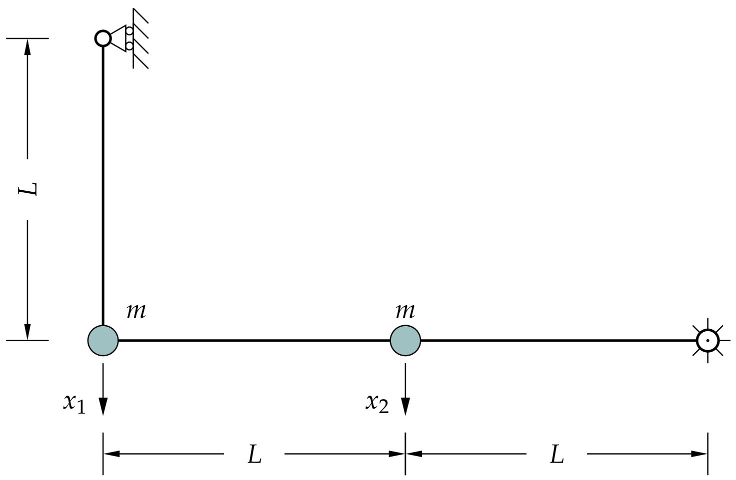
1 2 DOF System



The dynamic system in figure is composed of a single, simply supported uniform beam of negligible mass and two dimensionless bodies, both of mass equal to m; neglecting the axial deformability of the beam the dynamic degrees of freedom are the two vertical displacements of the bodies, x_1 and x_2.

The flexibility and stiffness matrices of the dynamic system, considering only the flexural deformability, are

$$\boldsymbol{F} = \frac{1}{6}\frac{L^3}{EJ}\begin{bmatrix}24 & 15\\15 & 10\end{bmatrix}, \qquad \boldsymbol{K} = \frac{2}{5}\frac{EJ}{L^3}\begin{bmatrix}10 & -15\\-15 & 24\end{bmatrix}$$

where EJ is the flexural stiffness of the beam and L is indicated in figure.

Compute the eigenvalues of the system in terms of the reference value $\omega_0^2 = EJ/(mL^3)$, i.e., $\omega_i^2 = \lambda_i^2\,\omega_0^2$.

Compute the eigenvectors of the system in the following format

$$\boldsymbol{\Psi} = \begin{bmatrix}1 & \psi_{12}\\ \psi_{21} & 1\end{bmatrix}.$$

The system is at rest when it is subjected to a vertical motion of its supports, $\ddot{v}_g$. The modal equations of motion can be written

$$\ddot{q}_i + \lambda_i^2\omega_0^2\,q_i = a_i\,\ddot{v}_g, \qquad i = 1,2.$$

Determine the numerical values of the coefficients a_1 and a_2.

1.1 Solution

The mass matrix of our system is $\boldsymbol{M} = m\boldsymbol{I}$ and the equation of free vibrations is

$$\left(\frac{EJ}{L^3}\begin{bmatrix}+4.000 & -6.000\\-6.000 & +9.600\end{bmatrix} - \omega^2 m\boldsymbol{I}\right)\boldsymbol{\psi} = \boldsymbol{0}$$

that, with the positions given in the text, is equivalent to

$$\begin{bmatrix}+4.000-\lambda^2 & -6.000\\-6.000 & +9.600-\lambda^2\end{bmatrix}\boldsymbol{\psi} = \boldsymbol{0}.$$

A non trivial solution is possible when

$$\det\begin{pmatrix}+4.000-\lambda^2 & -6.000\\ -6.000 & +9.600-\lambda^2\end{pmatrix} = {\lambda^2}^2-(4+9.6)\lambda^2+4\times 9.6-36=0.$$

The roots of ${\lambda^2}^2-2\times 6.8\lambda^2+2.4=0$ are $\lambda^2=6.8\mp\sqrt{6.8^2-2.4}$ and the system eigenvalues are

$$\omega_1^2=0.178822\omega_0^2\quad\text{and}\quad\omega_2^2=13.4212\omega_0^2$$

.

To determine the eigenvectors we will use the first equation,

$$(4-\lambda_i^2)\psi_{1i}=6\psi_{2i}.$$

Substituting, with $\psi_{11}=\psi_{22}=1$ we have

$$\begin{aligned} 4-0.178822=6\psi_{21} &\quad\rightarrow\quad \psi 22=+0.636863\\ (4-13.4212)\psi_{12}=6 &\quad\rightarrow\quad \psi 12=-0.636863 \end{aligned}$$

and we can write

$$\mathbf{\Psi}=\begin{bmatrix}1 & -0.636863\\ +0.636863 & 1\end{bmatrix}.$$

The equivalent force due to the ground acceleration is

$$p_{\text{eff}}=-m\boldsymbol{I}\begin{Bmatrix}1\\1\end{Bmatrix}\ddot{v}_{\text{g}}=-m\begin{Bmatrix}1\\1\end{Bmatrix}\ddot{v}_{\text{g}}$$

and the equation of motion, in modal coordinates, can be written

$$M_i^*\ddot{q}_i+\lambda_i^2M_i^*q_i=-m\mathbf{\Psi}^T\begin{Bmatrix}1\\1\end{Bmatrix}\ddot{v}_{\text{g}}=$$

The modal masses, for $\boldsymbol{M}=m\boldsymbol{I}$, are

$$M_i^*=m\sum_j\psi_{ji}^2$$

and it is $M_1^*=M_2^*=(1^2+0.636863^2)m=1.405595m$, so it is

$$\begin{aligned} a_1&=\frac{-(1+0.636863)m}{1.405595m}=-1.164534\\ a_2&=\frac{-(1-0.636863)m}{1.405595m}=-0.258351 \end{aligned}$$

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2 Single DOF Response

A SDOF system, $m=20\,\text{kg}$, $k=800\,\text{N}\,\text{m}^{-1}$ and $\zeta=11.4\%$ is at rest when it is excited by the load

$$p(t)=\begin{cases}\left(\frac{t}{t_0}-\frac{t^2}{t_0^2}\right)120\,\text{N} & 0\le t\le t_0=1\,\text{s}\\ 0 & \text{otherwise}\end{cases}.$$

Determine the particular integral $\xi=a_0+a_1\frac{t}{t_0}+a_2\frac{t^2}{t_0^2}$ and the displacement at $t=t_0$.

2.1 Solution

The damping coefficient is $c = 2\zeta\sqrt{km} = 28.839\,972\,\mathrm{N\,s\,m^{-1}}$ and the equation of dynamic equilibrium, written for $x(t) = \xi(t)$ is

$$k \times \left(a_0 + \frac{a_1 t}{t_0} + \frac{a_2 t^2}{t_0^2}\right) + c \times \left(\frac{a_1}{t_0} + \frac{2a_2 t}{t_0^2}\right) + m \times \left(\frac{2a_2}{t_0^2}\right) = p_0 \times \left(\frac{t}{t_0} - \frac{t^2}{t_0^2}\right).$$

Substituting the numerical values and collecting the powers of t, we have (all terms are forces)

$$\begin{aligned} -120 &= 800a_2 \\ +120 &= 800a_1 + 57.679945\frac{a_2}{1} \\ 0 &= 800a_0 + 28.839972\frac{a_1}{1} + 40\frac{a_2}{1^2} \end{aligned}$$

Solving (possibly in sequence) the three above equations gives

$$a_0 = 0.001\,703\,\mathrm{m}, \quad a_1 = 0.160\,815\,\mathrm{m}, \quad a_2 = -0.150\,000\,\mathrm{m}.$$

With $\omega_n = \sqrt{k/m} = 6.324\,555\,\mathrm{rad\,s^{-1}}$ and $\omega_D = \omega_n\sqrt{1-\zeta^2} = 6.283\,324\,\mathrm{rad\,s^{-1}}$ the general integral of the response, the sum of the particular integral and the homogeneous solution, is

$$\begin{aligned} x(t) &= \exp(-\zeta\omega_n t)\,(A\cos\omega_D t + B\sin\omega_D t) + \xi(t) \\ \dot{x}(t) &= \exp(-\zeta\omega_n t)\,(\omega_D(B\cos\omega_D t - A\sin\omega_D t) - \zeta\omega_n(A\cos\omega_D t + B\sin\omega_D t)) + \dot{\xi}(t) \end{aligned}$$

Imposing that the initial displacement and the initial velocity are equal to zero we have

$$\begin{aligned} 0 &= A + a_0 \\ 0 &= \omega_D B - \zeta\omega_n A + a_1/1 \end{aligned}$$

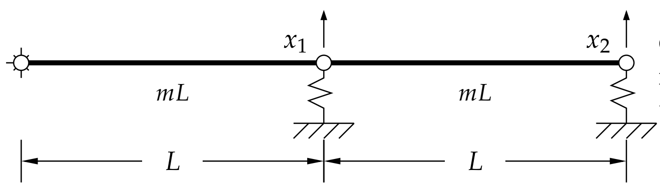
Substituting the numerical values and solving, it is

$$A = -0.001\,703\,\mathrm{m}, \quad B = -0.025\,789\,\mathrm{m}.$$

Substituting in the general integral and evaluating the response for $t = t_0$ it is

$$x(t_0) = 0.011\,688\,\mathrm{m}.$$

3 Mass Matrix



Compute the mass matrix of the system in figure.

3.1 Solution

For a rigid rod of mass M and rotatory inertia I moving in a plane the kinetic energy is

$$T = \frac{1}{2}\left(M(\dot{x}_G^2 + \dot{y}_G^2) + I\dot{\phi}^2\right),$$

where $\dot{x}_G, \dot{y}_G$ are the components of the velocity of the centroid and $\dot{\phi}$ is the angular velocity of the body.

With $M = mL$ and $I = mL^3/12$ we can write the kinetic energy of the two rigid bodies in the system as

$$T_1 = \frac{1}{2}\left(mL\left(\frac{\dot{x}_1}{2}\right)^2 + \frac{mL^3}{12}\left(\frac{\dot{x}_1}{L}\right)^2\right),$$
$$T_2 = \frac{1}{2}\left(mL\left(\frac{\dot{x}_2 + \dot{x}_1}{2}\right)^2 + \frac{mL^3}{12}\left(\frac{\dot{x}_2 - \dot{x}_1}{L}\right)^2\right)$$

summing and expanding

$$T = \frac{1}{2}\,mL\,\left(\dot{x}_1^2/4 + \dot{x}_1^2/12 + \dot{x}_1^2/4 + 2\dot{x}_1\dot{x}_2/4 + \dot{x}_2^2/4 + \dot{x}_1^2/12 - 2\dot{x}_1\dot{x}_2/12 + \dot{x}_2^2/12\right)$$

and simplifying

$$T = \frac{1}{2}\,mL\,\left(\left(\frac{2}{4} + \frac{2}{12}\right)\dot{x}_1^2 + \left(\frac{1}{4} - \frac{1}{12}\right)2\dot{x}_1\dot{x}_2\left(\frac{1}{4} + \frac{1}{12}\right)\dot{x}_2^2\right) = \frac{1}{2}\,mL\,\left(\frac{4}{6}\dot{x}_1^2 + 2\frac{1}{6}\dot{x}_1\dot{x}_2 + \frac{2}{6}\dot{x}_2^2\right)$$

By comparison with

$$T = \frac{1}{2}\,\boldsymbol{x}^T\boldsymbol{M}\,\boldsymbol{x}$$

it is evident that

$$\boldsymbol{M} = \frac{mL}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}.$$