Structural Matrices in *MDOF* Systems

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Outline

Introductory Remarks

Today we will study the properties of structural matrices, that is the operators that relate the vector of system coordinates $\boldsymbol{x}$ and its time derivatives $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ to the forces acting on the system nodes, $\boldsymbol{f}_{\mathrm{S}}$, $\boldsymbol{f}_{\mathrm{D}}$ and $\boldsymbol{f}_{\mathrm{I}}$, respectively.

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In the end, we will see again the solution of a *MDOF* problem by superposition, and in general today we will revisit many of the subjects of our previous class.

Structural Matrices

We already met the mass and the stiffness matrix, $\boldsymbol{M}$ and $\boldsymbol{K}$, and tangentially we introduced also the dampig matrix $\boldsymbol{C}$.
We have seen that these matrices express the linear relation that holds between the vector of system coordinates $\boldsymbol{x}$ and its time derivatives $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ to the forces acting on the system nodes, $\boldsymbol{f}_\text{S}$, $\boldsymbol{f}_\text{D}$ and $\boldsymbol{f}_\text{I}$, elastic, damping and inertial force vectors.

$$\begin{aligned} \boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} &= \boldsymbol{p}(t) \\ \boldsymbol{f}_\text{I} + \boldsymbol{f}_\text{D} + \boldsymbol{f}_\text{S} &= \boldsymbol{p}(t) \end{aligned}$$

Also, we know that $\boldsymbol{M}$ and $\boldsymbol{K}$ are symmetric and definite positive, and that it is possible to uncouple the equation of motion expressing the system coordinates in terms of the *eigenvectors*, $\boldsymbol{x}(t) = \sum q_i \boldsymbol{\psi}_i$, where the q_i are the *modal coordinates* and the eigenvectors ψ_i are the non-trivial solutions to the equation of free vibrations,

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right)\boldsymbol{\psi} = \boldsymbol{0}$$

Free Vibrations

From the homogeneous, undamped problem

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{0}$$

introducing separation of variables

$$\boldsymbol{x}(t) = \boldsymbol{\psi}\,(A \sin \omega t + B \cos \omega t)$$

we wrote the homogeneous linear system

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) \boldsymbol{\psi} = \boldsymbol{0}$$

whose non-trivial solutions $\boldsymbol{\psi}_i$ for ω_i^2 such that $\left\|\boldsymbol{K} - \omega_i^2 \boldsymbol{M}\right\| = 0$ are the eigenvectors.

It was demonstrated that, for each pair of distint *eigenvalues* ω_r^2 and ω_s^2, the corresponding eigenvectors obey the ortogonality condition,

$$\boldsymbol{\psi}_s^T \boldsymbol{M}\,\boldsymbol{\psi}_r = \delta_{rs} M_r, \quad \boldsymbol{\psi}_s^T \boldsymbol{K}\,\boldsymbol{\psi}_r = \delta_{rs} \omega_r^2 M_r.$$

Additional Orthogonality Relationships

From

$$\boldsymbol{K}\,\psi_s = \omega_s^2 \boldsymbol{M}\,\psi_s$$

premultiplying by $\psi_r^T \boldsymbol{K}\boldsymbol{M}^{-1}$ we have

$$\psi_r^T \boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}\,\psi_s = \omega_s^2 \psi_r^T \boldsymbol{K}\,\psi_s$$

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and, generalizing,

$$\psi_r^T \left(\boldsymbol{K}\boldsymbol{M}^{-1}\right)^b \boldsymbol{K}\,\psi_s = \delta_{rs}\left(\omega_r^2\right)^{b+1} M_r.$$

Additional Relationships, 2

From

$$\boldsymbol{M}\,\psi_s = \omega_s^{-2}\boldsymbol{K}\,\psi_s$$

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Additional Relationships, 3

Defining $X_{rs}(k) = \psi_r^T \boldsymbol{M} \left(\boldsymbol{M}^{-1}\boldsymbol{K}\right)^k \psi_s$ we have

$$\begin{cases} X_{rs}(0) = \psi_r^T \boldsymbol{M} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^0 M_s \\ X_{rs}(1) = \psi_r^T \boldsymbol{K} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^1 M_s \\ X_{rs}(2) = \psi_r^T \left(\boldsymbol{K}\boldsymbol{M}^{-1}\right)^1 \boldsymbol{K} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^2 M_s \\ \dots & \\ X_{rs}(n) = \psi_r^T \left(\boldsymbol{K}\boldsymbol{M}^{-1}\right)^{n-1} \boldsymbol{K} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^n M_s \end{cases}$$

Observing that $\left(\boldsymbol{M}^{-1}\boldsymbol{K}\right)^{-1} = \left(\boldsymbol{K}^{-1}\boldsymbol{M}\right)^1$

$$\begin{cases} X_{rs}(-1) = \psi_r^T \left(\boldsymbol{M}\boldsymbol{K}^{-1}\right)^1 \boldsymbol{M} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^{-1} M_s \\ \dots & \\ X_{rs}(-n) = \psi_r^T \left(\boldsymbol{M}\boldsymbol{K}^{-1}\right)^n \boldsymbol{M} \psi_s & = \delta_{rs} \left(\omega_s^2\right)^{-n} M_s \end{cases}$$

finally

$$X_{rs}(k) = \delta_{rs} \omega_s^{2k} M_s \quad \text{for } k = -\infty, \dots, \infty.$$

Flexibility

Given a system whose state is determined by the generalized displacements x_j of a set of nodes, we define the flexibility coefficient f_{jk} as the deflection, in direction of x_j, due to the application of a unit force in correspondance of the displacement x_k.

The matrix $\boldsymbol{F} = \left[f_{jk}\right]$ is the *flexibility matrix*.

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- presence of inertial forces.

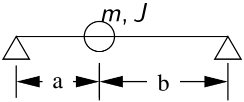
Given a load vector $\boldsymbol{p} = \left\{p_k\right\}$, the displacementent x_j is

$$x_j = \sum f_{jk} p_k$$

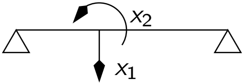
or, in vector notation,

$$\boldsymbol{x} = \boldsymbol{F}\,\boldsymbol{p}$$

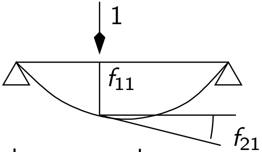
Example



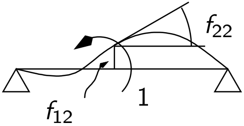
The dynamical system



The degrees of freedom



Displacements due to $p_1 = 1$



and due to $p_2 = 1$.

Elastic Forces

Momentarily disregarding inertial effects, each node shall be in equilibrium under the action of the external forces and the elastic forces, hence taking into accounts all the nodes, all the external forces and all the elastic forces it is possible to write the vector equation of equilibrium

$$\boldsymbol{p} = \boldsymbol{f}_\text{S}$$

and, substituting in the previos vector expression of the displacements

$$\boldsymbol{x} = \boldsymbol{F}\,\boldsymbol{f}_\text{S}$$

Stiffness Matrix

The *stiffness matrix* $\boldsymbol{K}$ can be simply defined as the inverse of the flexibility matrix $\boldsymbol{F}$,

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To understand our formal definition, we must consider an unary vector of displacements,

$$\boldsymbol{e}^{(i)} = \left\{\delta_{ij}\right\}, \qquad j = 1, \ldots, N,$$

and the vector of nodal forces $\boldsymbol{k}_i$ that, applied to the structure, produces the displacements $\boldsymbol{e}^{(i)}$

$$\boldsymbol{F}\,\boldsymbol{k}_i = \boldsymbol{e}^{(i)}, \qquad i = 1, \ldots, N.$$

Collecting all the ordered $\boldsymbol{e}^{(i)}$ in a matrix $\boldsymbol{E}$, it is clear that $\boldsymbol{E} \equiv \boldsymbol{I}$ and we have, writing all the equations at once,

$$\boldsymbol{F}\left[\boldsymbol{k}_i\right] = \left[\boldsymbol{e}^{(i)}\right] = \boldsymbol{E} = \boldsymbol{I}.$$

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Collecting the ordered force vectors in a matrix $\boldsymbol{K} = \left[\vec{k}_i\right]$ we have

$$\boldsymbol{F}\boldsymbol{K} = \boldsymbol{I}, \qquad \Rightarrow \quad \boldsymbol{K} = \boldsymbol{F}^{-1},$$

giving a physical interpretation to the columns of the stiffness matrix.

Finally, writing the nodal equilibrium, we have

$$\boldsymbol{p} = \boldsymbol{f}_S = \boldsymbol{K}\,\boldsymbol{x}.$$

Strain Energy

The elastic strain energy V can be written in terms of displacements and external forces,

$$V = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{x} = \frac{1}{2}\begin{cases} \boldsymbol{p}^T \underbrace{\boldsymbol{F}\,\boldsymbol{p}}_{\boldsymbol{x}}, \\ \underbrace{\boldsymbol{x}^T\boldsymbol{K}}_{\boldsymbol{p}^T}\,\boldsymbol{x}. \end{cases}$$

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Because the elastic strain energy of a stable system is always greater than zero, $\boldsymbol{K}$ is a positive definite matrix.

On the other hand, for an unstable system, think of a compressed beam, there are displacement patterns that are associated to zero strain energy.

Symmetry

Two sets of loads $\boldsymbol{p}^A$ and $\boldsymbol{p}^B$ are applied, one after the other, to an elastic system; the work done is

$$V_{AB} = \frac{1}{2}\boldsymbol{p}^{A\,T}\boldsymbol{x}^A + \boldsymbol{p}^{A\,T}\boldsymbol{x}^B + \frac{1}{2}\boldsymbol{p}^{B\,T}\boldsymbol{x}^B.$$

If we revert the order of application the work is

$$V_{BA} = \frac{1}{2}\boldsymbol{p}^{B\,T}\boldsymbol{x}^B + \boldsymbol{p}^{B\,T}\boldsymbol{x}^A + \frac{1}{2}\boldsymbol{p}^{A\,T}\boldsymbol{x}^A.$$

The total work being independent of the order of loading,

$$\boldsymbol{p}^{A\,T}\boldsymbol{x}^B = \boldsymbol{p}^{B\,T}\boldsymbol{x}^A.$$

Symmetry, 2

Expressing the displacements in terms of $\boldsymbol{F}$,

$$\boldsymbol{p}^{A^T} \boldsymbol{F} \, \boldsymbol{p}^B = \boldsymbol{p}^{B^T} \boldsymbol{F} \boldsymbol{p}^A,$$

both terms are scalars so we can write

$$\boldsymbol{p}^{A^T} \boldsymbol{F} \, \boldsymbol{p}^B = \left(\boldsymbol{p}^{B^T} \boldsymbol{F} \boldsymbol{p}^A \right)^T = \boldsymbol{p}^{A^T} \boldsymbol{F}^T \, \boldsymbol{p}^B.$$

Because this equation holds for every $\boldsymbol{p}$, we conclude that

$$\boldsymbol{F} = \boldsymbol{F}^T.$$

The inverse of a symmetric matrix is symmetric, hence

$$\boldsymbol{K} = \boldsymbol{K}^T.$$

A practical consideration

For the kind of *structures* we mostly deal with in our examples, problems, exercises and assignments, that is *simple structures*, it is usually convenient to compute first the flexibility matrix applying the Principle of Virtual Displacements and later the stiffness matrix, using inversion,

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The stiffness matrix for large, complex structures to construct different methods required are.
The most common procedure to compute the matrices that describe the behaviour of a complex system is the *Finite Element Method*, or *FEM*.

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Note that in the r-th *global* equation of equilibrium we have internal forces caused by the nodal displacements of the *FE* that have nodes i_{el} such that $i_{\text{el}} \mapsto r$, thus implying that global $\boldsymbol{K}$ is a *sparse* matrix.

Example

Consider a 2-D inextensible beam element, that has 4 *DOF*, namely two transverse end displacements x_1, x_2 and two end rotations, x_3, x_4. The element stiffness is computed using 4 shape functions ϕ_i, the transverse displacement being $v(s) = \sum_i \phi_i(s)\, x_i,\ 0 \leqslant s \leqslant L$, the different ϕ_i are such all end displacements or rotation are zero, except the one corresponding to index i.

The shape functions for a beam are

$$\phi_1(s) = 1 - 3\left(\frac{s}{L}\right)^2 + 2\left(\frac{s}{L}\right)^3, \qquad \phi_2(s) = 3\left(\frac{s}{L}\right)^2 - 2\left(\frac{s}{L}\right)^3,$$

$$\phi_3(s) = \left(\frac{s}{L}\right) - 2\left(\frac{s}{L}\right)^2 + \left(\frac{s}{L}\right)^3 \qquad \phi_4(s) = -\left(\frac{s}{L}\right)^2 + \left(\frac{s}{L}\right)^3.$$

Example, 2

The element stiffness coefficients can be computed using, what else, the PVD: we compute the external virtual work done by a variation δx_i by the force due to a unit displacement x_j, that is k_{ij},

$$\delta W_{\text{ext}} = \delta x_i \, k_{ij},$$

the virtual internal work is the work done by the variation of the curvature, $\delta x_i \phi_i''(s)$ by the bending moment associated with a unit x_j, $\phi_j''(s) EJ(s)$,

$$\delta W_{\text{int}} = \int_0^L \delta x_i \phi_i''(s) \phi_j''(s) EJ(s) \, \mathrm{d}s.$$

Example, 3

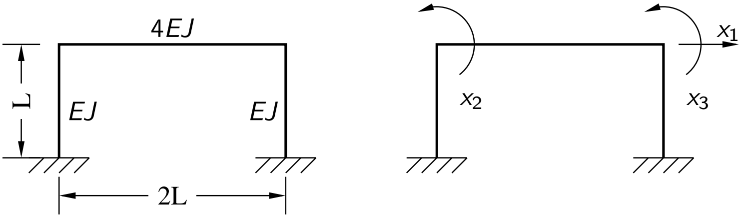
The equilibrium condition is the equivalence of the internal and external virtual works, so that simplifying δx_i we have

$$k_{ij} = \int_0^L \phi_i''(s)\phi_j''(s)EJ(s)\,\mathrm{d}s.$$

For $EJ = \text{const}$,

$$\boldsymbol{f}_\mathrm{S} = \frac{EJ}{L^3}\begin{bmatrix} 12 & -12 & 6L & 6L \\ -12 & 12 & -6L & -6L \\ 6L & -6L & 4L^2 & 2L^2 \\ 6L & -6L & 2L^2 & 4L^2 \end{bmatrix}\boldsymbol{x}$$

Blackboard Time!



Mass Matrix

The mass matrix maps the nodal accelerations to nodal inertial forces, and the most common assumption is to concentrate all the masses in nodal point masses, without rotational inertia, computed *lumping* a fraction of each element mass (or a fraction of the supported mass) on all its bounding nodes.
This procedure leads to a so called *lumped* mass matrix, a diagonal matrix with diagonal elements greater than zero for all the translational degrees of freedom and diagonal elements equal to zero for angular degrees of freedom.

Mass Matrix

The mass matrix is definite positive *only* if all the structure *DOF*'s are translational degrees of freedom, otherwise $\boldsymbol{M}$ is semi-definite positive and the eigenvalue procedure is not directly applicable. This problem can be overcome either by using a *consistent* mass matrix or using the *static condensation* procedure.

Consistent Mass Matrix

A consistent mass matrix is built using the rigorous *FEM* procedure, computing the nodal reactions that equilibrate the distributed inertial forces that develop in the element due to a linear combination of inertial forces.
Using our beam example as a reference, consider the inertial forces associated with a single nodal acceleration $\ddot{x}_j$, $f_{\mathrm{I},\mathrm{j}}(s) = m(s)\phi_j(s)\ddot{x}_j$ and denote with $m_{ij}\ddot{x}_j$ the reaction associated with the i-nth degree of freedom of the element, by the PVD

$$\delta x_i m_{ij} \ddot{x}_j = \int \delta x_i \phi_i(s) m(s) \phi_j(s) \,\mathrm{d}s\, \ddot{x}_j$$

simplifying

$$m_{ij} = \int m(s)\phi_i(s)\phi_j(s)\,\mathrm{d}s.$$

For $m(s) = \overline{m} = \mathrm{const.}$

$$\boldsymbol{f}_\mathrm{I} = \frac{\overline{m}L}{420}\begin{bmatrix} 156 & 54 & 22L & -13L \\ 54 & 156 & 13L & -22L \\ 22L & 13L & 4L^2 & -3L^2 \\ -13L & -22L & -3L^2 & 4L^2 \end{bmatrix} \ddot{\boldsymbol{x}}$$

Consistent Mass Matrix, 2

Pro

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- sligtly more accurate results,
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Contra

- $\boldsymbol{M}$ is no more diagonal, heavy computational aggravation,
- static condensation is computationally beneficial, inasmuch it *reduces* the global number of degrees of freedom.

Damping Matrix

For each element $c_{ij} = \int c(s)\phi_i(s)\phi_j(s)\,\mathrm{d}s$ and the damping matrix $\boldsymbol{C}$ can be assembled from element contributions.

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However, using the FEM $\boldsymbol{C}^\star = \boldsymbol{\Psi}^T\boldsymbol{C}\,\boldsymbol{\Psi}$ is not diagonal and the modal equations are no more uncoupled!
The alternative is to write directly the global damping matrix, in terms of the underdetermined coefficients $\mathfrak{c}_b$,

$$\boldsymbol{C} = \sum_b \mathfrak{c}_b \boldsymbol{M}\left(\boldsymbol{M}^{-1}\boldsymbol{K}\right)^b.$$

Damping Matrix

With our definition of $\boldsymbol{C}$,

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assuming normalized eigenvectors, we can write the individual component of $\boldsymbol{C}^\star = \boldsymbol{\Psi}^T \boldsymbol{C}\, \boldsymbol{\Psi}$

$$c^\star_{ij} = \boldsymbol{\psi}_i^T \boldsymbol{C}\, \boldsymbol{\psi}_j = \delta_{ij} \sum_b \mathfrak{c}_b \omega_j^{2b}$$

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Introducing the modal damping C_j we have

$$C_j = \boldsymbol{\psi}_j^T \boldsymbol{C} \boldsymbol{\psi}_j = \sum_b \mathfrak{c}_b \omega_j^{2b} = 2\zeta_j \omega_j$$

and we can write a system of linear equations in the $\mathfrak{c}_b$.

Example

We want a fixed, 5% damping ratio for the first three modes, taking note that the modal equation of motion is

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = p_i^\star$$

Using

$$\boldsymbol{C} = \mathfrak{c}_0\boldsymbol{M} + \mathfrak{c}_1\boldsymbol{K} + \mathfrak{c}_2\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}$$

we have

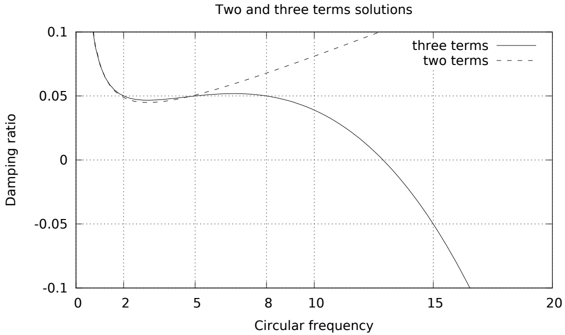
$$2\times 0.05\begin{Bmatrix}\omega_1\\\omega_2\\\omega_3\end{Bmatrix} = \begin{bmatrix}1 & \omega_1^2 & \omega_1^4\\1 & \omega_2^2 & \omega_2^4\\1 & \omega_3^2 & \omega_3^4\end{bmatrix}\begin{Bmatrix}\mathfrak{c}_0\\\mathfrak{c}_1\\\mathfrak{c}_2\end{Bmatrix}$$

Solving for the $\mathfrak{c}$'s and substituting above, the resulting damping matrix is orthogonal to every eigenvector of the system, for the first three modes, leads to a modal damping ratio that is equal to 5%.

Example

Computing the coefficients $\mathfrak{c}_0$, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ to have a 5% damping at frequencies $\omega_1 = 2$, $\omega_2 = 5$ and $\omega_3 = 8$ we have $\mathfrak{c}_0 = 1200/9100$, $\mathfrak{c}_1 = 159/9100$ and $\mathfrak{c}_2 = -1/9100$.

Writing $\zeta(\omega) = \dfrac{1}{2}\left(\dfrac{\mathfrak{c}_0}{\omega} + \mathfrak{c}_1\omega + \mathfrak{c}_2\omega^3\right)$ we can plot the above function, along with its two term equivalent ($\mathfrak{c}_0 = 10/70, \mathfrak{c}_1 = 1/70$).



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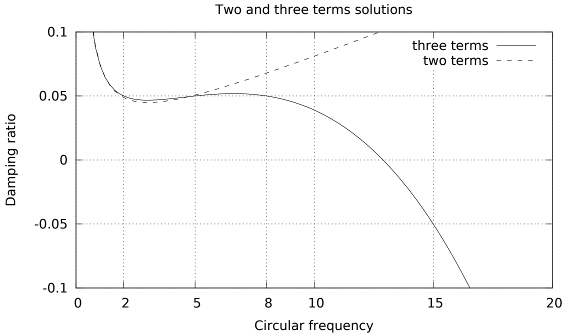
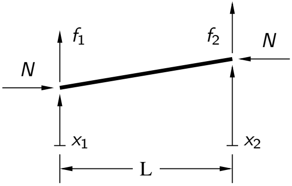


Negative damping? No, thank you: use only an even number of terms.

Geometric Stiffness

A common assumption is based on a linear approximation, for a beam element

$$\boldsymbol{f}_\mathrm{G} = \frac{N}{L}\begin{bmatrix} +1 & -1 & 0 & 0\\ -1 & +1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}$$



$$f_2 = -f_1$$
$$f_1 L = N\,(x_2 - x_1)$$

It is possible to compute the geometrical stiffness matrix using *FEM*, shape functions and PVD,

$$k_{\mathrm{G},ij} = \int N(s)\phi_i'(s)\phi_j'(s)\,\mathrm{d}s,$$

for constant N

$$\boldsymbol{K}_\mathrm{G} = \frac{N}{30L}\begin{bmatrix} 36 & -36 & 3L & 3L\\ -36 & 36 & -3L & -3L\\ 3L & -3L & 4L^2 & -L^2\\ 3L & -3L & -L^2 & 4L^2\end{bmatrix}$$

External Loadings

Following the same line of reasoning that we applied to find nodal inertial forces, by the PVD and the use of shape functions we have

$$p_i(t) = \int p(s,t)\phi_i(s)\,\mathrm{d}s.$$

For a constant, uniform load $p(s,t) = \overline{p} = \text{const}$, applied on a beam element,

$$\boldsymbol{p} = \overline{p}L \left\{ \frac{1}{2} \quad \frac{1}{2} \quad \frac{L}{12} \quad -\frac{L}{12} \right\}^T$$

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Some structural parameter is approximated, only translational *DOF*'s are retained in dynamic analysis.

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If we choose a simplified approach, we must use a procedure to remove unneeded structural *DOF*'s from the model that we use for the dynamic analysis.
Enter the *Static Condensation Method*.

Static Condensation

We have, from a *FEM* analysis, a stiffnes matrix that uses all nodal *DOF*'s, and from the lumped mass procedure a mass matrix were only translational (and maybe a few rotational) *DOF*'s are blessed with a non zero diagonal term.

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In this case, we can always rearrange and partition the displacement vector $\boldsymbol{x}$ in two subvectors: *a*) $\boldsymbol{x}_A$, all the *DOF*'s that are associated with inertial forces and *b*) $\boldsymbol{x}_B$, all the remaining *DOF*'s not associated with inertial forces.

$$\boldsymbol{x} = \left\{\boldsymbol{x}_A \quad \boldsymbol{x}_B\right\}^T$$

Static Condensation, 2

After rearranging the *DOF*'s, we must rearrange also the rows (equations) and the columns (force contributions) in the structural matrices, and eventually partition the matrices so that

$$\begin{Bmatrix}\boldsymbol{f}_I\\ \boldsymbol{0}\end{Bmatrix} = \begin{bmatrix}\boldsymbol{M}_{AA} & \boldsymbol{M}_{AB}\\ \boldsymbol{M}_{BA} & \boldsymbol{M}_{BB}\end{bmatrix}\begin{Bmatrix}\ddot{\boldsymbol{x}}_A\\ \ddot{\boldsymbol{x}}_B\end{Bmatrix}$$

$$\boldsymbol{f}_S = \begin{bmatrix}\boldsymbol{K}_{AA} & \boldsymbol{K}_{AB}\\ \boldsymbol{K}_{BA} & \boldsymbol{K}_{BB}\end{bmatrix}\begin{Bmatrix}\boldsymbol{x}_A\\ \boldsymbol{x}_B\end{Bmatrix}$$

with

$$\boldsymbol{M}_{BA} = \boldsymbol{M}_{AB}^T = \boldsymbol{0}, \quad \boldsymbol{M}_{BB} = \boldsymbol{0}, \quad \boldsymbol{K}_{BA} = \boldsymbol{K}_{AB}^T$$

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Finally we rearrange the loadings vector and write...

Static Condensation, 3

... the equation of dynamic equilibrium,

$$\begin{aligned}
\boldsymbol{p}_A &= \boldsymbol{M}_{AA}\ddot{\boldsymbol{x}}_A + \boldsymbol{M}_{AB}\ddot{\boldsymbol{x}}_B + \boldsymbol{K}_{AA}\boldsymbol{x}_A + \boldsymbol{K}_{AB}\boldsymbol{x}_B\\
\boldsymbol{p}_B &= \boldsymbol{M}_{BA}\ddot{\boldsymbol{x}}_A + \boldsymbol{M}_{BB}\ddot{\boldsymbol{x}}_B + \boldsymbol{K}_{BA}\boldsymbol{x}_A + \boldsymbol{K}_{BB}\boldsymbol{x}_B
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The terms in red are zero, so we can simplify

$$\begin{aligned} \boldsymbol{M}_{AA}\ddot{\boldsymbol{x}}_A + \boldsymbol{K}_{AA}\boldsymbol{x}_A + \boldsymbol{K}_{AB}\boldsymbol{x}_B &= \boldsymbol{p}_A \\ \boldsymbol{K}_{BA}\boldsymbol{x}_A + \boldsymbol{K}_{BB}\boldsymbol{x}_B &= \boldsymbol{p}_B \end{aligned}$$

solving for $\boldsymbol{x}_B$ in the 2nd equation and substituting

$$\begin{aligned} \boldsymbol{x}_B &= \boldsymbol{K}_{BB}^{-1}\boldsymbol{p}_B - \boldsymbol{K}_{BB}^{-1}\boldsymbol{K}_{BA}\boldsymbol{x}_A \\ \boldsymbol{p}_A - \boldsymbol{K}_{AB}\boldsymbol{K}_{BB}^{-1}\boldsymbol{p}_B &= \boldsymbol{M}_{AA}\ddot{\boldsymbol{x}}_A + \left(\boldsymbol{K}_{AA} - \boldsymbol{K}_{AB}\boldsymbol{K}_{BB}^{-1}\boldsymbol{K}_{BA}\right)\boldsymbol{x}_A \end{aligned}$$

Static Condensation, 4

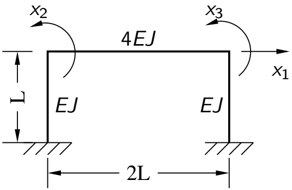
Going back to the homogeneous problem, with obvious positions we can write

$$\left(\overline{\boldsymbol{K}} - \omega^2 \overline{\boldsymbol{M}}\right) \boldsymbol{\psi}_A = \boldsymbol{0}$$

but the $\boldsymbol{\psi}_A$ are only part of the structural eigenvectors, because in essentially every application we must consider also the other *DOF*'s, so we write

$$\boldsymbol{\psi}_i = \begin{Bmatrix} \boldsymbol{\psi}_{A,i} \\ \boldsymbol{\psi}_{B,i} \end{Bmatrix}, \text{ with } \boldsymbol{\psi}_{B,i} = \boldsymbol{K}_{BB}^{-1} \boldsymbol{K}_{BA} \boldsymbol{\psi}_{A,i}$$

Example



$$\boldsymbol{K} = \frac{2EJ}{L^3}\begin{bmatrix} 12 & 3L & 3L \\ 3L & 6L^2 & 2L^2 \\ 3L & 2L^2 & 6L^2 \end{bmatrix}$$

Disregarding the factor $2EJ/L^3$,

$$\boldsymbol{K}_{BB} = L^2\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}, \boldsymbol{K}_{BB}^{-1} = \frac{1}{32L^2}\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}, \boldsymbol{K}_{AB} = \begin{bmatrix} 3L & 3L \end{bmatrix}$$

The matrix $\overline{\boldsymbol{K}}$ is

$$\overline{\boldsymbol{K}} = \frac{2EJ}{L^3}\left(12 - \boldsymbol{K}_{AB}\boldsymbol{K}_{BB}^{-1}\boldsymbol{K}_{AB}^{T}\right) = \frac{39EJ}{2L^3}$$