SDOF* system *exactly* as we found the s-s solution for a SDOF system.

Substituting $x(t) = \xi \sin \omega t$ in the equation of motion and simplifying $\sin \omega t$,

$$k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \xi - m\omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \xi = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

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dividing by $k$, with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

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dividing by $k$, with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

$$\left( \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \xi = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \xi = \Delta_{st} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

# The steady state solution

The determinant of the matrix of coefficients is

$$\text{Det} = 2\beta^4 - 5\beta^2 + 2$$

but we want to write the polynomial in $\beta$ in terms of its roots

$$\text{Det} = 2 \times (\beta^2 - 1/2) \times (\beta^2 - 2).$$

Solving for $\xi/\Delta_{st}$ in terms of the inverse of the coefficient matrix gives

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$$\text{Det} = 2 \times (\beta^2 - 1/2) \times (\beta^2 - 2).$$

Solving for $\xi/\Delta_{st}$ in terms of the inverse of the coefficient matrix gives

$$\begin{aligned}
\frac{\xi}{\Delta_{st}} &= \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{bmatrix} 1 - \beta^2 & 1 \\ 1 & 3 - 2\beta^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\
&= \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix}.
\end{aligned}$$

# The solution, graphically

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steady-state response for a 2 dof system, harmonic load
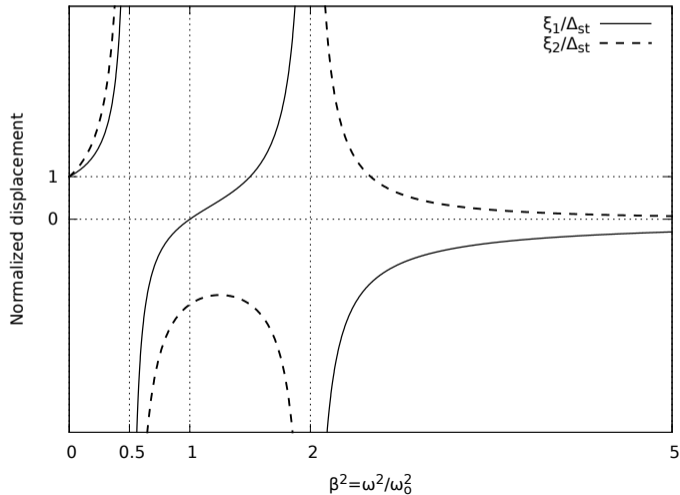
$\xi_1/\Delta_{st}$ ———
$\xi_2/\Delta_{st}$ – – –

Normalized displacement

$\beta^2=\omega^2/\omega_o^2$

## Comment to the Steady State Solution

The steady state solution is

$$x_{\text{s-s}} = \Delta_{\text{st}} \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \sin \omega t.$$

As it's apparent in the previous slide, we have two different values of the excitation frequency for which the *dynamic amplification factor* goes to infinity.

# Comment to the Steady State Solution

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For an undamped SDOF system, we had a single frequency of excitation that excites a *resonant response*, now for a *two* degrees of freedom system we have *two* different excitation frequencies that excite a resonant response.

## Comment to the Steady State Solution

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The steady state solution is

$$x_{\text{s-s}} = \Delta_{\text{st}} \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \left\{ \begin{matrix} 1 - \beta^2 \\ 1 \end{matrix} \right\} \sin \omega t.$$

As it's apparent in the previous slide, we have two different values of the excitation frequency for which the *dynamic amplification factor* goes to infinity.

For an undamped SDOF system, we had a single frequency of excitation that excites a *resonant response*, now for a *two* degrees of freedom system we have *two* different excitation frequencies that excite a resonant response.

> *We know how to compute a particular integral for a MDOF system (at least for a harmonic loading), what do we miss to be able to determine the integral of motion?*

# Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we have to study
the homogeneous solution.
Let's start writing the homogeneous equation of motion,

$$\mathbf{M}\ddot{x} + \mathbf{K}x = 0.$$

# Homogeneous equation of motion

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To understand the behaviour of a *MDOF* system, we have to study the homogeneous solution.

Let's start writing the homogeneous equation of motion,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* $\boldsymbol{\psi}$:

$$\mathbf{x}(t) = \boldsymbol{\psi}(A \sin \omega t + B \cos \omega t).$$

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$$\mathbf{x}(t) = \boldsymbol{\psi}(A\sin\omega t + B\cos\omega t).$$

Substituting in the equation of motion, we have

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\boldsymbol{\psi}(A\sin\omega t + B\cos\omega t) = 0$$

# Eigenvalues

The previous equation must hold for every value of t, so it can be simplified removing the time dependency:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \, \boldsymbol{\psi} = 0.$$

This is a homogeneous linear equation, with unknowns $\psi_i$ and the coefficients that depends on the parameter $\omega^2$.

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This is a homogeneous linear equation, with unknowns $\psi_i$ and the
coefficients that depends on the parameter $\omega^2$.

Speaking of homogeneous systems, we know that

- there is always a *trivial solution*, $\boldsymbol{\psi} = 0$, and

- *non-trivial solutions* are possible if the determinant of the matrix of
  coefficients is equal to zero,

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0$$

# Eigenvalues

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Speaking of homogeneous systems, we know that

- there is always a *trivial solution*, $\boldsymbol{\psi} = 0$, and

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  coefficients is equal to zero,

$$\det \left( \mathbf{K} - \omega^2 \mathbf{M} \right) = 0$$

The *eigenvalues* of the *MDOF* system are the values of $\omega^2$ for which the
above equation (the *equation of frequencies*) is verified or, in other words,
the frequencies of vibration associated with the shapes for which

$$\mathbf{K} \boldsymbol{\psi} \sin \omega t = \omega^2 \mathbf{M} \boldsymbol{\psi} \sin \omega t.$$

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# Eigenvalues, cont.

For a system with N degrees of freedom the expansion of $\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right)$ is an algebraic polynomial of degree N in $\omega^2$. A polynomial of degree N has exactly N roots, either real or complex conjugate.

In Dynamics of Structures those roots $\omega_i^2$, $i = 1, \ldots, N$ are all real because the structural matrices are symmetric matrices. Moreover, if both $\mathbf{K}$ and $\mathbf{M}$ are positive definite matrices (a condition that is always satisfied by stable structural systems) all the roots, all the *eigenvalues*, are strictly positive:

$$\omega_i^2 \geqslant 0, \qquad \text{for } i = 1, \ldots, N.$$

# Eigenvectors

Substituting one of the N roots $\omega_i^2$ in the characteristic equation,

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right) \boldsymbol{\psi}_i = 0$$

the resulting system of $N - 1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue $\omega_i^2$.

# Eigenvectors

The scale factor being arbitrary, you have to choose (arbitrarily) the value of one of the components and compute the values of all the other $N-1$ components using the $N-1$ linearly indipendent equations.

# Eigenvectors

The scale factor being arbitrary, you have to choose (arbitrarily) the value of one of the components and compute the values of all the other $N-1$ components using the $N-1$ linearly indipendent equations.

It is common to impose to each eigenvector a *normalisation with respect to the mass matrix*, so that

$$\boldsymbol{\psi}_i^\mathsf{T} \mathbf{M} \, \boldsymbol{\psi}_i = 1.$$

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It is common to impose to each eigenvector a *normalisation with respect to the mass matrix*, so that

$$\boldsymbol{\psi}_i^\mathsf{T} \mathbf{M}\, \boldsymbol{\psi}_i = 1.$$

*Please consider that, substituting **different eigenvalues** in the equation of free vibrations, you have **different linear systems**, leading to **different eigenvectors**.*

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The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\boldsymbol{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are 2N unknown *constants of integration*, that must be determined in terms of the initial conditions.

# Initial Conditions

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\mathbf{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \mathbf{x}_0, \qquad \dot{\mathbf{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\mathbf{x}}_0.$$

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$$\mathbf{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \mathbf{x}_0, \qquad \dot{\mathbf{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\mathbf{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the 2N constants of integration solving the 2N equations

$$\sum_{i=1}^{N} \psi_{ji} B_i = x_{0,j}, \qquad \sum_{i=1}^{N} \psi_{ji} \omega_i A_i = \dot{x}_{0,j}, \qquad j = 1, \dots, N.$$

# Orthogonality - 1

Take into consideration two distinct eigenvalues, $\omega_r^2$ and $\omega_s^2$, and write the characteristic equation for each eigenvalue:

$$\mathbf{K}\,\psi_r = \omega_r^2 \mathbf{M}\,\psi_r$$
$$\mathbf{K}\,\psi_s = \omega_s^2 \mathbf{M}\,\psi_s$$

# Orthogonality - 1

Take into consideration two distinct eigenvalues, $\omega_r^2$ and $\omega_s^2$, and write the characteristic equation for each eigenvalue:

$$\mathbf{K}\psi_r = \omega_r^2 \mathbf{M}\psi_r$$
$$\mathbf{K}\psi_s = \omega_s^2 \mathbf{M}\psi_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\psi_s^\mathsf{T}\mathbf{K}\psi_r = \omega_r^2 \psi_s^\mathsf{T}\mathbf{M}\psi_r$$
$$\psi_r^\mathsf{T}\mathbf{K}\psi_s = \omega_s^2 \psi_r^\mathsf{T}\mathbf{M}\psi_s$$

# Orthogonality - 2

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The term $\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r\right)^\mathsf{T} = \boldsymbol{\psi}_r^\mathsf{T} \mathbf{K}^\mathsf{T} \boldsymbol{\psi}_s$$

but $\mathbf{K}$ is symmetrical, $\mathbf{K}^\mathsf{T} = \mathbf{K}$ and we have

$$\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_r = \boldsymbol{\psi}_r^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_s.$$

# Orthogonality - 3

Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2) \, \boldsymbol{\psi}_r^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_s = 0$$

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Substituting our last identities in the previous equations, we have

$$\psi_r^\mathsf{T} K \psi_s = \omega_r^2 \psi_r^\mathsf{T} M \psi_s$$
$$\psi_r^\mathsf{T} K \psi_s = \omega_s^2 \psi_r^\mathsf{T} M \psi_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2)\, \psi_r^\mathsf{T} M \psi_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\psi_r^\mathsf{T} M \psi_s = 0, \qquad \text{for } r \neq s.$$

# Orthogonality - 4

The eigenvectors are orthogonal also with respect to the stiffness
matrix:

$$\boldsymbol{\psi}_s^T \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M} \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

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The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_i$$

and consequently

$$\boldsymbol{\psi}_i^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

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The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^\mathsf{T} \mathbf{M} \boldsymbol{\psi}_i$$

and consequently

$$\boldsymbol{\psi}_i^\mathsf{T} \mathbf{K} \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

$M_i$ is the *modal mass* associated with mode no. $i$ while $K_i \equiv \omega_i^2 M_i$ is the respective *modal stiffness*.

# Eigenvectors are a base

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The eigenvectors are linearly independent, so for every vector $x$ we can write

$$x = \sum_{j=1}^{N} \psi_j q_j.$$

The coefficients are readily given by premultiplication of $x$ by $\psi_i^T M$, because

$$\psi_i^T M x = \sum_{j=1}^{N} \psi_i^T M \psi_j q_j = \psi_i^T M \psi_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_j = \frac{\psi_j^T M x}{M_j}.$$

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# Eigenvectors are a base

Generalising our results for the displacement vector to the acceleration vector and expliciting the time dependency, it is

$$\mathbf{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\mathbf{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t),$$

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \psi_{ij} \ddot{q}_j(t).$$

Introducing $\mathbf{q}(t)$, the *vector of modal coordinates* and $\boldsymbol{\Psi}$, the *eigenvector matrix*, whose columns are the eigenvectors, we can write

$$\mathbf{x}(t) = \boldsymbol{\Psi}\,\mathbf{q}(t), \qquad\qquad \ddot{\mathbf{x}}(t) = \boldsymbol{\Psi}\,\ddot{\mathbf{q}}(t).$$

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## EoM in Modal Coordinates...

Substituting the last two equations in the equation of motion,

$$\mathbf{M}\,\boldsymbol{\Psi}\,\ddot{\mathbf{q}} + \mathbf{K}\,\boldsymbol{\Psi}\,\mathbf{q} = \mathbf{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^\mathsf{T}$

$$\boldsymbol{\Psi}^\mathsf{T}\mathbf{M}\,\boldsymbol{\Psi}\,\ddot{\mathbf{q}} + \boldsymbol{\Psi}^\mathsf{T}\mathbf{K}\,\boldsymbol{\Psi}\,\mathbf{q} = \boldsymbol{\Psi}^\mathsf{T}\mathbf{p}(t)$$

introducing the so called *starred matrices*, with $\mathbf{p}^\star(t) = \boldsymbol{\Psi}^\mathsf{T}\mathbf{p}(t)$, we can finally write

$$\mathbf{M}^\star\,\ddot{\mathbf{q}} + \mathbf{K}^\star\,\mathbf{q} = \mathbf{p}^\star(t)$$

The vector equation above corresponds to the set of scalar equations

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j, \qquad i = 1, \dots, N.$$

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# ... are N independent equations!

We must examine the structure of the starred symbols.
The generic element, with indexes $i$ and $j$, of the *starred* matrices can be expressed in terms of single eigenvectors,

$$
\begin{aligned}
m_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{M}\, \boldsymbol{\psi}_j &= \delta_{ij} M_i, \\
k_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{K}\, \boldsymbol{\psi}_j &= \omega_i^2 \delta_{ij} M_i.
\end{aligned}
$$

where $\delta_{ij}$ is the *Kroneker symbol*,

$$
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
$$

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# ... are N independent equations!

We must examine the structure of the starred symbols.
The generic element, with indexes i and j, of the *starred* matrices
can be expressed in terms of single eigenvectors,

$$m_{ij}^\star = \boldsymbol{\psi}_i^\mathsf{T} \mathbf{M}\, \boldsymbol{\psi}_j \qquad\qquad = \quad \delta_{ij} M_i,$$
$$k_{ij}^\star = \boldsymbol{\psi}_i^\mathsf{T} \mathbf{K}\, \boldsymbol{\psi}_j \qquad\qquad = \omega_i^2 \delta_{ij} M_i.$$

where $\delta_{ij}$ is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p_i^\star = \boldsymbol{\psi}_i^\mathsf{T} \mathbf{p}(t)$ we have
a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \dots, N$$

## Initial Conditions Revisited

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The initial displacements can be written in modal coordinates,

$$\mathbf{x}_0 = \mathbf{\Psi}\, \mathbf{q}_0$$

and premultiplying both members by $\mathbf{\Psi}^\mathsf{T}\mathbf{M}$ we have the following relationship:

$$\mathbf{\Psi}^\mathsf{T}\mathbf{M}\,\mathbf{x}_0 = \mathbf{\Psi}^\mathsf{T}\mathbf{M}\,\mathbf{\Psi}\,\mathbf{q}_0 = \mathbf{M}^\star \mathbf{q}_0.$$
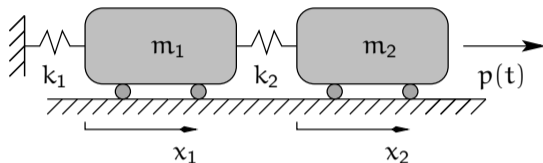
Premultiplying by the inverse of $\mathbf{M}^\star$ and taking into account that $\mathbf{M}^\star$ is diagonal,

$$\mathbf{q}_0 = (\mathbf{M}^\star)^{-1}\,\mathbf{\Psi}^\mathsf{T}\mathbf{M}\,\mathbf{x}_0 \quad \Rightarrow \quad q_{i0} = \frac{\mathbf{\psi}_i^\mathsf{T}\mathbf{M}\,\mathbf{x}_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\mathbf{\psi}_i^{\mathsf{T}}\mathbf{M}\,\dot{\mathbf{x}}_0}{M_i}$$

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# 2 DOF System



$k_1 = k, \quad k_2 = 2k; \qquad m_1 = 2m, \quad m_2 = m;$
$$p(t) = p_0 \sin \omega t.$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \ \mathbf{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\mathbf{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{K} = k \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}.$$

# Equation of frequencies

The equation of frequencies is

$$\|\mathbf{K} - \omega^2\mathbf{M}\| = \left\|\begin{matrix} 3k - 2\omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{matrix}\right\| = 0.$$

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# Equation of frequencies

The equation of frequencies is

$$\left\| \mathbf{K} - \omega^2 \mathbf{M} \right\| = \left\| \begin{matrix} 3k - 2\omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{matrix} \right\| = 0.$$

Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0$$

## Equation of frequencies

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Developing the determinant

$$(2m^2)\omega^4 - (7mk)\omega^2 + (2k^2)\omega^0 = 0$$

Solving the algebraic equation in $\omega^2$

$$\omega_1^2 = \frac{k}{m}\frac{7 - \sqrt{33}}{4} \qquad\qquad \omega_2^2 = \frac{k}{m}\frac{7 + \sqrt{33}}{4}$$
$$\omega_1^2 = 0.31386\frac{k}{m} \qquad\qquad \omega_2^2 = 3.18614\frac{k}{m}$$

# Eigenvectors

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Substituting $\omega_1^2$ for $\omega^2$ in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\,(3 - 2 \times 0.31386)\psi_{11} - 2k\psi_{21} = 0$$

while substituting $\omega_2^2$ gives

$$k\,(3 - 2 \times 3.18614)\psi_{12} - 2k\psi_{22} = 0.$$

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while substituting $\omega_2^2$ gives

$$k\,(3 - 2 \times 3.18614)\psi_{12} - 2k\psi_{22} = 0.$$

Solving with the arbitrary assignment $\psi_{21} = \psi_{22} = 1$ gives the *unnormalized* eigenvectors,

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} +0.84307 \\ +1.00000 \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -0.59307 \\ +1.00000 \end{Bmatrix}.$$

# Normalization

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We compute first $M_1$ and $M_2$,

$$M_1 = \boldsymbol{\psi}_1^\mathsf{T} \mathbf{M}\,\boldsymbol{\psi}_1$$

$$= \begin{Bmatrix} 0.84307, & 1 \end{Bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.68614m, & m \end{Bmatrix} \begin{Bmatrix} 0.84307 \\ 1 \end{Bmatrix} = 2.42153m$$

$$M_2 = 1.70346m$$

the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{2.42153}, \qquad \alpha_2 = \sqrt{1.70346}.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.54177 & -0.45440 \\ +0.64262 & +0.76618 \end{bmatrix}$$

# Modal Loadings

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The modal loading is

$$\begin{aligned}
\mathbf{p}^{\star}(t) &= \mathbf{\Psi}^{\mathsf{T}} \, \mathbf{p}(t) \\
&= p_0 \begin{bmatrix} +0.54177 & +0.64262 \\ -0.45440 & +0.76618 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \sin \omega t \\
&= p_0 \begin{Bmatrix} +0.64262 \\ +0.76618 \end{Bmatrix} \sin \omega t
\end{aligned}$$

# Modal EoM

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Substituting its modal expansion for $x$ into the equation of motion and premultiplying by $\mathbf{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} m\ddot{q}_1 + 0.31386k\, q_1 = +0.64262\, p_0 \sin \omega t \\ m\ddot{q}_2 + 3.18614k\, q_2 = +0.76618\, p_0 \sin \omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by $m$ both equations, we have

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = +0.64262\, \dfrac{p_0}{m} \sin \omega t \\ \ddot{q}_2 + \omega_2^2 q_2 = +0.76618\, \dfrac{p_0}{m} \sin \omega t \end{cases}$$

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# Particular Integral

We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 \left( \omega_1^2 - \omega^2 \right) \sin \omega t = \frac{p_1^\star}{m} \sin \omega t$$

solving for $C_1$

$$C_1 = \frac{p_1^\star}{m} \frac{1}{\omega_1^2 - \omega^2}$$

with $\omega_1^2 = K_1/m \Rightarrow m = K_1/\omega_1^2$:

$$C_1 = \frac{p_1^\star}{K_1} \frac{\omega_1^2}{\omega_1^2 - \omega^2} = \Delta_{st}^{(1)} \frac{1}{1 - \beta_1^2} \quad \text{with } \Delta_{st}^{(1)} = \frac{p_1^\star}{K_1} = 2.047 \frac{p_0}{k} \text{ and } \beta_1 = \frac{\omega}{\omega_1}$$

of course

$$C_2 = \Delta_{st}^{(2)} \frac{1}{1 - \beta_2^2} \quad \text{with } \Delta_{st}^{(2)} = \frac{p_2^\star}{K_2} = 0.2404 \frac{p_0}{k} \text{ and } \beta_2 = \frac{\omega}{\omega_2}$$

# Integrals

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The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + \Delta_{st}^{(1)} \dfrac{\sin \omega t}{1 - \beta_1^2} \\ q_2(t) = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t + \Delta_{st}^{(2)} \dfrac{\sin \omega t}{1 - \beta_2^2} \end{cases}$$

for a system initially at rest

$$\begin{cases} q_1(t) = \Delta_{st}^{(1)} \dfrac{1}{1 - \beta_1^2} \left( \sin \omega t - \beta_1 \sin \omega_1 t \right) \\ q_2(t) = \Delta_{st}^{(2)} \dfrac{1}{1 - \beta_2^2} \left( \sin \omega t - \beta_2 \sin \omega_2 t \right) \end{cases}$$

we are interested in structural degrees of freedom, too... disregarding transient

$$\begin{cases} x_1(t) = \left( \psi_{11} \dfrac{\Delta_{st}^{(1)}}{1 - \beta_1^2} + \psi_{12} \dfrac{\Delta_{st}^{(2)}}{1 - \beta_2^2} \right) \sin \omega t = \left( \dfrac{1.10926}{1 - \beta_1^2} - \dfrac{0.109271}{1 - \beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t \\ x_2(t) = \left( \psi_{21} \dfrac{\Delta_{st}^{(1)}}{1 - \beta_1^2} + \psi_{22} \dfrac{\Delta_{st}^{(2)}}{1 - \beta_2^2} \right) \sin \omega t = \left( \dfrac{1.31575}{1 - \beta_1^2} + \dfrac{0.184245}{1 - \beta_2^2} \right) \dfrac{p_0}{k} \sin \omega t \end{cases}$$

## The response in modal coordinates

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To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$, hence $\beta_1 = \frac{2.0}{\sqrt{0.31386}} = 6.37226$ and $\beta_2 = \frac{2.0}{\sqrt{3.18614}} = 0.62771$.
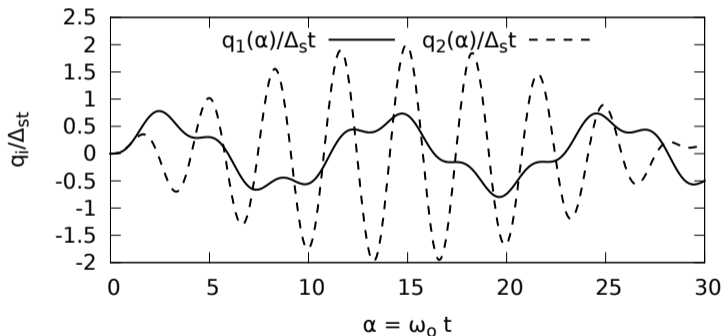


In the graph above, the responses are plotted against an adimensional time coordinate $\alpha$ with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{st} = \frac{p_0}{k}$

Generalized
SDOF's

Giacomo Boffi

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# The response in structural coordinates

Using the same normalisation factors, here are the response functions in
terms of $x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$: