Written Test

Dynamics of Structures

July 2, 2018

1 2 DoF system — Support Motion

$$\boldsymbol{K} = \frac{3}{8}\frac{EJ}{L^3}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}, \qquad \boldsymbol{M} = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = m\boldsymbol{I}.$$

The eq. of free vibrations (multiplied by a suitable constant) is

$$\frac{8}{3}\frac{L^3}{EJ}\left(\frac{3}{8}\frac{EJ}{L^3}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix} - m\omega^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\boldsymbol{\psi}\sin\omega t = \boldsymbol{0}$$

or, with $\Lambda = {}^8\!/_3\ {}^{\omega^2}\!/_{\omega_0^2}$

$$\left(\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix} - \Lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\boldsymbol{\psi} = \boldsymbol{0}$$

A non trivial solution is possible when

$$\det\begin{pmatrix}3-\Lambda & -1\\ -1 & 3-\Lambda\end{pmatrix} = \Lambda^2 - 6\Lambda + 8 = 0 \rightarrow \Lambda_1 = 2,\ \Lambda_2 = 4$$

or

$$\omega_1^2 = {}^3\!/_4\omega_0^2,\ \omega_2^2 = {}^3\!/_2\omega_0^2.$$

We can write, from the first of the equations of free vibrations,

$$\psi_{2i} = (3-\Lambda_i)\psi_{1i}$$

that, upon substitution of $\psi_{1i} \equiv 1$, gives

$$\boldsymbol{\Psi} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

The resulting modal mass matrix is

$$\boldsymbol{M}^* = \boldsymbol{\Psi}^T\boldsymbol{M}\,\boldsymbol{\Psi} = m\boldsymbol{\Psi}^T\boldsymbol{\Psi} = 2m\boldsymbol{I} \rightarrow \boldsymbol{M}^{*-1} = \frac{1}{2m}\boldsymbol{I}.$$

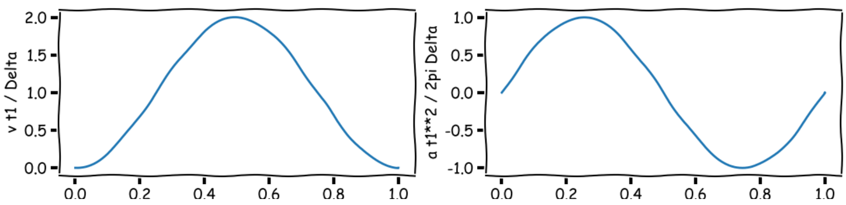
Writing our results only for $0 \le t \le t_1$, it is

$$\dot{u} = \frac{\Delta}{t_1}\left(1 - \cos(2\pi t/t_1)\right), \qquad \ddot{u} = \frac{2\pi\Delta}{t_1^2}\sin(2\pi t/t_1)$$

and, because ${}^{2\pi}\!/_{t_1} = 16\omega_0$ and ${}^{1}\!/_{t_1} = {}^{16\omega_0}\!/_{2\pi}$, it is

$$\ddot{u} = \frac{2\pi\Delta}{t_1^2}\sin(2\pi t/t_1) = \frac{256\Delta\omega_0^2}{2\pi}\sin(16\omega_0 t).$$

The sketches of the support velocity and acceleration are



The displacements of the mass are $\boldsymbol{x}_{\text{tot}} = \boldsymbol{x} + \boldsymbol{E}u_A$, where, by analysis of the rigid motion,

$$\boldsymbol{E} = \begin{Bmatrix}1\\0\end{Bmatrix}$$

so the equation of the motion is

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{E}\ddot{u} = -m\boldsymbol{E}\ddot{u}.$$

The modal equations are

$$\boldsymbol{M}^*\ddot{\boldsymbol{q}} + \boldsymbol{K}^*\boldsymbol{q} = -m\boldsymbol{\Psi}^T\boldsymbol{E}\ddot{u}$$

and, with $\lambda_i^2 = {}^3\!/_4, {}^3\!/_2$ and $\boldsymbol{\Psi}^T\boldsymbol{E} = \begin{Bmatrix}1 & 1\end{Bmatrix}^T$

$$2m\ddot{q}_i + 2m\lambda_i^2\omega_0^2 q_i = -m\ddot{u} = -\frac{128}{\pi}m\Delta\omega_0^2\sin(16\omega_0 t), \qquad i = 1, 2.$$

Simplifying

$$\ddot{q}_i + \lambda_i^2\omega_0^2 q_i = -\frac{64}{\pi}\Delta\omega_0^2\sin(16\omega_0 t)$$

and, considering that we start from rest conditions

$$q_i(t) = \Delta C_i\left(\sin(16\omega_0 t) - \frac{16}{\lambda_i}\sin(\lambda_i\omega_0 t)\right), \qquad C_i = \frac{64}{\pi}\frac{1}{256 - \lambda_i^2}$$

The displacement $x_1(t_1/2)$ is as follows,

$$x_1(t_1/2) = q_1(t_1/2) + q_2(t_1/2)$$

and because it is $16\,\omega_0 t_1/2 = \pi$ and $\lambda_i\omega_0 t_1/2 = {}^{\lambda_i}\!/_{16}\pi$

$$x_1(t_1/2) = \Delta\sum -C_i\frac{16}{\lambda_i}\sin(\frac{\lambda_i}{16}\pi)$$

2 Rayleigh Quotient

$$\boldsymbol{k} = k\{5,4,3\}, \quad \boldsymbol{m} = m\{5,4,4\},$$

$$\boldsymbol{x}_0 = Z_0\{1,2,3\}\sin\omega t, \dot{\boldsymbol{x}}_0 = \omega Z_0\{1,2,3\}\cos\omega t, \ddot{\boldsymbol{x}}_0 = -\omega^2 Z_0\{1,2,3\}\sin\omega t,$$

$$\boldsymbol{\delta}_0 = Z_0\{1,1,1\}\sin\omega t$$

$$V_{\max} = \frac{1}{2}\boldsymbol{\delta}_0^2 \cdot \boldsymbol{k} = \frac{1}{2}k(1\times 5 + 1\times 4 + 1\times 3)Z_0^2 = \frac{1}{2}\,12kZ_0^2$$

$$T_{\max} = \frac{1}{2}\dot{\boldsymbol{x}}_0^2 \cdot \boldsymbol{m} = \frac{1}{2}m\omega^2(1\times 5 + 4\times 4 + 9\times 4)Z_0^2 = \frac{1}{2}m\omega^2 57Z_0^2$$

It is

$$R_{00} = \frac{12}{57}\,\frac{k}{m} = \frac{4}{19}\,\frac{k}{m} = 0.21052631578947367\,\frac{k}{m}$$

The inertial forces associated with $\boldsymbol{\phi}_0$ are

$$\boldsymbol{f}_{\mathrm{I}} = \{-m_i\ddot{x}_i\} = m\omega^2\{5\times 1, 4\times 2, 4\times 3\}Z_0\sin\omega t = m\omega^2\{5,8,12\}Z_0\sin\omega t,$$

the storey shears are

$$\boldsymbol{s} = m\omega^2\{25,20,12\}Z_0\sin\omega t,$$

the storey deflections are

$$\boldsymbol{\delta}_1 = s_i/k_i = m\omega^2/k\{25/5, 20/4, 12/3\}Z_0\sin\omega t = m\omega^2/k\{5,5,4\}Z_0\sin\omega$$

and the displacements are

$$\boldsymbol{x}_1 = m\omega^2/k\{5,10,14\}Z_0\sin\omega.$$

The new approximation to $V_{\max}$ is

$$V_{\max} = \frac{1}{2}\sum\delta_{1,i}f_{\mathrm{I},i} = \frac{1}{2}\,m^2\omega^4(5\times 5 + 8\times 10 + 12\times 14)Z_0^2 = \frac{1}{2}\,m^2\omega^4 273Z_0^2$$

and equating to $T_{\max}$

$$R_{01} = \frac{57}{273}\,\frac{k}{m} = 0.2087912087912088\frac{k}{m}.$$

The new velocities are

$$\dot{\boldsymbol{x}}_1 = m\omega^3/k\{5,10,14\}Z_0\sin\omega t$$

and

$$T_{\max} = \frac{1}{2}\sum\dot{x}_{1,i}^2 m_i = \frac{1}{2}\,m^3\omega^6/k^2(5^2 5 + 10^2 4 + 14^2 4)Z_0^2 = \frac{1}{2}\,m^3\omega^6/k^2 1309Z_0^2$$

and, equating to $V_{\max}$, we have

$$R_{11} = \frac{273}{1309}\,\frac{k}{m} = 0.20855614973262032\frac{k}{m}.$$