# Written Test, July 21st 2017

There are many sub-problems, possibly more than you can manage in 4 hours, but I'm prepared to give full credit even to incomplete submissions.
Give priority to the correctness of your answers — I count correct answers only.

## Rigid System

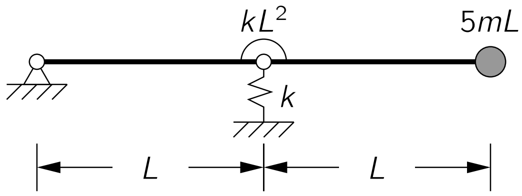


Figure 1: a 2 DoF rigid system.

The dynamic system in figure 1 is composed of two rigid, uniform bars each one of mass $mL$, a massive dimensionless body of mass $5mL$ and two springs.

The system has two degrees of freedom, namely $x_1$, the vertical displacement of the central hinge and $x_2$, the vertical displacement of the massive body.

- Compute an estimate of the natural frequency of vibration of the system using the Rayleigh Quotient method, in terms of its squared value $\omega^2$, using the initial trial vector $\boldsymbol{x}_0 = \{1 \quad 2\}^T$.
- Compute the structural matrices (hint: $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} = 2T$ and $\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} = 2V$).
- Compute a better estimate of said frequency refining your estimates of the strain and kinetic energies of the system.
- Compute the eigenvalues and the corresponding eigenvectors solving the equation of frequencies.

At time $t = 0$ the dynamic system is at rest, when it is excited by a vertical force $P(t) = k\delta \sin\omega_2 t$ applied to the central hinge, where $\omega_2$ is the natural frequency of the second mode of vibration.

- Write the equation of motion in modal coordinates.
- Integrate the equation of motion in modal coordinates.

### Solution

- Kinetic Energy, Mass Matrix

  We have three contributions to the kinetic energy $T$, the contribution from the lumped mass is

$$T_M = \frac{1}{2} \cdot 5mL \cdot \dot{x}_2^2$$

and the contributions from the two rigid bodies, that we can write in terms of the vertical velocity (the horizontal velocity is identicallly equal to zero) of the center of mass and the angular velocity for a generic body

$$T_B = \frac{1}{2} \cdot mL \cdot \dot{x}_G^2 + \frac{1}{2} \cdot mLL^2/12 \cdot \dot{\theta}^2$$

and specializing for our two bodies, whose state is described by $x_1$ and $x_2$, we have

$$T_1 = \frac{1}{2} \cdot mL \cdot (\dot{x}_{\frac{1}{2}})^2 + \frac{1}{2} \cdot mL^3/12 \cdot (\dot{x}_1/L)^2 = \frac{1}{2} \cdot mL \cdot (1/4 + 1/12 = 1/3)\dot{x}_1^2,$$
$$T_2 = \frac{1}{2} \cdot mL \cdot (\dot{x}_2/2 + \dot{x}_{\frac{1}{2}})^2 + \frac{1}{2} \cdot mL^3/12 \cdot (\dot{x}_2/L - \dot{x}_1/L)^2 = \frac{1}{2} \cdot mL(1/3\dot{x}_2^2 + 1/3\dot{x}_1^2 + (2/4 - 2/12 = 1/3)\dot{x}_2\dot{x}_1.$$

Putting it all together, we have

$$T = \frac{1}{2} \cdot mL \cdot ((1/3 + 1/3)\dot{x}_1^2 + 1/3\dot{x}_1\dot{x}_2 + (1/3 + 5)\dot{x}_2^2) = \frac{1}{2} \cdot mL \cdot (2/3\dot{x}_1^2 + 1/3\dot{x}_1\dot{x}_2 + 16/3\dot{x}_2^2).$$

While we are at it, the mass matrix is

$$M = \frac{mL}{6} \begin{bmatrix} 4 & 1 \\ 1 & 32 \end{bmatrix}.$$

- Strain Energy, Stiffness Matrix

  The strain energy is exclusively associated with the springs, the contribution of the extensional spring is $\frac{1}{2} \cdot k \cdot x_1^2$, while for the flexural spring we have to consider the *relative* rotation, $\frac{1}{2} \cdot kL^2 \cdot (\Delta\theta)^2$.

  The relative rotation is

$$\Delta\theta = \theta_2 - \theta 1 = \frac{x_2 - x_1}{L} - \frac{x_1}{L} = \frac{x_2 - 2x_1}{L}$$

  and the strain energy is

$$V = \frac{1}{2} \cdot k \cdot (x_1^2 + x_2^2 - 4x_2x_1 + 4x_1^2) = \frac{1}{2} \cdot k \cdot (5x_1^2 - 4x_2x_1 + x_2^2).$$

  The stiffness matrix is

$$K = k \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}.$$

- Rayleigh Quotient

  With

$$x = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \sin\omega t, \quad \dot{x} = \omega \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \cos\omega t$$

the maximum energies are it is

$$V = \frac{1}{2}\cdot k\cdot(5\cdot 1^2 - 4\cdot 1\cdot 2 + 2^2) = \frac{1}{2}\cdot k\cdot 1$$

and

$$T = \frac{1}{2}\omega^2\cdot mL\cdot(2/3\cdot 1^2 + 1/3\cdot 1\cdot 2 + 16/3\cdot 2^2) = \frac{1}{2}\omega^2\cdot mL\cdot\frac{68}{3}$$

Equating the maximum energies $T - V = 0$ and solving for $\omega^2$

$$\omega^2 = \frac{3}{68}\frac{k}{mL} = 0.0441\omega_0^2$$

- Better Strain Energy

  With $x$ as above, the inertial forces are $f = -M\ddot{x} = \omega^2 Mx\sin\omega t$ and the ensuing displacements, use the name $y$, are $y = K^{-1}f = \omega^2 K^{-1}Mx\sin\omega t$ so that a better approximation of the maximum strain energy is

  $$V = \frac{1}{2}y^T f = \frac{1}{2}\omega^4 x^T MK^{-1}Mx.$$

  Substituting and equating to max $T$,

  $$\omega^2 = 0.03591\omega_0^2.$$

- Better Kinetic Energy

  We use $\dot{y} = \omega K^{-1}Mx\cos\omega t$, the kinetic energy is

  $$T = \frac{1}{2}\dot{y}^T M\dot{y} = \frac{1}{2}\omega^6 x^T MFMFMx,$$

  substituting and equating to the max strain energy we have

  $$\omega^2 = 0.03588\omega_0^2.$$

- Response

  The equation of frequencies is

  $$\det\left(K - \omega^2 M\right) = 0$$

  Expanding the determinant we have

  $$127\Lambda^2 - 1008\Lambda + 36 = 0$$

  $$\Lambda_1 = \frac{504 - \sqrt{508^2 - 126\cdot 36}}{127} = \frac{504 - 78\sqrt{41}}{127} = 0.03587$$
  $$\Lambda_2 = \frac{504 + 78\sqrt{41}}{127} = 7.90113142141$$

  The corresponding, normalized eigenvectors are collected in the eigenvector matrix

$$\Psi = \begin{bmatrix} +0.17071738 & +1.21762297 \\ +0.42348370 & -0.09823497 \end{bmatrix}$$

The equation of motion is

$$mL\bar{M}\ddot{x} + k\bar{K}x = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} k\delta \sin\omega_2 t$$

applying the modal trasformation

$$mLM^*\ddot{q} + kK^*q = \begin{Bmatrix} \psi_{11} \\ \psi_{12} \end{Bmatrix} k\delta \sin\omega_2 qt$$

but eigenvectors are normalized, hence $M^* = I$, and we divide all terms by $mL$

$$\ddot{q} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} q = \begin{Bmatrix} \psi_{11} \\ \psi_{12} \end{Bmatrix} \omega_0^2 \delta \sin\omega_2 t$$

or, writing scalar equations

$$\ddot{q}_1 + \omega_1^2 q_1 = \psi_{11}\omega_0^2 \delta \sin\omega_2 t$$
$$\ddot{q}_2 + \omega_2^2 q_2 = \psi_{12}\omega_0^2 \delta \sin\omega_2 t$$

The integral is

$$q_1(t) = \psi_{11}\delta \frac{\omega_0^2}{\omega_1^2 - \omega_2^2}\left(\sin\omega_2 t - \frac{\omega_2}{\omega_1}\sin\omega_1 t\right)$$
$$q_2(t) = \psi_{12}\delta \frac{\omega_0^2}{\omega_2^2 - \omega_2^2}\left(\sin\omega_2 t - \frac{\omega_2}{\omega_2}\sin\omega_2 t\right) = \psi_{12}\delta \frac{\omega_0^2}{2\omega_2^2}(\sin\omega_2 t - \omega_2 t \cos\omega_2 t)$$

where the last term was derived applying the rule of De L'Hopital to the indetedeterminate form that results from tha standard solution for sine excitation.

# Deformable System

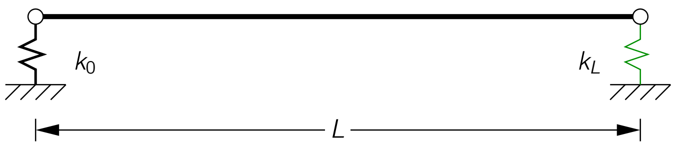


Figure 2: a deformable system.

A uniform beam, its unit mass $m$, its length $L$ and it flexural stiffness $EJ$ is supported at its ends by two springs of different stiffness: $k_0 = 12EJ/L^3$ and $k_L = EJ/12L^3$ (have you noticed that one of the springs is *much* softer than the other one?).

- Rayleigh Quotient

  Choose an appropriate shape function, motivating your choice (have you noticed that one of the springs is very soft?), and compute an approximation to the natural frequency of vibration of the dynamic system.

- Boundary Conditions

  Write the four boundary conditions needed to determine the frequencies and the shapes of vibration of the dynamic system — possibly in terms of the constants of integration.

## Solution

- Rayleigh Quotient

  If we consider the beam subjected to a uniform load $q$, its max displacement with respect to the segment that connects its supports is $\delta_{\max}=\frac{5}{584}\frac{qL^4}{EJ}$ while the support displacements are $\delta_0=\frac{qL/2}{12EJ/L^3}=\frac{1}{24}\frac{qL^4}{EJ}$, $\delta_L=\frac{qL/2}{EJ/12L^3}=6\frac{qL^4}{EJ}$.

  Comparing these values I choose the shape function $\phi=\frac{x}{L}$, i.e., a rigid rotation about the left end considered fixed.

  The maximum value of the strain energy is simply $V_{\max}=\frac{1}{2}kZ_0^2$, for the kinetic energy we write

$$T=\frac{1}{2}Z_0^2\omega^2\int_0^L m\left(\frac{x}{L}\right)^2 dx\cos^2\omega t=\frac{1}{2}Z_0^2\omega^2\frac{mL}{3}\cos^2\omega t$$

  and $T_{\max}=\frac{1}{2}Z_0^2\omega^2\frac{mL}{3}$ so that, equating the max values and solving for $\omega^2$ we have

$$\omega^2=\frac{k_L}{mL/3}=\frac{3EJ}{12mL^4}=\frac{EJ}{4mL^4}.$$

- Boundary Conditions

  The general integral is

$$\psi=A\sin\beta x+B\cos\beta x+C\sinh\beta x+D\cosh\beta x.$$

  The bending moment must be 0 in $x=0$ and $x=L$

$$-EJ\psi''(0)=(B-D)EJ=0\rightarrow D=B.$$
$$-EJ\psi''(L)=(A\sin\beta L+B\cos\beta L-C\sinh\beta L-B\cosh\beta)\beta^2EJ=0$$

  The shear must be equal to the spring reaction, that is directed upwards when $x>0$, hence for a shear positive if clockwise we have

$$V(0)=k_0x(0)\quad\text{and}\quad -V(L)=k_Lx(L).$$

  It is

$$V(x)=-EJ\psi'''(x)=EJ\beta^3(A\cos\beta x-B\sin\beta x-C\cosh\beta x-B\sinh\beta x)$$

  and the equilibrium at $x=0$ is

$$EJ\beta^3(A-C)=\frac{12EJ}{L^3}2B\rightarrow\frac{1}{12}\beta^3L^3(A-C)-2B=0$$

  while the equilibrium at $x=L$ requires that

$$EJ\beta^3(A\cos\beta L-B\sin\beta L-C\cosh\beta L-B\sinh\beta L)=$$
$$=\frac{EJ}{12L^3}(A\sin\beta L+B\cos\beta L+C\sinh\beta L+B\cosh\beta L).$$