

# Truncation Errors Numerical Integration Multiple Support Excitation

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# Part I

## How many eigenvectors?

Introduction

Modal participation factor

Dynamic magnification factor

Static Correction

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Num. Integration  
Support Exc.

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# How many eigenvectors?

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**Introduction**

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To understand how many eigenvectors we have to use in a modal analysis, we must consider two factors, the loading shape and the excitation frequency.

In the following, we'll consider *only* external loadings whose dependance on time and space can be separated, as in

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{r} f(t),$$

so that we can regard separately the two aspects of the problem.

It is worth noting that earthquake loadings are precisely of this type:

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{M}\tilde{\mathbf{r}}\ddot{u}_g$$

where the vector  $\tilde{\mathbf{r}}$  is used to choose the structural dof's that are *excited* by the ground motion component under consideration.

*$\tilde{\mathbf{r}}$  is an incidence vector, often simply a vector of ones and zeroes where the ones stay for the inertial forces that are excited by a specific component of the earthquake ground acceleration.*

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Multiplication of  $\mathbf{M}$  and division of  $\ddot{u}_g$  by  $g$ , acceleration of gravity, serves to show a dimensional load vector multiplied by an adimensional function.

$$\begin{aligned}\mathbf{p}(\mathbf{x}, t) &= g \mathbf{M}\tilde{\mathbf{r}} \frac{\ddot{u}_g(t)}{g} \\ &= \mathbf{r}^g f_g(t)\end{aligned}$$

# Modal participation factor

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Under the assumption of separability, we can write the  $i$ -th modal equation of motion as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2q_i = \begin{cases} \frac{\psi_i^T \mathbf{r}}{M_i} f(t) \\ \frac{g \psi_i^T \mathbf{M} \hat{\mathbf{r}}}{M_i} f_g(t) \end{cases} = \Gamma_i f(t)$$

with the modal mass  $M_i = \psi_i^T \mathbf{M} \psi_i$ .

It is apparent that the modal response amplitude depends

- ▶ on the characteristics of the time dependency of loading,  $f(t)$ ,
- ▶ on the so called *modal participation factor*  $\Gamma_i$ ,

$$\Gamma_i = \psi_i^T \mathbf{r} / M_i \quad \text{or} \quad \Gamma_i = g \psi_i^T \mathbf{M} \hat{\mathbf{r}} / M_i = \psi_i^T \mathbf{r}^g / M_i$$

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*Note that both the definitions of modal participation give it the dimensions of an acceleration.*



# Participation Factor Amplitudes

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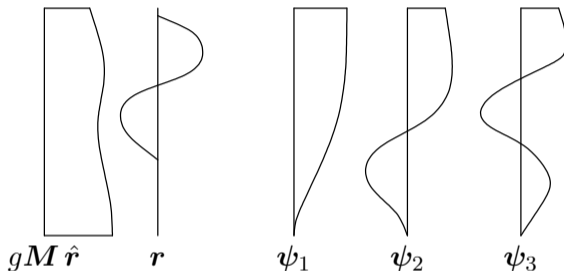
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For a given loading  $r$  the modal participation factor  $\Gamma_i$  is proportional to the work done by the modal displacement  $q_i \psi_i^T$  for the given loading  $r$ :

- ▶ if the mode shape and the loading shape are approximately equal (equal signs, component by component), the work (dot product) is maximized,
- ▶ if the mode shape is significantly different from the loading (different signs), there is some amount of cancellation and the value of the  $\Gamma$ 's will be reduced.

## Example

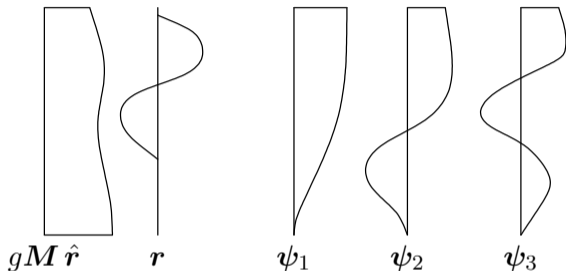


Consider a shear type building, its first 3 eigenvectors as sketched above, with mass distribution approximately constant over its height and its earthquake load shape vector

$$\hat{r} = \{1, 1, \dots, 1\}^T \quad \rightarrow \quad gM\hat{r} \approx mg\{1, 1, \dots, 1\}^T,$$

consider also an external, assigned load shape vector  $r$ .

## Example, cont.



For *EQ* loading,  $\Gamma_1$  is relatively large for the first mode, as loading components and displacements have the same sign, with respect to other  $\Gamma_i$ 's, where the oscillating nature of the higher eigenvectors will lead to increasing cancellation.

On the other hand, consider the external loading, whose peculiar shape is similar to the 3rd mode.  $\Gamma_3$  will be more relevant than  $\Gamma_i$ 's for lower or higher modes.

We define the modal load contribution as

$$\mathbf{r}_i = \mathbf{M} \boldsymbol{\psi}_i a_i$$

and express the load vector as a linear combination of the modal contributions

$$\mathbf{r} = \sum_i \mathbf{M} \boldsymbol{\psi}_i a_i = \sum_i \mathbf{r}_i.$$

Premultiplying by  $\boldsymbol{\psi}_j^T$  the above equation we have a relation that enables the computation of the coefficients  $a_i$ :

$$\boldsymbol{\psi}_j^T \mathbf{r} = \boldsymbol{\psi}_j^T \sum_i \mathbf{M} \boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i \quad \rightarrow \quad a_i = \boldsymbol{\psi}_i^T \mathbf{r} / M_i$$

1. A modal load component works *only* for the displacements associated with the corresponding eigenvector,

$$\boldsymbol{\psi}_j^T \mathbf{r}_i = a_i \boldsymbol{\psi}_j^T \mathbf{M} \boldsymbol{\psi}_i = \delta_{ij} a_i M_i.$$

2. Comparing  $\boldsymbol{\psi}_j^T \mathbf{r} = \boldsymbol{\psi}_j^T \sum_i \mathbf{M} \boldsymbol{\psi}_i a_i = \delta_{ij} M_i a_i$  with the definition of  $\Gamma_i = \boldsymbol{\psi}_i^T \mathbf{r} / M_i$ , we conclude that  $a_i \equiv \Gamma_i$  and finally write

$$\mathbf{r}_i = \Gamma_i \mathbf{M} \boldsymbol{\psi}_i.$$

3. The modal load contributions can be collected in a matrix: with  $\mathbf{\Gamma} = \text{diag} \Gamma_i$  we have

$$\mathbf{R} = \mathbf{M} \mathbf{\Psi} \mathbf{\Gamma}.$$

# Equivalent Static Forces

For mode  $i$ , the equation of motion is

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2q_i = \Gamma_i f(t)$$

with  $q_i = \Gamma_i D_i$ , we can write, to single out the dependency on the modulating function,

$$\ddot{D}_i + 2\zeta_i\omega_i\dot{D}_i + \omega_i^2D_i = f(t)$$

The modal contribution to displacement is

$$\mathbf{x}_i = \Gamma_i \boldsymbol{\psi}_i D_i(t)$$

and the modal contribution to elastic forces  $\mathbf{f}_i = \mathbf{K} \mathbf{x}_i$  can be written (being  $\mathbf{K}\boldsymbol{\psi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\psi}_i$ ) as

$$\mathbf{f}_i = \mathbf{K} \mathbf{x}_i = \Gamma_i \mathbf{K} \boldsymbol{\psi}_i D_i = \omega_i^2 (\Gamma_i \mathbf{M} \boldsymbol{\psi}_i) D_i = \mathbf{r}_i \omega_i^2 D_i$$

# Equivalent Static Forces

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The response can be determined by superposition of the effects of these pseudo-static forces  $\mathbf{f}_i = \mathbf{r}_i \omega_i^2 D_i(t)$ .

If a required response quantity (be it a nodal displacement, a bending moment in a beam, the total shear force in a building storey, etc etc) is indicated by  $s(t)$ , we can compute with a *static calculation* (usually using the *FEM* model underlying the dynamic analysis) the modal static contribution  $s_i^{\text{st}}$  and write

$$s(t) = \sum s_i^{\text{st}}(\omega_i^2 D_i(t)) = \sum s_i(t),$$

where the modal contribution to response  $s_i(t)$  is given by

1. static analysis using  $\mathbf{r}_i$  as the static load vector,
2. dynamic amplification using the factor  $\omega_i^2 D_i(t)$ .



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1. static analysis using  $\mathbf{r}_i$  as the static load vector,
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This formulation is particularly apt to our discussion of different contributions to response components.

Say that the static response due to  $\mathbf{r}$  is denoted by  $s^{\text{st}}$ , then  $s_i(t)$ , the modal contribution to response  $s(t)$ , can be written

$$s_i(t) = s_i^{\text{st}} \omega_i^2 D_i(t) = s^{\text{st}} \frac{s_i^{\text{st}}}{s^{\text{st}}} \omega_i^2 D_i(t) = \bar{s}_i s^{\text{st}} \omega_i^2 D_i(t).$$

We have introduced  $\bar{s}_i = \frac{s_i^{\text{st}}}{s^{\text{st}}}$ , the *modal contribution factor*, the ratio of the modal static contribution to the total static response. The  $\bar{s}_i$  are dimensionless, are independent from the eigenvector scaling procedure and their sum is unity,  $\sum \bar{s}_i = 1$ .

# Maximum Response

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Denote by  $D_{i0}$  the maximum absolute value (or *peak*) of the pseudo displacement time history,

$$D_{i0} = \max_t \{|D_i(t)|\}.$$

It will be

$$s_{i0} = \bar{s}_i s^{\text{st}} \omega_i^2 D_{i0}.$$

The dynamic response factor for mode  $i$ ,  $\mathfrak{R}_{di}$  is defined by

$$\mathfrak{R}_{di} = \frac{D_{i0}}{D_{i0}^{\text{st}}}$$

where  $D_{i0}^{\text{st}}$  is the peak value of the static pseudo displacement

$$D_i^{\text{st}} = \frac{f(t)}{\omega_i^2}, \quad \rightarrow \quad D_{i0}^{\text{st}} = \frac{f_0}{\omega_i^2}.$$

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With  $f_0 = \max\{|f(t)|\}$  the peak pseudo displacement is

$$D_{i0} = \mathfrak{R}_{di} f_0 / \omega_i^2$$

and the peak of the modal contribution is

$$s_{i0}(t) = \bar{s}_i s^{\text{st}} \omega_i^2 D_{i0}(t) = f_0 s^{\text{st}} \bar{s}_i \mathfrak{R}_{di}$$

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With  $f_0 = \max\{|f(t)|\}$  the peak pseudo displacement is

$$D_{i0} = \Re_{di} f_0 / \omega_i^2$$

and the peak of the modal contribution is

$$s_{i0}(t) = \bar{s}_i s^{\text{st}} \omega_i^2 D_{i0}(t) = f_0 s^{\text{st}} \bar{s}_i \Re_{di}$$

The first two terms are independent of the mode, the last are independent from each other and their product is the factor that influences the modal contributions.

Note that this product has the sign of  $\bar{s}_i$ , as the dynamic response factor is always positive.

## *MCF's* example

The following table (from Chopra, 2nd ed.) displays the  $\bar{s}_i$  and their partial sums for a shear-type, 5 floors building where all the storey masses are equal and all the storey stiffnesses are equal too.

The response quantities chosen are  $\bar{x}_{5n}$ , the *MCF's* to the top displacement and  $\bar{V}_n$ , the *MCF's* to the base shear, for two different load shapes.

## MCF's example

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The response quantities chosen are  $\bar{x}_{5n}$ , the MCF's to the top displacement and  $\bar{V}_n$ , the MCF's to the base shear, for two different load shapes.

$n$ or $J$	$\mathbf{r} = \{0, 0, 0, 0, 1\}^T$				$\mathbf{r} = \{0, 0, 0, -1, 2\}^T$			
	Top Displacement		Base Shear		Top Displacement		Base Shear	
	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$	$\bar{x}_{5n}$	$\sum^J \bar{x}_{5i}$	$\bar{V}_n$	$\sum^J \bar{V}_i$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	-0.362	0.890	0.123	0.915	-0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.043	1.172
4	0.008	0.998	-0.063	0.985	0.024	0.994	-0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

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Note that

1. for any given  $\mathbf{r}$ , the base shear is more influenced by higher modes, and
2. for any given response quantity, the second, *skewed*  $\mathbf{r}$  gives greater modal contributions for higher modes.



# Dynamic Response Ratios

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Dynamic Response Ratios are the same that we have seen for *SDOF* systems.  
Next page, for an undamped system, harmonically excited,

- ▶ solid line, the ratio of the modal elastic force  $F_{S,i} = K_i q_i \sin \omega t$  to the harmonic applied modal force,  $P_i \sin \omega t$ , plotted against the frequency ratio  $\beta = \omega/\omega_i$ .

For  $\beta = 0$  the ratio is 1, the applied load is fully balanced by the elastic resistance.

For fixed excitation frequency,  $\beta \rightarrow 0$  for high modal frequencies.

- ▶ dashed line, the ratio of the modal inertial force,  $F_{I,i} = -\beta^2 F_{S,i}$  to the load.

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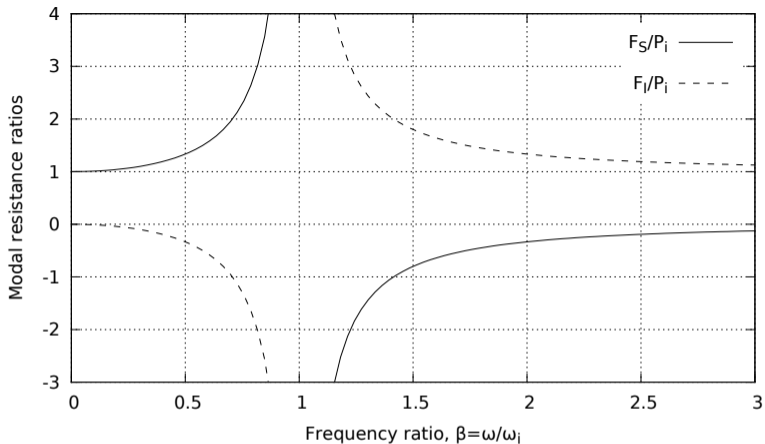
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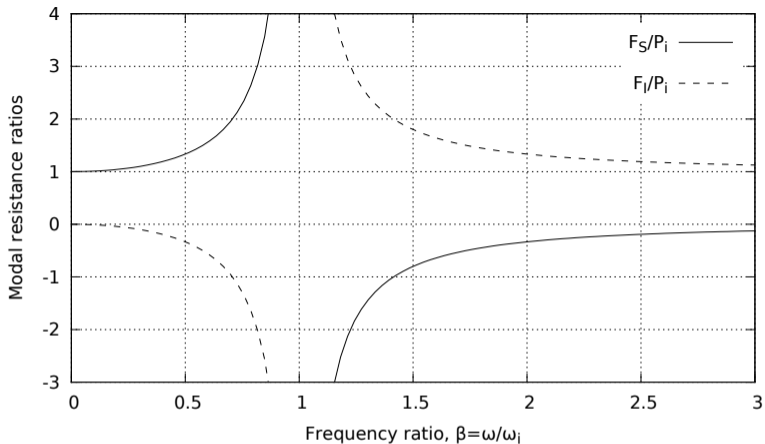
- ▶ dashed line, the ratio of the modal inertial force,  $F_{I,i} = -\beta^2 F_{S,i}$  to the load.

Note that for steady-state motion the sum of the elastic and inertial force ratios is constant and equal to 1, as in

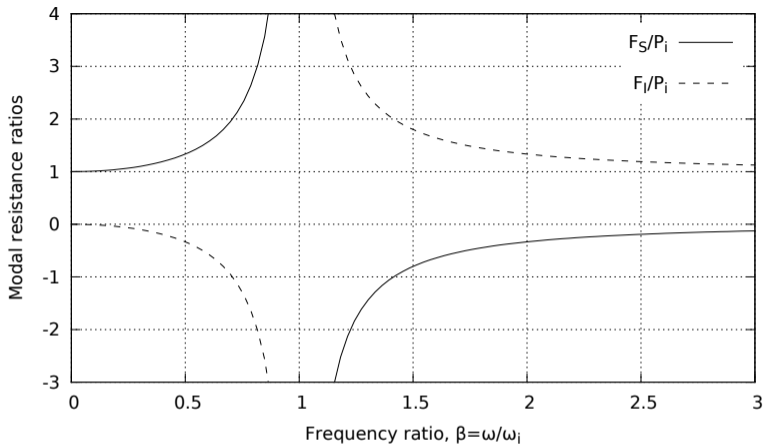
$$(F_{S,i} + F_{I,i}) \sin \omega t = P_i \sin \omega t.$$



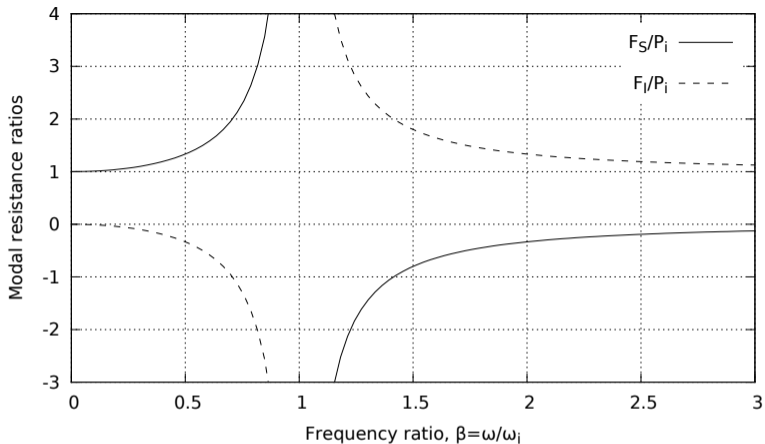
- For a fixed excitation frequency and high modal frequencies the frequency ratio  $\beta \rightarrow 0$ .



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- ▶ Hence, for higher modes the response is *pseudo-static*.



- ▶ For a fixed excitation frequency and high modal frequencies the frequency ratio  $\beta \rightarrow 0$ .
- ▶ For  $\beta \rightarrow 0$  the response is *quasi-static*.
- ▶ Hence, for higher modes the response is *pseudo-static*.
- ▶ On the contrary, for excitation frequencies high enough the lower modes respond with purely inertial forces.

The preceding discussion indicates that higher modes contributions to the response could be approximated with the static response, leading to a *Static Correction* of the dynamic response.

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For a system where  $q_i(t) \approx \frac{p_i(t)}{K_i}$  for  $i > n_{\text{dy}}$ ,

$n_{\text{dy}}$  being the number of dynamically responding modes, we can write

$$\mathbf{x}(t) \approx \mathbf{x}_{\text{dy}}(t) + \mathbf{x}_{\text{st}}(t) = \sum_1^{n_{\text{dy}}} \boldsymbol{\psi}_i q_i(t) + \sum_{n_{\text{dy}}+1}^N \boldsymbol{\psi}_i \frac{p_i(t)}{K_i}$$

where the response for each of the first  $n_{\text{dy}}$  modes can be computed as usual.



# Static Modal Components

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The static modal displacement component  $x_j, j > n_{dy}$  can be written

$$x_j(t) = \psi_j q_j(t) \approx \frac{\psi_j \psi_j^T}{K_j} \mathbf{p}(t) = \mathbf{F}_j \mathbf{p}(t)$$

The *modal flexibility matrix* is defined by

$$\mathbf{F}_j = \frac{\psi_j \psi_j^T}{K_j}$$

and is used to compute the  $j$ -th mode static deflections due to the applied load vector.

The total displacements, the dynamic contributions and the static correction, for  $\mathbf{p}(t) = \mathbf{r} f(t)$ , are then

$$\mathbf{x} \approx \sum_1^{n_{dy}} \psi_j q_j(t) + f(t) \sum_{n_{dy}+1}^N \mathbf{F}_j \mathbf{r}.$$

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Our last formula for static correction is

$$\mathbf{x} \approx \sum_1^{n_{\mathbf{dy}}} \psi_j q_j(t) + f(t) \sum_{n_{\mathbf{dy}}+1}^N \mathbf{F}_j \mathbf{r}.$$

To use the above formula all mode shapes, all modal stiffnesses and all modal flexibility matrices must be computed, undermining the efficiency of the procedure.

# Alternative Formulation

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This problem can be obviated computing the total static displacements,  $\mathbf{x}_{st}^{total} = \mathbf{K}^{-1}\mathbf{p}(t)$ , and subtracting the static displacements due to the first  $n_{dy}$  modes...

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$$\sum_{n_{dy}}^N \mathbf{F}_j \mathbf{r} f(t) = \mathbf{K}^{-1} \mathbf{r} f(t) - \sum_1^{n_{dy}} \mathbf{F}_j \mathbf{r} f(t) = f(t) \left( \mathbf{K}^{-1} - \sum_1^{n_{dy}} \mathbf{F}_j \right) \mathbf{r},$$

so that the corrected total displacements have the expression

$$\mathbf{x} \approx \sum_1^{n_{dy}} \boldsymbol{\psi}_i q_i(t) + f(t) \left( \mathbf{K}^{-1} - \sum_1^{n_{dy}} \mathbf{F}_i \right) \mathbf{r},$$

*The constant term (a generalized displacement vector) following  $f(t)$  can be computed with the information in our posses at the moment we begin the integration of the modal equations of motion.*

# Effectiveness of Static Correction

In these circumstances, few modes with static correction give results comparable to the results obtained using much more modes in a straightforward modal displacement superposition analysis.

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- ▶ An high number of modes is required to account for the spatial distribution of the loading but only a few lower modes are subjected to significant dynamic amplification.
- ▶ Refined stress analysis is required even if the dynamic response involves only a few lower modes.

# Part II

## Numerical Integration

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Constant Acceleration

Wilson's Theta Method

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**Giacomo Boffi**

Introduction

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When we reviewed the numerical integration methods, we said that some methods are unconditionally stable and others are conditionally stable, that is the response *blows-out* if the time step  $h$  is great with respect to the natural period of vibration,  $h > \frac{T_n}{a}$ , where  $a$  is a constant that depends on the numerical algorithm.

For *MDOF* systems, the relevant  $T$  is the one associated with the highest mode present in the structural model, so for moderately complex structures it becomes impossible to use a conditionally stable algorithm.

In the following, two unconditionally stable algorithms will be analysed, i.e., the constant acceleration method, that we already know, and the new Wilson's  $\theta$  method.



- ▶ The initial conditions are known:

$$\mathbf{x}_0, \quad \dot{\mathbf{x}}_0, \quad \mathbf{p}_0, \quad \rightarrow \quad \ddot{\mathbf{x}}_0 = \mathbf{M}^{-1}(\mathbf{p}_0 - \mathbf{C} \dot{\mathbf{x}}_0 - \mathbf{K} \mathbf{x}_0).$$

- ▶ With a fixed time step  $h$ , compute the constant matrices

$$\mathbf{A} = 2\mathbf{C} + \frac{4}{h}\mathbf{M}, \quad \mathbf{B} = 2\mathbf{M}, \quad \mathbf{K}^+ = \frac{2}{h}\mathbf{C} + \frac{4}{h^2}\mathbf{M}.$$

## Constant Acceleration, stepping

- ▶ Starting with  $i = 0$ , compute the effective force increment,

$$\Delta \hat{\mathbf{p}}_i = \mathbf{p}_{i+1} - \mathbf{p}_i + \mathbf{A} \dot{\mathbf{x}}_i + \mathbf{B} \ddot{\mathbf{x}}_i,$$

the tangent stiffness  $\mathbf{K}_i$  and the current incremental stiffness,

$$\hat{\mathbf{K}}_i = \mathbf{K}_i + \mathbf{K}^+.$$

- ▶ For linear systems, it is

$$\Delta \mathbf{x}_i = \hat{\mathbf{K}}_i^{-1} \Delta \hat{\mathbf{p}}_i,$$

for a non linear system  $\Delta \mathbf{x}_i$  is produced by the modified Newton-Raphson iteration procedure.

- ▶ The state vectors at the end of the step are

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta \mathbf{x}_i, \quad \dot{\mathbf{x}}_{i+1} = 2 \frac{\Delta \mathbf{x}_i}{h} + \dot{\mathbf{x}}_i$$

- ▶ Increment the step index,  $i = i + 1$ .
- ▶ Compute the accelerations using the equation of equilibrium,

$$\ddot{\mathbf{x}}_i = \mathbf{M}^{-1}(\mathbf{p}_i - \mathbf{C} \dot{\mathbf{x}}_i - \mathbf{K} \mathbf{x}_i).$$

- ▶ Repeat the substeps detailed in the previous slide.

# Modified Newton-Raphson

► Initialization

$$\mathbf{y}_0 = \mathbf{x}_i$$

$$\mathbf{f}_{S,0} = \mathbf{f}_S(\text{system state})$$

$$\Delta \mathbf{R}_1 = \Delta \hat{\mathbf{p}}_i$$

$$\mathbf{K}_T = \hat{\mathbf{K}}_i$$

► For  $j = 1, 2, \dots$

$$\mathbf{K}_T \Delta \mathbf{y}_j = \Delta \mathbf{R}_j$$

→  $\Delta \mathbf{y}_j$  (test for convergence)

$$\mathbf{y}_j = \mathbf{y}_{j-1} + \Delta \mathbf{y}_j,$$

$$\Delta \dot{\mathbf{y}}_j = \dots$$

$$\dot{\mathbf{y}}_j = \dot{\mathbf{y}}_{j-1} + \Delta \dot{\mathbf{y}}_j$$

$$\mathbf{f}_{S,j} = \mathbf{f}_S(\text{updated system state})$$

$$\Delta \mathbf{f}_{S,j} = \mathbf{f}_{S,j} - \mathbf{f}_{S,j-1} - (\mathbf{K}_T - \mathbf{K}_i) \Delta \mathbf{y}_j$$

$$\Delta \mathbf{R}_{j+1} = \Delta \mathbf{R}_j - \Delta \mathbf{f}_{S,j}$$

► Return the value  $\Delta \mathbf{x}_i = \mathbf{y}_j - \mathbf{x}_i$

A suitable convergence test is

$$\frac{\Delta \mathbf{R}_j^T \Delta \mathbf{y}_j}{\Delta \hat{\mathbf{p}}_i^T \Delta \mathbf{x}_{i,j}} \leq \text{tol}$$

The linear acceleration method is significantly more accurate than the constant acceleration method, meaning that it is possible to use a longer time step to compute the response of a *SDOF* system within a required accuracy.

On the other hand, the method is not safely applicable to *MDOF* systems due to its numerical instability.

# Wilson's Theta Method

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The linear acceleration method is significantly more accurate than the constant acceleration method, meaning that it is possible to use a longer time step to compute the response of a *SDOF* system within a required accuracy.

On the other hand, the method is not safely applicable to *MDOF* systems due to its numerical instability.

Professor Ed Wilson demonstrated that simple variations of the linear acceleration method can be made unconditionally stable and found the most accurate in this family of algorithms, collectively known as *Wilson's  $\theta$  methods*.

# Wilson's $\theta$ method

Wilson's idea is very simple: the results of the linear acceleration algorithm are *good enough* only in a fraction of the time step.

Wilson demonstrated that his idea was correct, too...

The procedure is really simple,

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1. solve the incremental equation of equilibrium using the linear acceleration algorithm, with an extended time step

$$\hat{h} = \theta h, \quad \theta \geq 1,$$



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2. compute the extended acceleration increment  $\hat{\Delta}\ddot{\mathbf{x}}$  at  $\hat{t} = t_i + \hat{h}$ ,

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3. scale the extended acceleration increment under the assumption of linear acceleration,  $\Delta\ddot{\mathbf{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\mathbf{x}}$ ,

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3. scale the extended acceleration increment under the assumption of linear acceleration,  $\Delta\ddot{\mathbf{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\mathbf{x}}$ ,
4. compute the velocity and displacements increment using the reduced value of the increment of acceleration.

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# Wilson's $\theta$ method description

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Using the same symbols used for constant acceleration.

First of all, for given initial conditions  $\mathbf{x}_0$  and  $\dot{\mathbf{x}}_0$ , initialise the procedure computing the constants (matrices) used in the following procedure and the initial acceleration,

$$\ddot{\mathbf{x}}_0 = \mathbf{M}^{-1}(\mathbf{p}_0 - \mathbf{C} \dot{\mathbf{x}}_0 - \mathbf{K} \mathbf{x}_0),$$

$$\mathbf{A} = 6\mathbf{M}/\hat{h} + 3\mathbf{C},$$

$$\mathbf{B} = 3\mathbf{M} + \hat{h}\mathbf{C}/2,$$

$$\mathbf{K}^+ = 3\mathbf{C}/\hat{h} + 6\mathbf{M}/\hat{h}^2.$$

# Wilson's $\theta$ method description

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Starting with  $i = 0$ ,

1. update the tangent stiffness,  $\mathbf{K}_i = \mathbf{K}(\mathbf{x}, \dot{\mathbf{x}}_i)$  and the effective stiffness,  $\hat{\mathbf{K}}_i = \mathbf{K}_i + \mathbf{K}^+$ ,  
compute  $\hat{\Delta \mathbf{p}}_i = \theta \Delta \mathbf{p}_i + \mathbf{A} \dot{\mathbf{x}}_i + \mathbf{B} \ddot{\mathbf{x}}_i$ ,  
with  $\Delta \mathbf{p}_i = \mathbf{p}(t_i + h) - \mathbf{p}(t_i)$

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compute  $\hat{\Delta}\hat{\mathbf{p}}_i = \theta\Delta\mathbf{p}_i + \mathbf{A}\dot{\mathbf{x}}_i + \mathbf{B}\ddot{\mathbf{x}}_i$ ,  
with  $\Delta\mathbf{p}_i = \mathbf{p}(t_i + h) - \mathbf{p}(t_i)$
2. solve  $\hat{\mathbf{K}}_i\hat{\Delta}\mathbf{x} = \hat{\Delta}\hat{\mathbf{p}}_i$ , compute

$$\hat{\Delta}\ddot{\mathbf{x}} = 6\frac{\hat{\Delta}\mathbf{x}}{\hat{h}^2} - 6\frac{\dot{\mathbf{x}}_i}{\hat{h}} - 3\ddot{\mathbf{x}}_i \rightarrow \Delta\ddot{\mathbf{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\mathbf{x}}$$

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2. solve  $\hat{\mathbf{K}}_i\hat{\Delta}\mathbf{x} = \hat{\Delta}\hat{\mathbf{p}}_i$ , compute

$$\hat{\Delta}\ddot{\mathbf{x}} = 6\frac{\hat{\Delta}\mathbf{x}}{h^2} - 6\frac{\dot{\mathbf{x}}_i}{h} - 3\ddot{\mathbf{x}}_i \rightarrow \Delta\ddot{\mathbf{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\mathbf{x}}$$

3. compute

$$\Delta\dot{\mathbf{x}} = (\ddot{\mathbf{x}}_i + \frac{1}{2}\Delta\ddot{\mathbf{x}})h$$

$$\Delta\mathbf{x} = \dot{\mathbf{x}}_i h + (\frac{1}{2}\ddot{\mathbf{x}}_i + \frac{1}{6}\Delta\ddot{\mathbf{x}})h^2$$

# Wilson's $\theta$ method description

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2. solve  $\hat{\mathbf{K}}_i\hat{\Delta}\mathbf{x} = \hat{\Delta}\hat{\mathbf{p}}_i$ , compute

$$\hat{\Delta}\ddot{\mathbf{x}} = 6\frac{\hat{\Delta}\mathbf{x}}{\hat{h}^2} - 6\frac{\dot{\mathbf{x}}_i}{\hat{h}} - 3\ddot{\mathbf{x}}_i \rightarrow \Delta\ddot{\mathbf{x}} = \frac{1}{\theta}\hat{\Delta}\ddot{\mathbf{x}}$$

3. compute

$$\Delta\dot{\mathbf{x}} = (\ddot{\mathbf{x}}_i + \frac{1}{2}\Delta\ddot{\mathbf{x}})h$$

$$\Delta\mathbf{x} = \dot{\mathbf{x}}_i h + (\frac{1}{2}\ddot{\mathbf{x}}_i + \frac{1}{6}\Delta\ddot{\mathbf{x}})h^2$$

4. update state,  $\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta\mathbf{x}$ ,  $\dot{\mathbf{x}}_{i+1} = \dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}$ ,  $i = i + 1$ ,  
iterate restarting from 1.



## A final remark

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The Theta Method is unconditionally stable for  $\theta > 1.37$  and it achieves the maximum accuracy for  $\theta = 1.42$ .

# Part III

## Multiple Support Excitation

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Consider the case of a structure where the supports are subjected to *assigned* displacements histories,  $u_i = u_i(t)$ .

To solve this problem, we start with augmenting the degrees of freedom with the support displacements.

We denote the superstructure *DOF* with  $\mathbf{x}_T$ , the support *DOF* with  $\mathbf{x}_g$  and we have a global displacement vector  $\mathbf{x}$ ,

$$\mathbf{x} = \begin{Bmatrix} \mathbf{x}_T \\ \mathbf{x}_g \end{Bmatrix}.$$

Damping effects will be introduced at the end of our manipulations.  
The equation of motion is

$$\begin{bmatrix} \mathbf{M} & \mathbf{M}_g \\ \mathbf{M}_g^T & \mathbf{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_T \\ \ddot{\mathbf{x}}_g \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{K}_g \\ \mathbf{K}_g^T & \mathbf{K}_{gg} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_T \\ \mathbf{x}_g \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p}_g \end{Bmatrix}$$

where  $\mathbf{M}$  and  $\mathbf{K}$  are the usual structural matrices, while  $\mathbf{M}_g$  and  $\mathbf{M}_{gg}$  are, in the common case of a lumped mass model, zero matrices.

We decompose the vector of displacements into two contributions, a static contribution and a dynamic contribution, attributing the *given* support displacements to the static contribution.

$$\begin{Bmatrix} \mathbf{x}_T \\ \mathbf{x}_g \end{Bmatrix} = \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{x}_g \end{Bmatrix} + \begin{Bmatrix} \mathbf{x} \\ \mathbf{0} \end{Bmatrix}$$

where  $\mathbf{x}$  is the usual *relative displacements* vector.

# Determination of static components

Because the  $\mathbf{x}_g$  are given, we can write two matricial equations that give us the static superstructure displacements and the forces we must apply to the supports,

$$\begin{aligned}\mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g &= \mathbf{0} \\ \mathbf{K}_g^T\mathbf{x}_s + \mathbf{K}_{gg}\mathbf{x}_g &= \mathbf{p}_g\end{aligned}$$

From the first equation we have

$$\mathbf{x}_s = -\mathbf{K}^{-1}\mathbf{K}_g\mathbf{x}_g$$

and from the second we have

$$\mathbf{p}_g = (\mathbf{K}_{gg} - \mathbf{K}_g^T\mathbf{K}^{-1}\mathbf{K}_g)\mathbf{x}_g$$

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$$\mathbf{x}_s = -\mathbf{K}^{-1}\mathbf{K}_g\mathbf{x}_g$$

and from the second we have

$$\mathbf{p}_g = (\mathbf{K}_{gg} - \mathbf{K}_g^T\mathbf{K}^{-1}\mathbf{K}_g)\mathbf{x}_g$$

The support forces are zero when the structure is isostatic or the structure is subjected to a rigid motion.

# Going back to the EOM

We need the first row of the two matrix equation of equilibrium,

$$\begin{bmatrix} \mathbf{M} & \mathbf{M}_g \\ \mathbf{M}_g^T & \mathbf{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_T \\ \ddot{\mathbf{x}}_g \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{K}_g \\ \mathbf{K}_g^T & \mathbf{K}_{gg} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_T \\ \mathbf{x}_g \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p}_g \end{Bmatrix}$$

substituting  $\mathbf{x}_T = \mathbf{x}_s + \mathbf{x}$  in the first row

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{M}\ddot{\mathbf{x}}_s + \mathbf{M}_g\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} + \mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g = \mathbf{0}$$

by the equation of static equilibrium,  $\mathbf{K}\mathbf{x}_s + \mathbf{K}_g\mathbf{x}_g = \mathbf{0}$  we can simplify

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{M}\ddot{\mathbf{x}}_s + \mathbf{M}_g\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} = \mathbf{M}\ddot{\mathbf{x}} + (\mathbf{M}_g - \mathbf{M}\mathbf{K}^{-1}\mathbf{K}_g)\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} = \mathbf{0}.$$



The equation of motion is

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{M}_g - \mathbf{M}\mathbf{K}^{-1}\mathbf{K}_g)\ddot{\mathbf{x}}_g + \mathbf{K}\mathbf{x} = \mathbf{0}.$$

We define the *influence matrix*  $\mathbf{E}$  by

$$\mathbf{E} = -\mathbf{K}^{-1}\mathbf{K}_g,$$

and write, reintroducing the damping effects,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -(\mathbf{M}\mathbf{E} + \mathbf{M}_g)\ddot{\mathbf{x}}_g - (\mathbf{C}\mathbf{E} + \mathbf{C}_g)\dot{\mathbf{x}}_g$$

# Simplification of the EOM

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For a lumped mass model,  $M_g = \mathbf{0}$  and also the efficace forces due to damping are really small with respect to the inertial ones, and with this understanding we write

$$M\ddot{x} + C\dot{x} + Kx = -ME\ddot{x}_g.$$

# Significance of $\mathbf{E}$

$\mathbf{E}$  can be understood as a collection of vectors  $\mathbf{e}_i$ ,  $i = 1, \dots, N_g$  ( $N_g$  being the number of *DOF* associated with the support motion),

$$\mathbf{E} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_{N_g}]$$

where the individual  $\mathbf{e}_i$  collects the displacements in all the *DOF* of the superstructure due to imposing a unit displacement to the support *DOF* number  $i$ .

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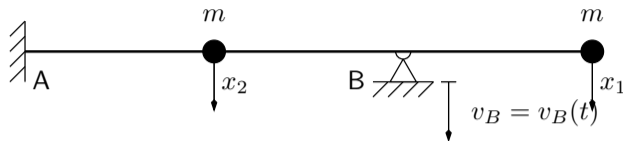
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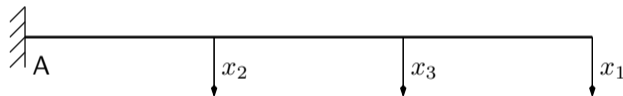
This understanding means that the influence matrix can be computed column by column,

- ▶ in the general case by releasing one support *DOF*, applying a unit force to the released *DOF*, computing all the displacements and scaling the displacements so that the support displacement component is made equal to 1,
- ▶ or in the case of an isostatic component by examining the instantaneous motion of the 1 *DOF* rigid system that we obtain by releasing one constraint.

# A First Example



We want to determine the influence matrix  $\mathbf{E}$  for the structure in the figure above, subjected to an assigned motion in B.



First step, put in evidence another degree of freedom  $x_3$  corresponding to the assigned vertical motion of the support in B and compute, using e.g. the PVD, the flexibility matrix:

$$\mathbf{F} = \frac{L^3}{3EJ} \begin{bmatrix} 54.0000 & 8.0000 & 28.0000 \\ 8.0000 & 2.0000 & 5.0000 \\ 28.0000 & 5.0000 & 16.0000 \end{bmatrix} .$$

## Example, cont.

The stiffness matrix is found by inversion,

$$\mathbf{K} = \frac{3EJ}{13L^3} \begin{bmatrix} +7.0000 & +12.0000 & -16.0000 \\ +12.0000 & +80.0000 & -46.0000 \\ -16.0000 & -46.0000 & +44.0000 \end{bmatrix}.$$

We are interested in the partitions  $\mathbf{K}_{xx}$  and  $\mathbf{K}_{xg}$ :

$$\mathbf{K}_{xx} = \frac{3EJ}{13L^3} \begin{bmatrix} +7.0000 & +12.0000.0000 \\ +12.0000 & +80.0000.0000 \end{bmatrix}, \quad \mathbf{K}_{xg} = \frac{3EJ}{13L^3} \begin{bmatrix} -16 \\ -46 \end{bmatrix}.$$

The influence matrix is

$$\mathbf{E} = -\mathbf{K}_{xx}^{-1} \mathbf{K}_{xg} = \frac{1}{16} \begin{bmatrix} 28.0000 \\ 5.0000 \end{bmatrix},$$

please compare  $\mathbf{E}$  with the last column of the flexibility matrix,  $\mathbf{F}$ .

Consider the vector of support accelerations,

$$\ddot{\mathbf{x}}_g = \{ \ddot{x}_{gl}, \quad l = 1, \dots, N_g \}$$

and the effective load vector

$$\mathbf{p}_{eff} = -\mathbf{M}\mathbf{E}\ddot{\mathbf{x}}_g = -\sum_{l=1}^{N_g} \mathbf{M}\mathbf{e}_l \ddot{x}_{gl}(t).$$

We can write the modal equation of motion for mode number  $n$

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2q_n = -\sum_{l=1}^{N_g} \Gamma_{nl}\ddot{x}_{gl}(t)$$

where

$$\Gamma_{nl} = \frac{\boldsymbol{\psi}_n^T \mathbf{M}\mathbf{e}_l}{M_n^*}$$

The solution  $q_n(t)$  is hence, with the notation of last lesson,

$$q_n(t) = \sum_{l=1}^{N_g} \Gamma_{nl} D_{nl}(t),$$

$D_{nl}$  being the response function for  $\zeta_n$  and  $\omega_n$  due to the ground excitation  $\ddot{x}_{gl}$ .



The total displacements  $\mathbf{x}_T$  are given by two contributions,  $\mathbf{x}_T = \mathbf{x}_s + \mathbf{x}$ , the expression of the contributions are

$$\mathbf{x}_s = \mathbf{E}\mathbf{x}_g(t) = \sum_{l=1}^{N_g} \mathbf{e}_l x_{gl}(t),$$

$$\mathbf{x} = \sum_{n=1}^N \sum_{l=1}^{N_g} \psi_n \Gamma_{nl} D_{nl}(t),$$

and finally we have

$$\mathbf{x}_T = \sum_{l=1}^{N_g} \mathbf{e}_l x_{gl}(t) + \sum_{n=1}^N \sum_{l=1}^{N_g} \psi_n \Gamma_{nl} D_{nl}(t).$$

For a computer program, the easiest way to compute the nodal forces is

- a) compute, element by element, the nodal displacements by  $\mathbf{x}_T$  and  $\mathbf{x}_g$ ,
- b) use the element stiffness matrix compute nodal forces,
- c) assemble element nodal loads into global nodal loads.

That said, let's see the analytical development...

The forces on superstructure nodes due to deformations are

$$\mathbf{f}_s = \sum_{n=1}^N \sum_{l=1}^{N_g} \Gamma_{nl} \mathbf{K} \boldsymbol{\psi}_n D_{nl}(t)$$

$$\mathbf{f}_s = \sum_{n=1}^N \sum_{l=1}^{N_g} (\Gamma_{nl} \mathbf{M} \boldsymbol{\psi}_n) (\omega_n^2 D_{nl}(t)) = \sum \sum r_{nl} A_{nl}(t)$$

the forces on support

$$\mathbf{f}_{gs} = \mathbf{K}_g^T \mathbf{x}_T + \mathbf{K}_{gg} \mathbf{x}_g = \mathbf{K}_g^T \mathbf{x} + \mathbf{p}_g$$

or, using  $\mathbf{x}_s = \mathbf{E} \mathbf{x}_g$

$$\mathbf{f}_{gs} = \left( \sum_{l=1}^{N_g} \mathbf{K}_g^T \mathbf{e}_l + \mathbf{K}_{gg,l} \right) \mathbf{x}_{gl} + \sum_{n=1}^N \sum_{l=1}^{N_g} \Gamma_{nl} \mathbf{K}_g^T \boldsymbol{\psi}_n D_{nl}(t)$$

The structure response components must be computed considering the structure loaded by all the nodal forces,

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{f}_{gs} \end{Bmatrix}.$$

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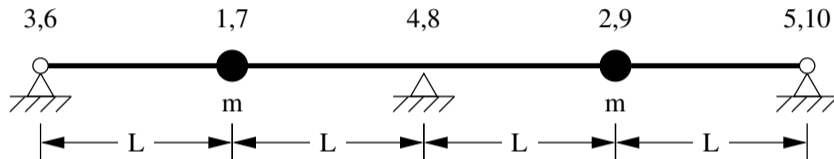
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Response  
Analysis Example



The dynamic *DOF* are  $x_1$  and  $x_2$ , vertical displacements of the two equal masses,  $x_3, x_4, x_5$  are the imposed vertical displacements of the supports,  $x_6, \dots, x_{10}$  are the rotational degrees of freedom (removed by static condensation).

# Example

The stiffness matrix for the 10x10 model is

$$\mathbf{K}_{10 \times 10} = \frac{EJ}{L^3} \begin{bmatrix} 12 & -12 & 0 & 0 & 0 & 6L & 6L & 0 & 0 & 0 \\ -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 & 0 \\ 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L & 0 \\ 0 & 0 & -12 & 24 & -12 & 0 & 0 & -6L & 0 & 6L \\ 0 & 0 & 0 & -12 & 12 & 0 & 0 & 0 & -6L & -6L \\ 6L & -6L & 0 & 0 & 0 & 4L^2 & 2L^2 & 0 & 0 & 0 \\ 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 & 0 \\ 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 & 0 \\ 0 & 0 & 6L & 0 & -6L & 0 & 0 & 2L^2 & 8L^2 & 2L^2 \\ 0 & 0 & 0 & 6L & -6L & 0 & 0 & 0 & 2L^2 & 4L^2 \end{bmatrix}$$

## Example, cont.

The first product of the static condensation procedure is the linear mapping between translational and rotational degrees of freedom, given by

$$\vec{\phi} = \frac{1}{56L} \begin{bmatrix} 71 & -90 & 24 & -6 & 1 \\ 26 & 12 & -48 & 12 & -2 \\ -7 & 42 & 0 & -42 & 7 \\ 2 & -12 & 48 & -12 & -26 \\ -1 & 6 & -24 & 90 & -71 \end{bmatrix} \vec{x}.$$

## Example, cont.

Following static condensation and reordering rows and columns, the partitioned stiffness matrices are

$$\mathbf{K} = \frac{EJ}{28L^3} \begin{bmatrix} 276 & 108 \\ 108 & 276 \end{bmatrix},$$

$$\mathbf{K}_g = \frac{EJ}{28L^3} \begin{bmatrix} -102 & -264 & -18 \\ -18 & -264 & -102 \end{bmatrix},$$

$$\mathbf{K}_{gg} = \frac{EJ}{28L^3} \begin{bmatrix} 45 & 72 & 3 \\ 72 & 384 & 72 \\ 3 & 72 & 45 \end{bmatrix}.$$

The influence matrix is

$$\mathbf{E} = \mathbf{K}^{-1} \mathbf{K}_g = \frac{1}{32} \begin{bmatrix} 13 & 22 & -3 \\ -3 & 22 & 13 \end{bmatrix}.$$



## Example, cont.

The eigenvector matrix is

$$\Psi = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

the matrix of modal masses is

$$M^* = \Psi^T M \Psi = m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

the matrix of the non normalized modal participation coefficients is

$$L = \Psi^T M E = m \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{16} & \frac{11}{8} & \frac{5}{16} \end{bmatrix}$$

and, finally, the matrix of modal participation factors,

$$\Gamma = (M^*)^{-1} L = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{5}{32} & \frac{11}{16} & \frac{5}{32} \end{bmatrix}$$

Denoting with  $D_{ij} = D_{ij}(t)$  the response function for mode  $i$  due to ground excitation  $\ddot{x}_{gj}$ , the response can be written

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} \psi_{11}\left(-\frac{1}{4}D_{11} + \frac{1}{4}D_{13}\right) + \psi_{12}\left(\frac{5}{32}D_{21} + \frac{5}{32}D_{23} + \frac{11}{16}D_{22}\right) \\ \psi_{21}\left(-\frac{1}{4}D_{11} + \frac{1}{4}D_{13}\right) + \psi_{22}\left(\frac{5}{32}D_{21} + \frac{5}{32}D_{23} + \frac{11}{16}D_{22}\right) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}D_{13} + \frac{1}{4}D_{11} + \frac{5}{32}D_{21} + \frac{5}{32}D_{23} + \frac{11}{16}D_{22} \\ -\frac{1}{4}D_{11} + \frac{1}{4}D_{13} + \frac{5}{32}D_{21} + \frac{5}{32}D_{23} + \frac{11}{16}D_{22} \end{pmatrix}. \end{aligned}$$