Truncation Errors
Numerical Integration
Multiple Support Excitation

Giacomo Boffi

http://intranet.dica.polimi.it/people/boffi-giacomo

Dipartimento di Ingegneria Civile Ambientale e Territoriale
Politecnico di Milano

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Part I

How many eigenvectors?

How many eigenvectors?

To understand how many eigenvectors we have to use in a modal analysis, we must consider two factors, the loading shape and the excitation frequency.

Introduction

In the following, we'll consider *only* external loadings whose dependance on time and space can be separated, as in

$$\boldsymbol{p}(\boldsymbol{x}, t) = \boldsymbol{r}\, f(t),$$

so that we can regard separately the two aspects of the problem.

Introduction

It is worth noting that earthquake loadings are precisely of this type:

$$\boldsymbol{p}(\boldsymbol{x}, t) = \boldsymbol{M}\tilde{\boldsymbol{r}}\, \ddot{u}_\text{g}$$

where the vector $\tilde{\boldsymbol{r}}$ is used to choose the structural dof's that are *excited* by the ground motion component under consideration.

> *$\tilde{\boldsymbol{r}}$ is an incidence vector, often simply a vector of ones and zeroes where the ones stay for the inertial forces that are excited by a specific component of the earthquake ground acceleration.*

Multiplication of $\boldsymbol{M}$ and division of $\ddot{u}_g$ by g, acceleration of gravity, serves to show a dimensional load vector multiplied by an adimensional function.

$$\begin{aligned}\boldsymbol{p}(\boldsymbol{x}, t) &= g\,\boldsymbol{M}\tilde{\boldsymbol{r}}\, \frac{\ddot{u}_\text{g}(t)}{g}\\ &= \boldsymbol{r}^\text{g} f_\text{g}(t)\end{aligned}$$

Modal partecipation factor

Under the assumption of separability, we can write the i-th modal equation of motion as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \begin{cases} \frac{\boldsymbol{\psi}_i^T\boldsymbol{r}}{M_i} f(t) \\ \frac{g\,\boldsymbol{\psi}_i^T\boldsymbol{M}\hat{\boldsymbol{r}}}{M_i} f_\text{g}(t)\end{cases} = \Gamma_i f(t)$$

with the modal mass $M_i = \boldsymbol{\psi}_i^T\boldsymbol{M}\boldsymbol{\psi}_i$.
It is apparent that the modal response amplitude depends

- on the characteristics of the time dependency of loading, $f(t)$,
- on the so called *modal partecipation factor* Γ_i,

$$\Gamma_i = \boldsymbol{\psi}_i^T\boldsymbol{r}/M_i \qquad \text{or} \qquad \Gamma_i = g\,\boldsymbol{\psi}_i^T\boldsymbol{M}\hat{\boldsymbol{r}}/M_i = \boldsymbol{\psi}_i^T\boldsymbol{r}^\text{g}/M_i$$

> *Note that both the definitions of modal partecipation give it the dimensions of an acceleration.*

Partecipation Factor Amplitudes

For a given loading $\boldsymbol{r}$ the modal partecipation factor Γ_i is proportional to the work done by the modal displacement $q_i\boldsymbol{\psi}_i^T$ for the given loading $\boldsymbol{r}$:

- if the mode shape and the loading shape are approximately equal (equal signs, component by component), the work (dot product) is maximized,
- if the mode shape is significantly different from the loading (different signs), there is some amount of cancellation and the value of the Γ's will be reduced.

Truncation
Num. Integration
Support Exc.

Giacomo Boffi

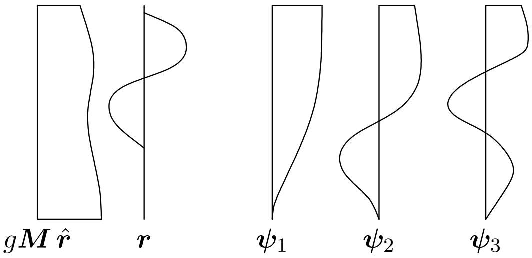
Introduction

Modal partecipation factor

Dynamic magnification factor

Static Correction

Example



Consider a shear type building, its first 3 eigenvectors as sketched above, with mass distribution approximately constant over its height and its earthquake load shape vector

$$\hat{\boldsymbol{r}} = \{1, 1, \dots, 1\}^T \quad \rightarrow \quad g\,\boldsymbol{M}\hat{\boldsymbol{r}} \approx mg\{1, 1, \dots, 1\}^T,$$

consider also an external, assigned load shape vector $\boldsymbol{r}$.

Truncation
Num. Integration
Support Exc.

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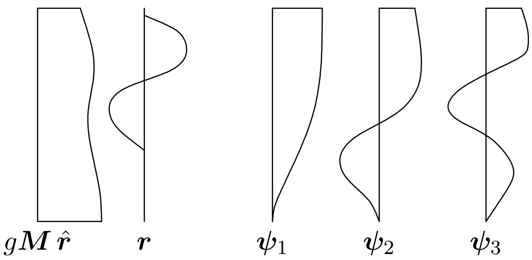
Introduction

Modal partecipation factor

Dynamic magnification factor

Static Correction

Example, cont.



For *EQ* loading, Γ_1 is relatively large for the first mode, as loading components and displacements have the same sign, with respect to other Γ_i's, where the oscillating nature of the higher eigenvectors will lead to increasing cancellation.
On the other hand, consider the external loading, whose peculiar shape is similar to the 3rd mode. Γ_3 will be more relevant than Γ_i's for lower or higher modes.

Modal Loads Expansion

We define the modal load contribution as

$$\boldsymbol{r}_i = \boldsymbol{M}\,\boldsymbol{\psi}_i a_i$$

and express the load vector as a linear combination of the modal contributions

$$\boldsymbol{r} = \sum_i \boldsymbol{M}\,\boldsymbol{\psi}_i a_i = \sum_i \boldsymbol{r}_i.$$

Premultiplying by $\boldsymbol{\psi}_j^T$ the above equation we have a relation that enables the computation of the coefficients a_i:

$$\boldsymbol{\psi}_j^T \boldsymbol{r} = \boldsymbol{\psi}_j^T \sum_i \boldsymbol{M}\,\boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i \quad \rightarrow \quad a_i = {}^{\boldsymbol{\psi}_i^T \boldsymbol{r}}/_{M_i}$$

Modal Loads Expansion

1. A modal load component works *only* for the displacements associated with the corresponding eigenvector,
$$\boldsymbol{\psi}_j^T \boldsymbol{r}_i = a_i\,\boldsymbol{\psi}_j^T \boldsymbol{M}\boldsymbol{\psi}_i = \delta_{ij} a_i M_i.$$
2. Comparing $\boldsymbol{\psi}_j^T \boldsymbol{r} = \boldsymbol{\psi}_j^T \sum_i \boldsymbol{M}\,\boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i$ with the definition of $\Gamma_i = \boldsymbol{\psi}_i^T \boldsymbol{r}/M_i$, we conclude that $a_i \equiv \Gamma_i$ and finally write
$$\boldsymbol{r}_i = \Gamma_i \boldsymbol{M}\,\boldsymbol{\psi}_i.$$
3. The modal load contributions can be collected in a matrix: with $\boldsymbol{\Gamma} = \operatorname{diag}\Gamma_i$ we have
$$\boldsymbol{R} = \boldsymbol{M}\,\boldsymbol{\Psi}\,\boldsymbol{\Gamma}.$$

Equivalent Static Forces

For mode i, the equation of motion is

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \Gamma_i f(t)$$

with $q_i = \Gamma_i D_i$, we can write, to single out the dependency on the modulating function,

$$\ddot{D}_i + 2\zeta_i\omega_i\dot{D}_i + \omega_i^2 D_i = f(t)$$

The modal contribution to displacement is

$$\boldsymbol{x}_i = \Gamma_i \boldsymbol{\psi}_i D_i(t)$$

and the modal contribution to elastic forces $\boldsymbol{f}_i = \boldsymbol{K}\,\boldsymbol{x}_i$ can be written (being $\boldsymbol{K}\boldsymbol{\psi}_i = \omega_i^2 \boldsymbol{M}\boldsymbol{\psi}_i$) as

$$\boldsymbol{f}_i = \boldsymbol{K}\,\boldsymbol{x}_i = \Gamma_i \boldsymbol{K}\,\boldsymbol{\psi}_i D_i = \omega_i^2(\Gamma_i \boldsymbol{M}\,\boldsymbol{\psi}_i) D_i = \boldsymbol{r}_i \omega_i^2 D_i$$

Equivalent Static Response

The response can be determined by superposition of the effects of these pseudo-static forces $\boldsymbol{f}_i = \boldsymbol{r}_i \omega_i^2 D_i(t)$.
If a required response quantity (be it a nodal displacement, a bending moment in a beam, the total shear force in a building storey, etc etc) is indicated by $s(t)$, we can compute with a *static calculation* (usually using the *FEM* model underlying the dynamic analysis) the modal static contribution s_i^{st} and write

$$s(t) = \sum s_i^{\mathsf{st}}(\omega_i^2 D_i(t)) = \sum s_i(t),$$

where the modal contribution to response $s_i(t)$ is given by

1. static analysis using $\boldsymbol{r}_i$ as the static load vector,
2. dynamic amplification using the factor $\omega_i^2 D_i(t)$.

This formulation is particularly apt to our discussion of different contributions to response components.

Modal Contribution Factors

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Num. Integration
Support Exc.

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Introduction
Modal partecipation factor
Dynamic magnification factor
Static Correction

Say that the static response due to $\boldsymbol{r}$ is denoted by s^{st}, then $s_i(t)$, the modal contribution to response $s(t)$, can be written

$$s_i(t) = s_i^{\mathsf{st}} \omega_i^2 D_i(t) = s^{\mathsf{st}} \frac{s_i^{\mathsf{st}}}{s^{\mathsf{st}}} \omega_i^2 D_i(t) = \bar{s}_i s^{\mathsf{st}} \omega_i^2 D_i(t).$$

We have introduced $\bar{s}_i = \frac{s_i^{\mathsf{st}}}{s^{\mathsf{st}}}$, the *modal contribution factor*, the ratio of the modal static contribution to the total static response. The $\bar{s}_i$ are dimensionless, are indipendent from the eigenvector scaling procedure and their sum is unity, $\sum \bar{s}_i = 1$.

Maximum Response

Truncation
Num. Integration
Support Exc.

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Introduction
Modal partecipation factor
Dynamic magnification factor
Static Correction

Denote by D_{i0} the maximum absolute value (or *peak*) of the pseudo displacement time history,

$$D_{i0} = \max_t\{|D_i(t)|\}.$$

It will be

$$s_{i0} = \bar{s}_i s^{\mathsf{st}} \omega_i^2 D_{i0}.$$

The dynamic response factor for mode i, $\mathfrak{R}_{di}$ is defined by

$$\mathfrak{R}_{di} = \frac{D_{i0}}{D_{i0}^{\mathsf{st}}}$$

where D_{i0}^{st} is the peak value of the static pseudo displacement

$$D_i^{\mathsf{st}} = \frac{f(t)}{\omega_i^2}, \quad \rightarrow \quad D_{i0}^{\mathsf{st}} = \frac{f_0}{\omega_i^2}.$$

Maximum Response

With $f_0 = \max\{|f(t)|\}$ the peak pseudo displacement is

$$D_{i0} = \mathfrak{R}_{di} f_0/\omega_i^2$$

and the peak of the modal contribution is

$$s_{i0}(t) = \bar{s}_i s^{\mathsf{st}} \omega_i^2 D_{i0}(t) = f_0 s^{\mathsf{st}} \ \bar{s}_i \mathfrak{R}_{di}$$

The first two terms are independent of the mode, the last are independent from each other and their product is the factor that influences the modal contributions.
Note that this product has the sign of $\bar{s}_i$, as the dynamic response factor is always positive.

MCF's example

The following table (from Chopra, 2nd ed.) displays the $\bar{s}_i$ and their partial sums for a shear-type, 5 floors building where all the storey masses are equal and all the storey stiffnesses are equal too.
The response quantities chosen are $\bar{x}_{5n}$, the *MCF*'s to the top displacement and $\bar{V}_n$, the *MCF* 's to the base shear, for two different load shapes.

	$\boldsymbol{r} = \{0,0,0,0,1\}^T$				$\boldsymbol{r} = \{0,0,0,-1,2\}^T$			
	Top Displacement		Base Shear		Top Displacement		Base Shear	
n or J	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	-0.362	0.890	0.123	0.915	-0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.043	1.172
4	0.008	0.998	-0.063	0.985	0.024	0.994	-0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

Note that

1. for any given $\boldsymbol{r}$, the base shear is more influenced by higher modes, and
2. for any given reponse quantity, the second, *skewed* $\boldsymbol{r}$ gives greater modal contributions for higher modes.

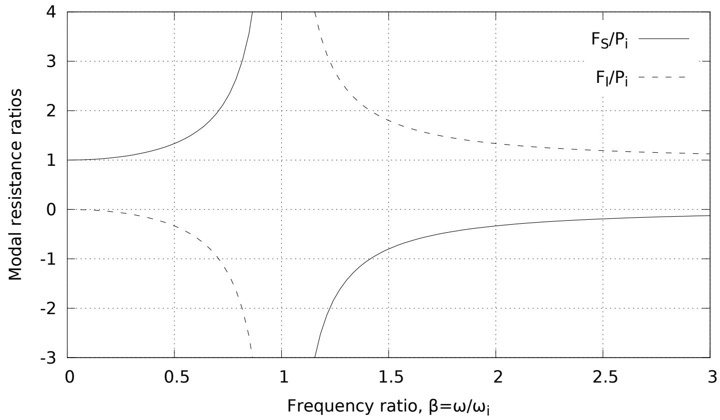
Dynamic Response Ratios

Dynamic Response Ratios are the same that we have seen for *SDOF* systems.
Next page, for an undamped system, harmonically excited,

- solid line, the ratio of the modal elastic force $F_{S,i} = K_i q_i \sin\omega t$ to the harmonic applied modal force, $P_i \sin\omega t$, plotted against the frequency ratio $\beta = \omega/\omega_i$.
  For $\beta = 0$ the ratio is 1, the applied load is fully balanced by the elastic resistance.
  For fixed excitation frequency, $\beta \to 0$ for high modal frequencies.
- dashed line,the ratio of the modal inertial force, $F_{I,i} = -\beta^2 F_{S,i}$ to the load.

Note that for steady-state motion the sum of the elastic and inertial force ratios is constant and equal to 1, as in

$$(F_{S,i} + F_{I,i}) \sin\omega t = P_i \sin\omega t.$$



- For a fixed excitation frequency and high modal frequencies the frequency ratio $\beta \to 0$.
- For $\beta \to 0$ the response is *quasi-static*.
- Hence, for higher modes the response is *pseudo-static*.
- On the contrary, for excitation frequencies high enough the lower modes respond with purely inertial forces.

Static Correction

The preceding discussion indicates that higher modes contributions to the response could be approximated with the static response, leading to a *Static Correction* of the dynamic response.
For a system where $q_i(t) \approx \dfrac{p_i(t)}{K_i}$ for $i > n_{\text{dy}}$,
n_{dy} being the number of dynamically responding modes,
we can write

$$\boldsymbol{x}(t) \approx \boldsymbol{x}_{\text{dy}}(t) + \boldsymbol{x}_{\text{st}}(t) = \sum_1^{n_{\text{dy}}} \boldsymbol{\psi}_i q_i(t) + \sum_{n_{\text{dy}}+1}^{N} \boldsymbol{\psi}_i \frac{p_i(t)}{K_i}$$

where the response for each of the first n_{dy} modes can be computed as usual.

Static Modal Components

Truncation
Num. Integration
Support Exc.
Giacomo Boffi
Introduction
Modal partecipation factor
Dynamic magnification factor
Static Correction

The static modal displacement component $\boldsymbol{x}_j, j > n_{\text{dy}}$ can be written

$$x_j(t) = \boldsymbol{\psi}_j q_j(t) \approx \frac{\boldsymbol{\psi}_j \boldsymbol{\psi}_j^T}{K_j} \boldsymbol{p}(t) = \boldsymbol{F}_j \boldsymbol{p}(t)$$

The *modal flexibility matrix* is defined by

$$\boldsymbol{F}_j = \frac{\boldsymbol{\psi}_j \boldsymbol{\psi}_j^T}{K_j}$$

and is used to compute the j-th mode static deflections due to the applied load vector.
The total displacements, the dynamic contributions and the static correction, for $\boldsymbol{p}(t) = \boldsymbol{r}\, f(t)$, are then

$$\boldsymbol{x} \approx \sum_1^{n_{\text{dy}}} \boldsymbol{\psi}_j q_j(t) + f(t) \sum_{n_{\text{dy}}+1}^{N} \boldsymbol{F}_j \boldsymbol{r}.$$

Alternative Formulation

Our last formula for static correction is

$$\boldsymbol{x} \approx \sum_{1}^{n_{\mathsf{dy}}} \boldsymbol{\psi}_j q_j(t) + f(t) \sum_{n_{\mathsf{dy}}+1}^{N} \boldsymbol{F}_j \boldsymbol{r}.$$

To use the above formula all mode shapes, all modal stiffnesses and all modal flexibility matrices must be computed, undermining the efficiency of the procedure.

Alternative Formulation

Truncation
Num. Integration
Support Exc.

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Introduction
Modal partecipation factor
Dynamic magnification factor
Static Correction

This problem can be obviated computing the total static displacements, $\boldsymbol{x}_{\mathsf{st}}^{\mathsf{total}} = \boldsymbol{K}^{-1}\boldsymbol{p}(t)$, and subtracting the static displacements due to the first n_{dy} modes...

$$\sum_{n_{\mathsf{dy}}}^{N} \boldsymbol{F}_j \boldsymbol{r} f(t) = \boldsymbol{K}^{-1}\boldsymbol{r} f(t) - \sum_{1}^{n_{\mathsf{dy}}} \boldsymbol{F}_j \boldsymbol{r} f(t) = f(t) \left(\boldsymbol{K}^{-1} - \sum_{1}^{n_{\mathsf{dy}}} \boldsymbol{F}_j \right) \boldsymbol{r},$$

so that the corrected total displacements have the expression

$$\boldsymbol{x} \approx \sum_{1}^{n_{\mathsf{dy}}} \boldsymbol{\psi}_i q_i(t) + f(t) \left(\boldsymbol{K}^{-1} - \sum_{1}^{n_{\mathsf{dy}}} \boldsymbol{F}_i \right) \boldsymbol{r},$$

> *The constant term (a generalized displacement vector) following $f(t)$ can be computed with the information in our posses at the moment we begin the integration of the modal equations of motion.*

Effectiveness of Static Correction

Truncation
Num. Integration
Support Exc.

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Introduction
Modal partecipation factor
Dynamic magnification factor
Static Correction

In these circumstances, few modes with static correction give results comparable to the results obtained using much more modes in a straightforward modal displacement superposition analysis.

- An high number of modes is required to account for the spatial distribution of the loading but only a few lower modes are subjected to significant dynamic amplification.
- Refined stress analysis is required even if the dynamic response involves only a few lower modes.

Part II

Numerical Integration

Truncation
Num. Integration
Support Exc.

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Introduction
Constant Acceleration
Wilson's Theta Method

Introduction to Numerical Integration

When we reviewed the numerical integration methods, we said that some methods are unconditionally stable and others are conditionally stable, that is the response *blows-out* if the time step h is great with respect to the natural preriod of vibration, $h > \frac{T_n}{a}$, where a is a constant that depends on the numerical algorithm.
For *MDOF* systems, the relevant T is the one associated with the highest mode present in the structural model, so for moderately complex structures it becomes impossibile to use a conditionally stable algorithm.
In the following, two unconditionally stable algorithms will be analysed, i.e., the constant acceleration method, that we already know, and the new Wilson'sθ method.

Truncation
Num. Integration
Support Exc.

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Introduction
Constant Acceleration
Wilson's Theta Method

Constant Acceleration, preliminaries

- The initial conditions are known:

$$\boldsymbol{x}_0, \quad \dot{\boldsymbol{x}}_0, \quad \boldsymbol{p}_0, \quad \rightarrow \quad \ddot{\boldsymbol{x}}_0 = \boldsymbol{M}^{-1}(\boldsymbol{p}_0 - \boldsymbol{C}\,\dot{\boldsymbol{x}}_0 - \boldsymbol{K}\,\boldsymbol{x}_0).$$

- With a fixed time step h, compute the constant matrices

$$\boldsymbol{A} = 2\boldsymbol{C} + \frac{4}{h}\boldsymbol{M}, \qquad \boldsymbol{B} = 2\boldsymbol{M}, \qquad \boldsymbol{K}^+ = \frac{2}{h}\boldsymbol{C} + \frac{4}{h^2}\boldsymbol{M}.$$

Constant Acceleration, stepping

- Starting with $i = 0$, compute the effective force increment,

$$\Delta\hat{\boldsymbol{p}}_i = \boldsymbol{p}_{i+1} - \boldsymbol{p}_i + \boldsymbol{A}\dot{\boldsymbol{x}}_i + \boldsymbol{B}\ddot{\boldsymbol{x}}_i,$$

  the tangent stiffness $\boldsymbol{K}_i$ and the current incremental stiffness,

$$\hat{\boldsymbol{K}}_i = \boldsymbol{K}_i + \boldsymbol{K}^+.$$

- For linear systems, it is

$$\Delta\boldsymbol{x}_i = \hat{\boldsymbol{K}}_i^{-1}\Delta\hat{\boldsymbol{p}}_i,$$

  for a non linear system $\Delta\boldsymbol{x}_i$ is produced by the modified Newton-Raphson iteration procedure.
- The state vectors at the end of the step are

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \Delta\boldsymbol{x}_i, \qquad \dot{\boldsymbol{x}}_{i+1} = 2\frac{\Delta\boldsymbol{x}_i}{h} - \dot{\boldsymbol{x}}_i$$

Constant Acceleration, new step

Truncation
Num. Integration
Support Exc.

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Introduction
Constant Acceleration
Wilson's Theta Method

- Increment the step index, $i = i + 1$.
- Compute the accelerations using the equation of equilibrium,

$$\ddot{\boldsymbol{x}}_i = \boldsymbol{M}^{-1}(\boldsymbol{p}_i - \boldsymbol{C}\,\dot{\boldsymbol{x}}_i - \boldsymbol{K}\,\boldsymbol{x}_i).$$

- Repeat the substeps detailed in the previous slide.

Modified Newton-Raphson

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Num. Integration
Support Exc.

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Introduction
Constant Acceleration
Wilson's Theta Method

- Initialization

$$\boldsymbol{y}_0 = \boldsymbol{x}_i \qquad \boldsymbol{f}_{\mathsf{S},0} = \boldsymbol{f}_{\mathsf{S}}(\text{system state})$$
$$\Delta\boldsymbol{R}_1 = \Delta\hat{\boldsymbol{p}}_i \qquad \boldsymbol{K}_{\mathsf{T}} = \hat{\boldsymbol{K}}_i$$

- For $j = 1, 2, \ldots$

$$\boldsymbol{K}_{\mathsf{T}}\Delta\boldsymbol{y}_j = \Delta\boldsymbol{R}_j \qquad \to \Delta\boldsymbol{y}_j \text{ (test for convergence)}$$
$$\Delta\dot{\boldsymbol{y}}_j = \cdots$$
$$\boldsymbol{y}_j = \boldsymbol{y}_{j-1} + \Delta\boldsymbol{y}_j, \qquad \dot{\boldsymbol{y}}_j = \dot{\boldsymbol{y}}_{j-1} + \Delta\dot{\boldsymbol{y}}_j$$
$$\boldsymbol{f}_{\mathsf{S},j} = \boldsymbol{f}_{\mathsf{S}}(\text{updated system state})$$
$$\Delta\boldsymbol{f}_{\mathsf{S},j} = \boldsymbol{f}_{\mathsf{S},j} - \boldsymbol{f}_{\mathsf{S},j-1} - (\boldsymbol{K}_{\mathsf{T}} - \boldsymbol{K}_i)\Delta\boldsymbol{y}_j$$
$$\Delta\boldsymbol{R}_{j+1} = \Delta\boldsymbol{R}_j - \Delta\boldsymbol{f}_{\mathsf{S},j}$$

- Return the value $\Delta\boldsymbol{x}_i = \boldsymbol{y}_j - \boldsymbol{x}_i$

A suitable convergence test is

$$\frac{\Delta\boldsymbol{R}_j^T\,\Delta\boldsymbol{y}_j}{\Delta\hat{\boldsymbol{p}}_i^T\,\Delta\boldsymbol{x}_{i,j}} \le \text{tol}$$

Wilson's Theta Method

The linear acceleration method is significantly more accurate than the constant acceleration method, meaning that it is possible to use a longer time step to compute the response of a *SDOF* system within a required accuracy.
On the other hand, the method is not safely applicable to *MDOF* systems due to its numerical instability.
Professor Ed Wilson demonstrated that simple variations of the linear acceleration method can be made unconditionally stable and found the most accurate in this family of algorithms, collectively known as *Wilson's θ methods*.

Wilson's θ method

Wilson's idea is very simple: the results of the linear acceleration algorithm are *good enough* only in a fraction of the time step.
Wilson demonstrated that his idea was correct, too...
The procedure is really simple,

1. solve the incremental equation of equilibrium using the linear acceleration algorithm, with an extended time step

$$\hat{h} = \theta\, h, \qquad \theta \geq 1,$$

2. compute the extended acceleration increment $\hat{\Delta}\ddot{\boldsymbol{x}}$ at $\hat{t} = t_i + \hat{h}$,
3. scale the extended acceleration increment under the assumption of linear acceleration, $\Delta\ddot{\boldsymbol{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\boldsymbol{x}}$,
4. compute the velocity and displacements increment using the reduced value of the increment of acceleration.

Wilson's θ method description

Using the same symbols used for constant acceleration.
First of all, for given initial conditions $\boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}_0$, initialise the procedure computing the constants (matrices) used in the following procedure and the initial acceleration,

$$\begin{aligned}
\ddot{\boldsymbol{x}}_0 &= \boldsymbol{M}^{-1}(\boldsymbol{p}_0 - \boldsymbol{C}\,\dot{\boldsymbol{x}}_0 - \boldsymbol{K}\,\boldsymbol{x}_0),\\
\boldsymbol{A} &= 6\boldsymbol{M}/\hat{h} + 3\boldsymbol{C},\\
\boldsymbol{B} &= 3\boldsymbol{M} + \hat{h}\boldsymbol{C}/2,\\
\boldsymbol{K}^+ &= 3\boldsymbol{C}/\hat{h} + 6\boldsymbol{M}/\hat{h}^2.
\end{aligned}$$

Wilson's θ method description

Starting with $i = 0$,

1. update the tangent stiffness, $\boldsymbol{K}_i = \boldsymbol{K}(\boldsymbol{x},\dot{\boldsymbol{x}}_i)$ and the effective stiffness, $\hat{\boldsymbol{K}}_i = \boldsymbol{K}_i + \boldsymbol{K}^+$,
   compute $\hat{\Delta}\hat{\boldsymbol{p}}_i = \theta\Delta\boldsymbol{p}_i + \boldsymbol{A}\dot{\boldsymbol{x}}_i + \boldsymbol{B}\ddot{\boldsymbol{x}}_i$,
   with $\Delta\boldsymbol{p}_i = \boldsymbol{p}(t_i + h) - \boldsymbol{p}(t_i)$
2. solve $\hat{\boldsymbol{K}}_i\hat{\Delta}\boldsymbol{x} = \hat{\Delta}\hat{\boldsymbol{p}}_i$, compute
$$\hat{\Delta}\ddot{\boldsymbol{x}} = 6\frac{\hat{\Delta}\boldsymbol{x}}{\hat{h}^2} - 6\frac{\dot{\boldsymbol{x}}_i}{\hat{h}} - 3\ddot{\boldsymbol{x}}_i \rightarrow \Delta\ddot{\boldsymbol{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\boldsymbol{x}}$$
3. compute
$$\Delta\dot{\boldsymbol{x}} = (\ddot{\boldsymbol{x}}_i + \frac{1}{2}\Delta\ddot{\boldsymbol{x}})h$$
$$\Delta\boldsymbol{x} = \dot{\boldsymbol{x}}_i h + (\frac{1}{2}\ddot{\boldsymbol{x}}_i + \frac{1}{6}\Delta\ddot{\boldsymbol{x}})h^2$$
4. update state, $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \Delta\boldsymbol{x}$, $\dot{\boldsymbol{x}}_{i+1} = \dot{\boldsymbol{x}}_i + \Delta\dot{\boldsymbol{x}}$, $i = i + 1$, iterate restarting from 1.

A final remark

The Theta Method is unconditionally stable for $\theta > 1.37$ and it achieves the maximum accuracy for $\theta = 1.42$.

Part III

Multiple Support Excitation

Definitions

Consider the case of a structure where the supports are subjected to *assigned* displacements histories, $u_i = u_i(t)$.
To solve this problem, we start with augmenting the degrees of freedom with the support displacements.
We denote the superstructure *DOF* with $\boldsymbol{x}_T$, the support *DOF* with $\boldsymbol{x}_g$ and we have a global displacement vector $\boldsymbol{x}$,

$$\boldsymbol{x} = \begin{Bmatrix} \boldsymbol{x}_T \\ \boldsymbol{x}_g \end{Bmatrix}.$$

The Equation of Motion

Truncation
Num. Integration
Support Exc.

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Introduction

The Equation of Motion

An Example

Response Analysis

Response Analysis Example

Damping effects will be introduced at the end of our manipulations.
The equation of motion is

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}_g \\ \boldsymbol{M}_g^T & \boldsymbol{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}_T \\ \ddot{\boldsymbol{x}}_g \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}_g \\ \boldsymbol{K}_g^T & \boldsymbol{K}_{gg} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_T \\ \boldsymbol{x}_g \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p}_g \end{Bmatrix}$$

where $\boldsymbol{M}$ and $\boldsymbol{K}$ are the usual structural matrices, while $\boldsymbol{M}_g$ and $\boldsymbol{M}_{gg}$ are, in the common case of a lumped mass model, zero matrices.

Static Components

Truncation
Num. Integration
Support Exc.

Giacomo Boffi

Introduction

The Equation of Motion

An Example

Response Analysis

Response Analysis Example

We decompose the vector of displacements into two contributions, a static contribution and a dynamic contribution, attributing the *given* support displacements to the static contribution.

$$\begin{Bmatrix} \boldsymbol{x}_T \\ \boldsymbol{x}_g \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{x}_s \\ \boldsymbol{x}_g \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{Bmatrix}$$

where $\boldsymbol{x}$ is the usual *relative displacements* vector.

Determination of static components

Because the $\boldsymbol{x}_g$ are given, we can write two matricial equations that give us the static superstructure displacements and the forces we must apply to the supports,

$$\boldsymbol{K}\boldsymbol{x}_s + \boldsymbol{K}_g\boldsymbol{x}_g = \boldsymbol{0}$$
$$\boldsymbol{K}_g^T\boldsymbol{x}_s + \boldsymbol{K}_{gg}\boldsymbol{x}_g = \boldsymbol{p}_g$$

From the first equation we have

$$\boldsymbol{x}_s = -\boldsymbol{K}^{-1}\boldsymbol{K}_g\boldsymbol{x}_g$$

and from the second we have

$$\boldsymbol{p}_g = (\boldsymbol{K}_{gg} - \boldsymbol{K}_g^T\boldsymbol{K}^{-1}\boldsymbol{K}_g)\boldsymbol{x}_g$$

The support forces are zero when the structure is isostatic or the structure is subjected to a rigid motion.

Going back to the EOM

We need the first row of the two matrix equation of equilibrium,

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}_g \\ \boldsymbol{M}_g^T & \boldsymbol{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}_T \\ \ddot{\boldsymbol{x}}_g \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}_g \\ \boldsymbol{K}_g^T & \boldsymbol{K}_{gg} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_T \\ \boldsymbol{x}_g \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{p}_g \end{Bmatrix}$$

substituting $\boldsymbol{x}_T = \boldsymbol{x}_s + \boldsymbol{x}$ in the first row

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{M}\ddot{\boldsymbol{x}}_s + \boldsymbol{M}_g\ddot{\boldsymbol{x}}_g + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{K}\boldsymbol{x}_s + \boldsymbol{K}_g\boldsymbol{x}_g = \boldsymbol{0}$$

by the equation of static equilibrium, $\boldsymbol{K}\boldsymbol{x}_s + \boldsymbol{K}_g\boldsymbol{x}_g = \boldsymbol{0}$ we can simplify

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{M}\ddot{\boldsymbol{x}}_s+\boldsymbol{M}_g\ddot{\boldsymbol{x}}_g+\boldsymbol{K}\boldsymbol{x} = \boldsymbol{M}\ddot{\boldsymbol{x}}+(\boldsymbol{M}_g-\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{K}_g)\ddot{\boldsymbol{x}}_g+\boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}.$$

Influence matrix

The equation of motion is

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + (\boldsymbol{M}_g - \boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{K}_g)\ddot{\boldsymbol{x}}_g + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}.$$

We define the *influence matrix* $\boldsymbol{E}$ by

$$\boldsymbol{E} = -\boldsymbol{K}^{-1}\boldsymbol{K}_g,$$

and write, reintroducing the damping effects,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -(\boldsymbol{M}\boldsymbol{E} + \boldsymbol{M}_g)\ddot{\boldsymbol{x}}_g - (\boldsymbol{C}\boldsymbol{E} + \boldsymbol{C}_g)\dot{\boldsymbol{x}}_g$$

Simplification of the EOM

For a lumped mass model, $\boldsymbol{M}_g = \boldsymbol{0}$ and also the efficace forces due to damping are really small with respect to the inertial ones, and with this understanding we write

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{E}\ddot{\boldsymbol{x}}_g.$$

Significance of $\boldsymbol{E}$

$\boldsymbol{E}$ can be understood as a collection of vectors $\boldsymbol{e}_i$, $i = 1, \ldots, N_g$ (N_g being the number of *DOF* associated with the support motion),

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{N_g} \end{bmatrix}$$

where the individual $\boldsymbol{e}_i$ collects the displacements in all the *DOF* of the superstructure due to imposing a unit displacement to the support *DOF* number i.

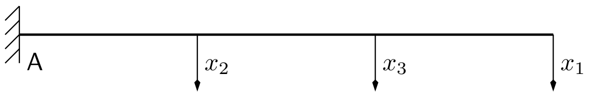
Significance of $\boldsymbol{E}$

This understanding means that the influence matrix can be computed column by column,

- in the general case by releasing one support *DOF*, applying a unit force to the released *DOF*, computing all the displacements and scaling the displacements so that the support displacement component is made equal to 1,
- or in the case of an isostatic component by examining the instantaneous motion of the 1 *DOF* rigid system that we obtain by releasing one constraint.

A First Example

We want to determine the influence matrix $\boldsymbol{E}$ for the structure in the figure above, subjected to an assigned motion in B.



First step, put in evidence another degree of freedom x_3 corresponding to the assigned vertical motion of the support in B and compute, using e.g. the PVD, the flexibility matrix:

$$\boldsymbol{F} = \frac{L^3}{3EJ}\begin{bmatrix} 54.0000 & 8.0000 & 28.0000 \\ 8.0000 & 2.0000 & 5.0000 \\ 28.0000 & 5.0000 & 16.0000 \end{bmatrix}.$$

Example, cont.

The stiffness matrix is found by inversion,

$$\boldsymbol{K} = \frac{3EJ}{13L^3}\begin{bmatrix} +7.0000 & +12.0000 & -16.0000 \\ +12.0000 & +80.0000 & -46.0000 \\ -16.0000 & -46.0000 & +44.0000 \end{bmatrix}.$$

We are interested in the partitions $\boldsymbol{K}_{xx}$ and $\boldsymbol{K}_{xg}$:

$$\boldsymbol{K}_{xx} = \frac{3EJ}{13L^3}\begin{bmatrix} +7.0000 & +12.0000.0000 \\ +12.0000 & +80.0000.0000 \end{bmatrix}, \ \boldsymbol{K}_{xg} = \frac{3EJ}{13L^3}\begin{bmatrix} -16 \\ -46 \end{bmatrix}.$$

The influence matrix is

$$\boldsymbol{E} = -\boldsymbol{K}_{xx}^{-1}\boldsymbol{K}_{xg} = \frac{1}{16}\begin{bmatrix} 28.0000 \\ 5.0000 \end{bmatrix},$$

please compare $\boldsymbol{E}$ with the last column of the flexibility matrix, $\boldsymbol{F}$.

Response Analysis

Consider the vector of support accelerations,

$$\ddot{\boldsymbol{x}}_g = \{\ddot{x}_{gl}, \qquad l = 1, \ldots, N_g\}$$

and the effective load vector

$$\boldsymbol{p}_{eff} = -\boldsymbol{M}\boldsymbol{E}\ddot{\boldsymbol{x}}_g = -\sum_{l=1}^{N_g} \boldsymbol{M}\boldsymbol{e}_l \ddot{x}_{gl}(t).$$

We can write the modal equation of motion for mode number n

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = -\sum_{l=1}^{N_g} \Gamma_{nl}\ddot{x}_{gl}(t)$$

where

$$\Gamma_{nl} = \frac{\boldsymbol{\psi}_n^T \boldsymbol{M}\boldsymbol{e}_l}{M_n^*}$$

Response Analysis, cont.

The solution $q_n(t)$ is hence, with the notation of last lesson,

$$q_n(t) = \sum_{l=1}^{N_g} \Gamma_{nl} D_{nl}(t),$$

D_{nl} being the response function for ζ_n and ω_n due to the ground excitation $\ddot{x}_{gl}$.

Response Analysis, cont.

The total displacements $\boldsymbol{x}_T$ are given by two contributions, $\boldsymbol{x}_T = \boldsymbol{x}_s + \boldsymbol{x}$, the expression of the contributions are

$$\boldsymbol{x}_s = \boldsymbol{E}\boldsymbol{x}_g(t) = \sum_{l=1}^{N_g} \boldsymbol{e}_l x_{gl}(t),$$

$$\boldsymbol{x} = \sum_{n=1}^{N} \sum_{l=1}^{N_g} \boldsymbol{\psi}_n \Gamma_{nl} D_{nl}(t),$$

and finally we have

$$\boldsymbol{x}_T = \sum_{l=1}^{N_g} \boldsymbol{e}_l x_{gl}(t) + \sum_{n=1}^{N} \sum_{l=1}^{N_g} \boldsymbol{\psi}_n \Gamma_{nl} D_{nl}(t).$$

Response in terms of Forces

For a computer program, the easiest way to compute the nodal forces is

a) compute, element by element, the nodal displacements by $\boldsymbol{x}_T$ and $\boldsymbol{x}_g$,

b) use the element stiffness matrix compute nodal forces,

c) assemble element nodal loads into global nodal loads.

That said, let's see the analytical development...

Forces, cont.

The forces on superstructure nodes due to deformations are

$$\boldsymbol{f}_s = \sum_{n=1}^{N} \sum_{l=1}^{N_g} \Gamma_{nl} \boldsymbol{K} \boldsymbol{\psi}_n D_{nl}(t)$$

$$\boldsymbol{f}_s = \sum_{n=1}^{N} \sum_{l=1}^{N_g} (\Gamma_{nl} \boldsymbol{M} \boldsymbol{\psi}_n)(\omega_n^2 D_{nl}(t)) = \sum \sum r_{nl} A_{nl}(t)$$

the forces on support

$$\boldsymbol{f}_{gs} = \boldsymbol{K}_g^T \boldsymbol{x}_T + \boldsymbol{K}_{gg} \boldsymbol{x}_g = \boldsymbol{K}_g^T \boldsymbol{x} + \boldsymbol{p}_g$$

or, using $\boldsymbol{x}_s = \boldsymbol{E} \boldsymbol{x}_g$

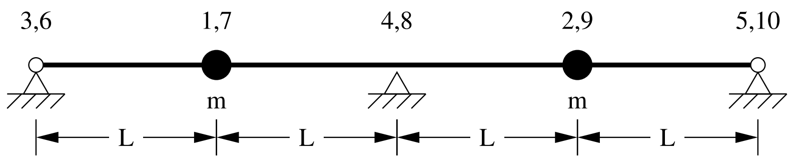
$$\boldsymbol{f}_{gs} = (\sum_{l=1}^{N_g} \boldsymbol{K}_g^T \boldsymbol{e}_l + \boldsymbol{K}_{gg,l}) x_{gl} + \sum_{n=1}^{N} \sum_{l=1}^{N_g} \Gamma_{nl} \boldsymbol{K}_g^T \boldsymbol{\psi}_n D_{nl}(t)$$

Forces

The structure response components must be computed considering the structure loaded by all the nodal forces,

$$\boldsymbol{f} = \begin{Bmatrix} \boldsymbol{f}_s \\ \boldsymbol{f}_{gs} \end{Bmatrix}.$$

Example



The dynamic *DOF* are x_1 and x_2, vertical displacements of the two equal masses, x_3, x_4, x_5 are the imposed vertical displacements of the supports, $x_6, \dots, x_{10}$ are the rotational degrees of freedom (removed by static condensation).

Example

The stiffness matrix for the 10x10 model is

$$\boldsymbol{K}_{10\times10} = \frac{EJ}{L^3}\begin{bmatrix} 12 & -12 & 0 & 0 & 0 & 6L & 6L & 0 & 0 & 0 \\ -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 & 0 \\ 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 \\ 0 & 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L \\ 0 & 0 & 0 & -12 & 12 & 0 & 0 & 0 & -6L & -6L \\ 6L & -6L & 0 & 0 & 0 & 4L^2 & 2L^2 & 0 & 0 & 0 \\ 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 & 0 \\ 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 \\ 0 & 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 \\ 0 & 0 & 0 & 6L & -6L & 0 & 0 & 0 & 2L^2 & 4L^2 \end{bmatrix}$$

Example, cont.

Truncation
Num. Integration
Support Exc.

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Introduction
The Equation of Motion
An Example
Response Analysis
Response Analysis Example

The first product of the static condensation procedure is the linear mapping between translational and rotational degrees of freedom, given by

$$\vec{\phi} = \frac{1}{56L}\begin{bmatrix} 71 & -90 & 24 & -6 & 1 \\ 26 & 12 & -48 & 12 & -2 \\ -7 & 42 & 0 & -42 & 7 \\ 2 & -12 & 48 & -12 & -26 \\ -1 & 6 & -24 & 90 & -71 \end{bmatrix}\vec{x}.$$

Example, cont.

Truncation
Num. Integration
Support Exc.

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Introduction
The Equation of Motion
An Example
Response Analysis
Response Analysis Example

Following static condensation and reordering rows and columns, the partitioned stiffness matrices are

$$\boldsymbol{K} = \frac{EJ}{28L^3}\begin{bmatrix} 276 & 108 \\ 108 & 276 \end{bmatrix},$$

$$\boldsymbol{K}_\mathrm{g} = \frac{EJ}{28L^3}\begin{bmatrix} -102 & -264 & -18 \\ -18 & -264 & -102 \end{bmatrix},$$

$$\boldsymbol{K}_\mathrm{gg} = \frac{EJ}{28L^3}\begin{bmatrix} 45 & 72 & 3 \\ 72 & 384 & 72 \\ 3 & 72 & 45 \end{bmatrix}.$$

The influence matrix is

$$\boldsymbol{E} = \boldsymbol{K}^{-1}\boldsymbol{K}_\mathrm{g} = \frac{1}{32}\begin{bmatrix} 13 & 22 & -3 \\ -3 & 22 & 13 \end{bmatrix}.$$

Example, cont.

The eigenvector matrix is

$$\mathbf{\Psi} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

the matrix of modal masses is

$$\boldsymbol{M}^\star = \mathbf{\Psi}^T \boldsymbol{M} \mathbf{\Psi} = m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

the matrix of the non normalized modal partecipation coefficients is

$$\boldsymbol{L} = \mathbf{\Psi}^T \boldsymbol{M} \boldsymbol{E} = m \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{16} & \frac{11}{8} & \frac{5}{16} \end{bmatrix}$$

and, finally, the matrix of modal partecipation factors,

$$\mathbf{\Gamma} = (\boldsymbol{M}^\star)^{-1} \boldsymbol{L} = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{5}{32} & \frac{11}{16} & \frac{5}{32} \end{bmatrix}$$

Example, cont.

Denoting with $D_{ij} = D_{ij}(t)$ the response function for mode i due to ground excitation $\ddot{x}_{\mathrm{g}j}$, the response can be written

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} \psi_{11}\left(-\frac{1}{4}D_{11}+\frac{1}{4}D_{13}\right)+\psi_{12}\left(\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22}\right) \\ \psi_{21}\left(-\frac{1}{4}D_{11}+\frac{1}{4}D_{13}\right)+\psi_{22}\left(\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22}\right) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}D_{13}+\frac{1}{4}D_{11}+\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22} \\ -\frac{1}{4}D_{11}+\frac{1}{4}D_{13}+\frac{5}{32}D_{21}+\frac{5}{32}D_{23}+\frac{11}{16}D_{22} \end{pmatrix}. \end{aligned}$$