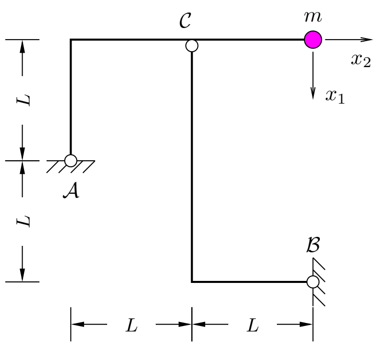
2 DOF System

Giacomo Boffi



The dynamical system in figure is composed of two massless, uniform beams supporting a dimensionless body of mass m.

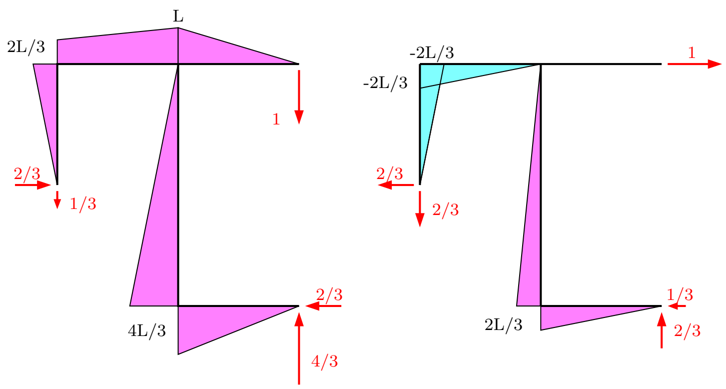
With $\omega_0^2 = (EJ)/(mL^3)$ determine the system's eigenvalues $\omega_i^2 = \lambda_i^2\,\omega_0^2$ and the mass normalized eigenvectors.

The system is subjected to a horizontal motion $u_{\mathcal{B}} = u_{\mathcal{B}}(t)$, determine the mass displacements $\boldsymbol{x}_{\mathcal{B}} = \boldsymbol{r}\,u_{\mathcal{B}}$ and write the modal equations of dynamic equilibrium as

$$\ddot{q}_i + \lambda_i^2\omega_0^2 q_i = \alpha_i\,\ddot{u}_{\mathcal{B}}$$

determining the numerical values of the α_i. ## Structural matrices

The flexibility is computed using the Principle of Virtual Displacements,



the stiffness is computed by inversion and the mass matrix is the unit matrix multiplied by m, $\boldsymbol{M} = m\,\boldsymbol{I}$.

```
l = [1, 1, 1, 2, 1]
m = [[p( 2/3, 0), p(-1/3, 1), p(1, 0), p(2/3, 0), p(4/3, 0)],
     [p(-2/3, 0), p(-2/3, 0), p(0, 0), p(1/3, 0), p(2/3, 0)]]

F = array([[vw(emme, chi, l) for emme in m] for chi in m])
K = inv(F)
M = eye(2)

dl(dmat(r'\boldsymbol{F}=\frac{1}{27}\frac{L^3}{EJ}', F*27, r','))
dl(dmat(r'\boldsymbol{K}=\frac{1}{53}\frac{EJ}{L^3}', K*53, r','))
dl(dmat(r'\boldsymbol{M}=m', M,'.', fmt='%d'))
```

$$\boldsymbol{F} = \frac{1}{27}\frac{L^3}{EJ}\begin{bmatrix} +80 & +13 \\ +13 & +20 \end{bmatrix}, \tag{1}$$

$$\boldsymbol{K} = \frac{1}{53}\frac{EJ}{L^3}\begin{bmatrix}+20 & -13\\ -13 & +80\end{bmatrix}, \tag{2}$$

$$\boldsymbol{M} = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{3}$$

0.1 The eigenvalues problem

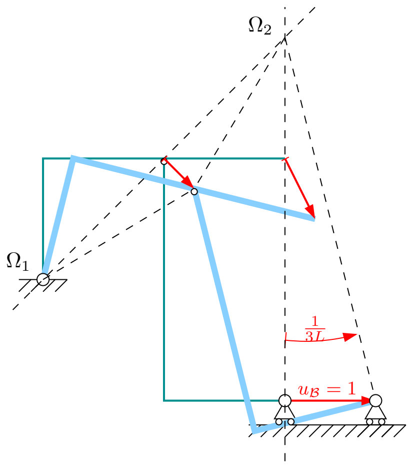
```
wn2, Psi = eigh(K, M) ; Psi[:,0] *= -1
Lambda2 = diag(wn2)

dl(dmat(r'\boldsymbol{\Lambda^2}=', Lambda2, r'.'))
dl(dmat(r'\boldsymbol{\Psi}=', Psi,  r'.'))
```

$$\boldsymbol{\Lambda^2} = \begin{bmatrix}+0.326499 & +0\\ +0 & +1.56029\end{bmatrix}. \tag{4}$$

$$\boldsymbol{\Psi} = \begin{bmatrix}+0.979172 & -0.203032\\ +0.203032 & +0.979172\end{bmatrix}. \tag{5}$$

0.2 Mass Displacements and Inertial Forces

We downgrade the hinge in $\mathscr{B}$ to permit a unit horizontal displacement and observe that the centre of instantaneous rotation for the lower beam is at the intersection of the vertical in $\mathscr{B}$ and the line connecting $\mathscr{A}$ and $\mathscr{C}$.



The angle of rotation of the lower beam is $\theta_2 = 1/3L$, anti-clockwise and the angle of rotation of the upper beam (that rotates about the hinge in $\mathscr{A}$) is equal but clockwise, due to the continuity of displacements in the internal hinge $\mathscr{C}$.

Knowing the angle of rotation of the upper beam, the mass displacements are

$$x_1 = 2L\frac{1}{3L} = \frac{2}{3} \quad \text{and} \quad x_2 = L\frac{1}{3L} = \frac{1}{3}.$$

We can eventually write

$$\boldsymbol{x}_{\text{tot}} = \boldsymbol{x} + \begin{Bmatrix}2/3\\ 1/3\end{Bmatrix} u_{\mathscr{B}}$$

and the inertial force can be written as

$$\boldsymbol{f}_{\text{I}} = \boldsymbol{M}\,\ddot{\boldsymbol{x}}_{\text{tot}} = \boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{M}\begin{Bmatrix}2/3\\ 1/3\end{Bmatrix} \ddot{u}_{\mathscr{B}}$$

0.3 The Modal Equations of Motion

The equation of motion, in structural coordinates, is

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = -\boldsymbol{M}\begin{Bmatrix}2/3\\1/3\end{Bmatrix}\ddot{u}_{\mathcal{B}} \tag{6}$$

or, because $\boldsymbol{M} = m\,\boldsymbol{I}$,

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = -m\begin{Bmatrix}2/3\\1/3\end{Bmatrix}\ddot{u}_{\mathcal{B}}$$

Using the modal expansion, $\boldsymbol{x} = \boldsymbol{\Psi}\,\boldsymbol{q}$ and premultiplying term by term by $\boldsymbol{\Psi}^T$ we have, because the eigenvectors are normalized w/r to the mass matrix,

$$m\,\boldsymbol{I}\,\ddot{\boldsymbol{q}} + m\omega_0^2\,\boldsymbol{\Lambda}^2\,\boldsymbol{q} = -m\,\boldsymbol{\Psi}^T\begin{Bmatrix}2/3\\1/3\end{Bmatrix}\ddot{u}_{\mathcal{B}}.$$

```
r = array((2/3, 1/3))
a = -Psi.T@r
print('The modal equations of motion are')

for i, (ai, l2i) in enumerate(zip(a, wn2), 1):
    dl(r'$$\ddot q_{%d} %+.6f\,\omega_0^2\,q_{%d} = %+.6f\,\ddot u_\mathcal{B}$$' %
        (i, l2i, i, ai))
```

```
The modal equations of motion are
```

$$\ddot{q}_1 + 0.326499\,\omega_0^2\,q_1 = -0.720459\,\ddot{u}_{\mathcal{B}}$$

$$\ddot{q}_2 + 1.560294\,\omega_0^2\,q_2 = -0.191036\,\ddot{u}_{\mathcal{B}}$$

```
r = array((1, 1))
a = -Psi.T@r
print('The modal equations of motion are not')

for i, (ai, l2i) in enumerate(zip(a, wn2), 1):
    dl(r'$$\ddot q_{%d} %+.6f\,\omega_0^2\,q_{%d} = %+.6f\,\ddot u_\mathcal{B}$$' %
        (i, l2i, i, ai))
```

```
The modal equations of motion are not
```

$$\ddot{q}_1 + 0.326499\,\omega_0^2\,q_1 = -1.182204\,\ddot{u}_{\mathcal{B}}$$

$$\ddot{q}_2 + 1.560294\,\omega_0^2\,q_2 = -0.776140\,\ddot{u}_{\mathcal{B}}$$

0.4 Initialization

```
from numpy import array, diag, eye, poly1d
from scipy.linalg import eigh, inv

def p(*l): return poly1d(l)
def vw(M, X, L):
    return sum(p(l)-p(0) for (m, χ, l) in zip(M, X, L) for p in ((m*χ).integ(),))
def dmat(pre, mat, post, mattype='b', fmt='%+.6g'):
    s = r'\begin{align}' + pre + r'\begin{%smatrix}'%mattype
    s += r'\\'.join('&'.join(fmt%val for val in row) for row in mat)
    s += r'\end{%smatrix}'%mattype + post + r'\end{align}'
    return s
def dl(ls):
    display(Latex(ls))
    return None
```