Written Test, July 7th 2017

Dynamics of Structures 2016-2017

1 Estimation of Structural Parameters

A single storey structure is tested to determine its characteristics.

- First, the structure is statically loaded with an hydraulic jack until the storey displacement is equal to 2 mm; the applied force is equal to 320 kN.
- Next, the static force is instantaneously released and the displacement is recorded at a sampling rate of 250 samples per second — the measurements are collected in a vector $\boldsymbol{x} = \{x_i\}$, $i = 0, 1, 2, \ldots$, where $x_i = x(t_i)$ and $t_i = i/250$.

It is apparent that $x_0 = 2.0\,\text{mm}$ is a maximum, because $\dot{x}_0 \equiv 0$ — but due to sampling effects we don't know *exactly* neither the value nor the location of successive maxima...

What we can do is to list, in the following table, all the measurements x_i such that $x_{i-1} < x_i$ and $x_i > x_{i+1}$; time is in seconds, displacements (given with 3 digits of accuracy) are in millimetres.

i	t_i	x_{i-1}	x_i	x_{i+1}
33	0.132	1.694	1.741	1.726
67	0.268	1.512	1.512	1.460
100	0.400	1.309	1.322	1.288
133	0.532	1.131	1.153	1.134
166	0.664	0.975	1.003	0.996
200	0.800	0.871	0.873	0.843

1. Estimate the damped period of vibration of the system.
2. Estimate the stiffness, the damping ratio and the mass of the system.

Solution

Period of Vibration

From the table we know the position on the time axis of six *sampled maxima*, in particular the 6th one occurs at $t = 800\,\text{ms}$ hence

$$T_D = \frac{800\,\text{ms}}{6} = 133.33\,\text{ms}$$

If we look at the detail of the last maximum, we could recognize that $x_{199} \approx x_{200}$ and consequently a better approximation of the time location of the 6th maximum is $t \approx 798\,\mathrm{ms}$ and a better approximation of the damped period of vibration is $T_D = 133.00\,\mathrm{ms}$.

Stiffness

The static displacement is $x_0 = 2\,\mathrm{mm} = 0.002\,\mathrm{m}$, the equation of static equilibrium is $k\,x_0 = P = 320\,\mathrm{kN} = 320\,000\,\mathrm{N}$ and solving the equation of equilibrium with respect to k gives

$$k = \frac{320 \times 10^3}{2 \times 10^{-3}}\mathrm{N\,m^{-1}} = 160 \times 10^6\,\mathrm{N\,m^{-1}}.$$

Damping ratio

The response of a damped SDoF system is the product of a periodic function, its period T_D and a decaying exponential, $\exp(-\zeta\omega_n t)$, hence

$$\frac{x(0)}{x(mT_D)} = \frac{1}{\exp(-\zeta\omega_n mT_D)} = \exp(\zeta\omega_n m\frac{2\pi}{\omega_D}) = \exp(\frac{\zeta}{\sqrt{1-\zeta^2}}\,2m\pi).$$

Taking the logarithm of the first and the last term, introducing the logarithmic decrement $\delta_m = \log\frac{x(0)}{x(mT_D)}$, we have

$$\delta_m = 2m\pi\,\frac{\zeta}{\sqrt{1-\zeta^2}}$$

and solving formally with respect to ζ

$$\zeta = \frac{\delta_m}{2m\pi}\,\sqrt{1-\zeta^2}.$$

Using the 6th maximum, $x_{200} = 0.873\,\mathrm{mm}$,

$$\delta_6 = \log\frac{2.000}{0.873} = \log 2.29096 = 0.82897$$

and with $m = 6$

$$\zeta = 0.021989030064\,\sqrt{1-\zeta^2}$$

and we can generate the sequence

$$\begin{aligned}\zeta_0 &= 0.000000,\\ \zeta_1 &= 0.021989,\\ \zeta_2 &= 0.021983,\\ \zeta_3 &= 0.021983.\end{aligned}$$

Our estimate for the damping ratio is hence $\zeta = 21.983\,‰$

Mass

From the equation of frequencies we have $m\omega_n^2 = k = 160 \times 10^6\,\mathrm{N\,m^{-1}}$ and we can derive ω_n^2 from $\omega_D^2 = \left(\frac{2\pi}{T_D}\right)^2 = (1-\zeta^2)\,\omega_n^2$.

It is $\omega_D = 47.124\,\mathrm{rad\,s^{-1}}$, $\omega_n = 47.135\,\mathrm{rad\,s^{-1}}$ and $\omega_n^2 = 2221.735\,\mathrm{rad^2\,s^{-2}}$ and eventually

$$m = \frac{160 \times 10^6}{2221.735}\,\mathrm{kg} = 72\,016\,\mathrm{kg}.$$

2 Free Vibrations of a 2 DOF System

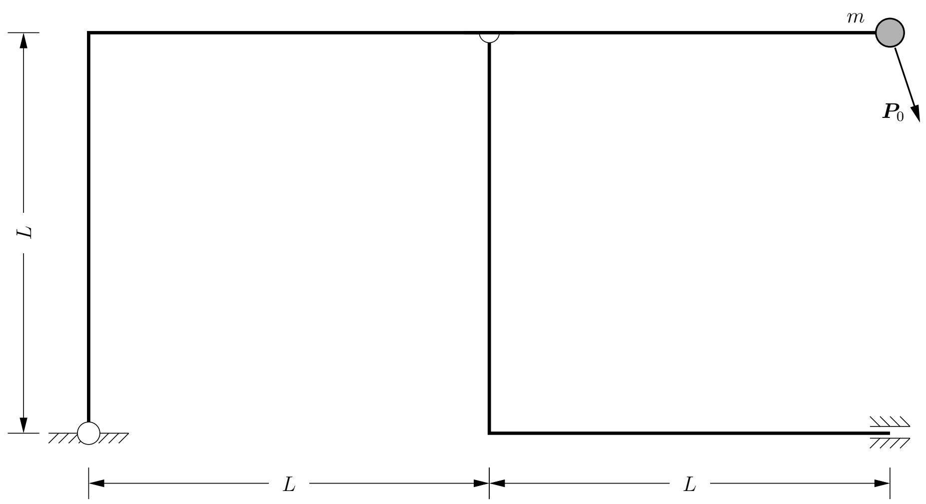

Figure 1: the 2 DOF system.

The undamped structure in figure 1 consists of two uniform beams of negligible mass and supports a body of mass m, hence the dynamic system can be modeled as a 2 DOF system (use x_1 for the downward component of the body displacement and x_2 for the rightward cpmponent).

1. Compute the 2×2 structural matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ in terms of m, L and the flexural stiffness EJ, neglecting the axial and shear deformabilities.

An unknown static force $\boldsymbol{P}_0$ is applied to the mass, deforming the structure, and at time $t = 0$ the static force is suddenly removed, causing the free vibrations of the structure.

At a later time, $t = \pi/\omega_0$ (with $\omega_0^2 = 2EJ/7mL^3$), the displacements are measured and it is

$$\boldsymbol{x}\left(\frac{\pi}{\omega_0}\right) = \begin{Bmatrix} -3 \\ \sqrt{2}-1 \end{Bmatrix}\delta$$

with

$$\delta = \frac{PL^3}{EJ}.$$

2. Determine the static force $\boldsymbol{p}_0$ initially applied to the system.

Solution

1. Initially we have an unknown static load $\boldsymbol{P}_0$ and consequently the initial displacements are $\boldsymbol{x}_0 = \boldsymbol{F}\boldsymbol{P}_0$ or, conversely, $\boldsymbol{P}_0 = \boldsymbol{K}\boldsymbol{x}_0$.

2. Introducing the unknown initial value of the modal coordinates $\boldsymbol{q}_0$ it is $\boldsymbol{x}_0 = \boldsymbol{\Psi}\boldsymbol{q}_0$ and $\boldsymbol{P}_0 = \boldsymbol{K}\boldsymbol{\Psi}\boldsymbol{q}_0$.

3. With the notations $\boldsymbol{x}_\pi = \boldsymbol{x}({}^\pi/_{\omega_0})$ and $\boldsymbol{q}_\pi = \boldsymbol{q}({}^\pi/_{\omega_0})$ it is $\boldsymbol{q}_\pi = \boldsymbol{\Psi}^{-1}\boldsymbol{x}_\pi$ — note that $\boldsymbol{x}_\pi$ is a known quantity.

4. Writing $\omega_i = \lambda_i\omega_0$, because our system starts with $\dot{\boldsymbol{q}}_0 = 0$ the modal responses are given by the expression
$$q_i(t) = q_{i,0}\cos\lambda_i\omega_0 t$$
and substituting $t = \pi/\omega_0$ it is
$$q_{i,\pi} = q_{i,0}\cos\lambda_i\pi$$
or, in other words,
$$\boldsymbol{q}_\pi = \begin{bmatrix}\cos\lambda_1\pi & 0\\ 0 & \cos\lambda_2\pi\end{bmatrix}\boldsymbol{q}_0 \quad\Rightarrow\quad \boldsymbol{q}_0 = \begin{bmatrix}\cos\lambda_1\pi & 0\\ 0 & \cos\lambda_2\pi\end{bmatrix}^{-1}\boldsymbol{\Psi}^{-1}\boldsymbol{x}_\pi.$$

5. Eventually
$$\boldsymbol{P}_0 = \boldsymbol{K}\boldsymbol{\Psi}\begin{bmatrix}\cos\lambda_1\pi & 0\\ 0 & \cos\lambda_2\pi\end{bmatrix}^{-1}\boldsymbol{\Psi}^{-1}\boldsymbol{x}_\pi.$$

Structural Matrices

Using the PVD, it's easy to compute the flexibility matrix,
$$\boldsymbol{F} = \frac{1}{2}\frac{L^3}{EJ}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}$$
and by inversion
$$\boldsymbol{K} = \boldsymbol{F}^{-1} = \frac{2}{7}\frac{EJ}{L^3}\begin{bmatrix}2 & -1\\ -1 & 4\end{bmatrix}.$$

The mass matrix is
$$\boldsymbol{M} = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Equation of Frequencies, Eigenvectors

For free vibrations, $\boldsymbol{x} = \boldsymbol{\psi}\sin\omega t$ and substituting in the equation of motion
$$\left(\frac{2}{7}\frac{EJ}{L^3}\begin{bmatrix}2 & -1\\ -1 & 4\end{bmatrix} - m\omega^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\boldsymbol{\psi}\cos\omega t = \boldsymbol{0}.$$

With $\omega_0^2 = {}^{2EJ}/_{7mL^3}$ and writing $\omega^2 = \lambda^2\omega^2$ the preceding equation can be simplified,

$$\begin{bmatrix} 2-\lambda^2 & -1 \\ -1 & 4-\lambda^2 \end{bmatrix} \boldsymbol{\psi} = \mathbf{0}$$

that has non trivial solutions when

$$\begin{vmatrix} 2-\lambda^2 & -1 \\ -1 & 4-\lambda^2 \end{vmatrix} = \lambda^4 - 6\lambda^2 + 7 = 0,$$

that gives us the eigenvalues of our system

$$\lambda_{1,2}^2 = 3 \mp \sqrt{2}.$$

Substituting in the 2nd of the two equations of free vibrations, it is

$$\psi_{1i} = (1 \pm \sqrt{2})\,\psi_{2i}$$

and collecting the eigenvectors in the eigenvector matrix, with $\psi_{2i} = 1$

$$\mathbf{\Psi} = \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{bmatrix}$$

and

$$\mathbf{\Psi}^{-1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} +1 & \sqrt{2}-1 \\ -1 & \sqrt{2}+1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} +\sqrt{2} & 2-\sqrt{2} \\ -\sqrt{2} & 2+\sqrt{2} \end{bmatrix}$$

2.0.1 Drawing the Conclusion

Substituting the numerical values, we have

$$\lambda^2 = \begin{Bmatrix} 1.58578644 \\ 4.41421356 \end{Bmatrix}, \qquad \lambda = \begin{Bmatrix} 1.25928013 \\ 2.10100299 \end{Bmatrix}, \qquad \lambda\pi = \begin{Bmatrix} 3.9561452 \\ 6.60049556 \end{Bmatrix}$$

$$\cos\lambda\pi = \begin{Bmatrix} -0.68619395 \\ 0.95007809 \end{Bmatrix}, \quad \frac{1}{\cos\lambda\pi} = \begin{Bmatrix} -1.45731392 \\ 1.05254506 \end{Bmatrix},$$

$$\mathbf{\Psi} = \begin{bmatrix} 2.41421356 & -0.41421356 \\ 1.0 & 1.0 \end{bmatrix}, \qquad \mathbf{\Psi}^{-1} = \begin{bmatrix} 0.35355339 & 0.14644661 \\ -0.35355339 & 0.85355339 \end{bmatrix}$$

and finally

$$\boldsymbol{p}_0 = \frac{2}{7}\frac{EJ}{L^3} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2.4142 & -0.4142 \\ 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} -1.4573 & 0 \\ 0 & 1.0525 \end{bmatrix} \begin{bmatrix} 0.3536 & 0.1464 \\ -0.3536 & 0.8536 \end{bmatrix} \begin{Bmatrix} -3 \\ 0.4142 \end{Bmatrix} \frac{PL^3}{EJ}$$
$$= \begin{Bmatrix} 0.8164 \\ 2.5376 \end{Bmatrix} P.$$

A numerical mistake in the last passage is, of course, not relevant...